Diffusion determines the compact manifold

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Abstract

We provide a short proof for the theorem that two compact Riemannian manifolds are isomorphic if and only there exists an order isomorphism which intertwines between the heat semigroups on the manifolds.

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1 Introduction

Two years before the publication of Kac's famous paper [Kac] 'Can one hear the shape of a drum' Milnor [Mil] gave a counter example showing that one cannot hear the shape of a compact Riemannian manifold. Milnor presented two 16-dimensional Riemannian manifolds for which the associated Laplace–Beltrami operators have the same spectrum, i.e. are isospectral. The latter is equivalent with the existence of a unitary operator U which intertwines the heat semigroups on the compact manifolds. The heat semigroups are positive, which means that they map positive functions (i.e. positive heat) to positive functions on the L_2-spaces of the compact manifolds. In this paper we replace the unitary operator by an order isomorphism, i.e. a linear bijective mapping U such that $U\varphi \geq 0$ if and only if $\varphi \geq 0$. Then we show that the manifolds are indeed isomorphic. This may be interpreted in the following way. The heat semigroups are positive, which means that positive functions (heat densities) are mapped to positive functions. The orbit corresponding to a positive initial value describes the propagation of the heat density, i.e. the diffusion. Thus to say that an order isomorphism intertwines between two heat semigroups means that the positive orbits are mapped to positive orbits. So our result may be rephrased by saying that diffusion determines the compact manifold. For open connected subsets of $\mathbf{R}^d$ satisfying a weak smoothness condition Arendt [Are2] proved that diffusion determines the body (see also [Are1]). In a recent paper [ABE] this was extended to connected Riemannian manifolds satisfying the same smoothness condition. Every compact connected Riemannian manifold satisfies this smoothness condition.

The aim of this paper is to give a direct and short proof that diffusion determines the body for compact Riemannian manifolds. The compact Riemannian manifolds do not have to be connected.

Let (M, g) be a compact Riemannian manifold of dimension d. Then M has a natural Radon measure with respect to which we define the L_p-spaces on M. Set

$$H^1(M) = \{\varphi \in L_2(M) : \varphi \circ x^{-1} \in H^1(x(V)) \text{ for every chart } (V, x)\} \quad .$$

If $\varphi \in H^1(M)$ and (V, x) is a chart on M then set $\frac{\partial}{\partial x^i}\varphi = (D_i(\varphi \circ x^{-1})) \circ x \in L_2(V)$, where D_i denotes the partial derivative in $\mathbf{R}^d$. Moreover, for all $\varphi, \psi \in H^1(M)$ there exists a unique element $\nabla\varphi \cdot \nabla\psi \in L_1(M)$ such that

$$\nabla\varphi \cdot \nabla\psi\Big|_V = \sum_{i,j=1}^{d} g^{ij} \Big(\frac{\partial}{\partial x^i}\varphi\Big)\Big(\frac{\partial}{\partial x^j}\psi\Big)$$

for every chart (V, x) on M. Set $|\nabla\varphi| = (\nabla\varphi \cdot \nabla\varphi)^{1/2}$. We provide $H^1(M)$ with the norm $\varphi \mapsto (\|\varphi\|_2^2 + \| \, |\nabla\varphi| \, \|_2^2)^{1/2}$. Then $H^1(M)$ is a Hilbert space. Define the bilinear form $a\colon H^1(M) \times H^1(M) \to \mathbf{R}$ by $a(\psi, \varphi) = \int \nabla\psi \cdot \nabla\varphi$. Then a is a closed and positive form in $L_2(M)$. The **Dirichlet Laplace–Beltrami operator** Δ on M is the associated self-adjoint operator. If (V, x) is a chart on M then

$$\Delta\,\varphi = -\sum_{i,j=1}^{d} \frac{1}{\sqrt{g}}\,\frac{\partial}{\partial x^i}\, g^{ij}\,\sqrt{g}\,\frac{\partial}{\partial x^j}\,\varphi$$

for all $\varphi \in C_c^\infty(V)$. Let S be the semigroup on $L_2(M)$ generated by $-\Delta$ and let $p \in [1, \infty)$. By the Beurling–Deny criteria the operator $S_t|_{L_2(M) \cap L_p(M)}$ extends to a positive contraction operator on $L_p(M)$ for all $t > 0$. Moreover, $S^{(p)}$ is a C_0-semigroup. Since the semigroup S has a smooth kernel satisfying Gaussian bounds ([Sal] Theorem 5.4.12), it follows that $S_t C(M) \subset C(M)$ and $S|_{C(M)}$ is a C_0-semigroup on $C(M)$.

If (M_1, g_1) and (M_2, g_2) are two compact Riemannian manifolds then a map $\tau\colon M_1 \to M_2$ is called an **isometry** if it is a C^∞-diffeomorphism and

$$g_2|_{\tau(p)}(\tau_*(v), \tau_*(w)) = g_1|_p(v, w)$$

for all $p \in M_1$ and $v, w \in T_p M_1$. The Riemannian manifolds (M_1, g_1) and (M_2, g_2) are called **isomorphic** if there exists an isometry from M_1 onto M_2. If $\tau\colon M_1 \to M_2$ is an isometry and $p \in [1, \infty]$ then $\varphi \circ \tau \in L_p(M_1)$ and

$$\|\varphi \circ \tau\|_{L_p(M_1)} = \|\varphi\|_{L_p(M_2)} \tag{1}$$

for all $\varphi \in L_p(M_2)$.

A linear operator $U\colon E \to F$ between two Riesz spaces is said to be a **lattice homomorphism** if

$$U(\varphi \wedge \psi) = (U\varphi) \wedge (U\psi)$$

for all $\varphi, \psi \in E$. For alternative equivalent definitions see [AlB] Theorem 7.2. Each lattice homomorphism U is positive, i.e. $\varphi \geq 0$ implies $U\varphi \geq 0$. An **order isomorphism** $U\colon E \to F$ is a bijective mapping such that $U\varphi \geq 0$ if and only if $\varphi \geq 0$. Equivalently, U is an order isomorphism if and only if U is a bijective lattice homomorphism. Then also U^{-1} is an order isomorphism. Recall also that each positive operator between L_p-spaces, or from $C(M_1)$ into $C(M_2)$ where M_1 and M_2 are compact Hausdorff spaces, is continuous by [AlB] Theorem 12.3.

The main theorem of this paper is the following.

Theorem 1.1 *Let (M_1, g_1) and (M_2, g_2) be two compact Riemannian manifolds. Let $p \in [1, \infty)$. For all $j \in \{1, 2\}$ let Δ_j be the Laplace–Beltrami operator on M_j and let $S^{(j)}$ and $T^{(j)}$ be the associated semigroups on $L_p(M_j)$ and $C(M_j)$. Then the following three conditions are equivalent.*

I. *(M_1, g_1) and (M_2, g_2) are isomorphic.*

II. *There exists an order isomorphism $U\colon L_p(M_1) \to L_p(M_2)$ such that*

$$U S_t^{(1)} = S_t^{(2)} U$$

for all $t > 0$.

III. *There exists an order isomorphism $U\colon C(M_1) \to C(M_2)$ such that*

$$U T_t^{(1)} = T_t^{(2)} U$$

for all $t > 0$.

Moreover, if the manifolds are connected and if U is an order isomorphism as in Condition II *or* III *then there exist $c > 0$ and a (surjective) isometry $\tau\colon M_2 \to M_1$ such that $U\varphi = c\, \varphi \circ \tau$ for all $\varphi \in L_p(M_1)$.*

The implications I⇒II and I⇒III are an easy consequence of (1).

2 Proof of Theorem 1.1

The first part in the proof of Theorem 1.1 is the observation that C^∞-functions are invariant under intertwining operators.

Lemma 2.1 *Let (M_1, g_1) and (M_2, g_2) be two compact Riemannian manifolds. Let $p \in [1, \infty)$. For all $j \in \{1, 2\}$ let Δ_j be the Laplace–Beltrami operator on M_j and let $S^{(j)}$ and $T^{(j)}$ be the associated semigroups on $L_p(M_j)$ and $C(M_j)$. Let either $U \colon L_p(M_1) \to L_p(M_2)$ be an order isomorphism such that*

$$US_t^{(1)} = S_t^{(2)} U$$

for all $t > 0$, or $U \colon C(M_1) \to C(M_2)$ be an order isomorphism such that

$$UT_t^{(1)} = T_t^{(2)} U \tag{2}$$

for all $t > 0$. Then

(i) $UC^\infty(M_1) = C^\infty(M_2)$.

(ii) $U\varphi \geq 0$ *if and only if* $\varphi \geq 0$, *for all* $\varphi \in C^\infty(M_1)$.

(iii) $(U\varphi)(U\psi) = 0$ *for all* $\varphi, \psi \in C^\infty(M_1)$ *with* $\varphi\,\psi = 0$.

(iv) $\Delta_2 U\varphi = U\Delta_1\varphi$ *for all* $\varphi \in C^\infty(M_1)$.

Proof Suppose U is an order isomorphism from $C(M_1)$ onto $C(M_2)$. Let H_j be the generator of $T^{(j)}$ for all $j \in \{1, 2\}$. If $\varphi \in D(H_1)$ then it follows from (2) that

$$\tfrac{1}{t}(I - T_t^{(2)})U\varphi = \tfrac{1}{t}U(I - T^{(1)})\varphi$$

for all $t > 0$. Since U is continuous one deduces that $U\varphi \in D(H^{(2)})$. So $UD(H_1) \subset D(H_2)$ and $H_2 U\varphi = UH_1\varphi$ for all $\varphi \in D(\Delta_1)$. Similarly $U^{-1}D(H_2) \subset D(H_1)$ and therefore $UD(H_1) = D(H_2)$. Hence by iteration $U \bigcap_{n=1}^\infty D(H_1^n) = \bigcap_{n=1}^\infty D(H_2^n)$. But $C^\infty(M_j) = \bigcap_{n=1}^\infty D(H_j^n)$ for all $j \in \{1, 2\}$ by elliptic regularity. Here we use that the manifolds are compact. This shows (i) and (iv). Property (ii) follows since U is an order isomorphism. Moreover, $|U\varphi| = U|\varphi|$ for all $\varphi \in C(M_1)$. Hence if $\varphi, \psi \in C(M_1)$ and $\varphi\,\psi = 0$ then $|\varphi| \wedge |\psi| = 0$ and $|U\varphi| \wedge |U\psi| = U|\varphi| \wedge U|\psi| = U(|\varphi| \wedge |\psi|) = 0$. Therefore $|(U\varphi)(U\psi)| = |U\varphi|\,|U\psi| = 0$ and $(U\varphi)(U\psi) = 0$. This implies Property (iii).

The proof on the L_p-spaces is similar. □

The next lemma is a C^∞-version of the Riesz representation theorem. (Cf. [EvG] Corollary 1.8.1.)

Lemma 2.2 *Let M be a compact Riemannian manifold and $F \colon C^\infty(M) \to \mathbf{R}$ a positive linear functional such that*

$$F(\varphi)\,F(\psi) = 0 \text{ for all } \varphi, \psi \in C^\infty(M) \text{ with } \varphi\,\psi = 0. \tag{3}$$

Then there exist $c \in [0, \infty)$ and $p \in M$ such that $F(\varphi) = c\,\varphi(p)$ for all $\varphi \in C^\infty(M)$.

Proof Let $\varphi \in C^\infty(M)$. Then $\|\varphi\|_\infty \mathbb{1} - \varphi \geq 0$, so it follows from positivity that $F(\varphi) \leq F(\mathbb{1}) \|\varphi\|_\infty$. Since $C^\infty(M)$ is dense in $C(M)$ one can extend F to a continuous linear function from $C(M)$ into $\mathbf{R}$. This extension is again positive since positive functions in $C(M)$ can be approximated uniformly by positive functions in $C^\infty(M)$. By the Riesz representation theorem there exists a unique Radon measure μ on M such that $F(\varphi) = \int \varphi \, d\mu$ for all $\varphi \in C^\infty(M)$. Then it follows from (3) that μ is a point measure. Hence there exist $p \in M$ and $c \in [0, \infty)$ such that $F(\varphi) = c\, \varphi(p)$ for all $\varphi \in C^\infty(M)$. □

Proposition 2.3 *Let (M_1, g_1) and (M_2, g_2) be two compact Riemannian manifolds. Suppose there exists a linear bijection $U \colon C^\infty(M_1) \to C^\infty(M_2)$ such that*

(i) $U\varphi \geq 0$ *if and only if* $\varphi \geq 0$, *for all* $\varphi \in C^\infty(M_1)$.

(ii) $(U\varphi)(U\psi) = 0$ *if and only if* $\varphi \, \psi = 0$, *for all* $\varphi, \psi \in C^\infty(M_1)$.

(iii) $\Delta_2 U\varphi = U \Delta_1 \varphi$ *for all* $\varphi \in C^\infty(M_1)$.

Then the Riemannian manifolds (M_1, g_1) and (M_2, g_2) are isomorphic.

Proof Let $q \in M_2$. Then the map $\varphi \mapsto (U\varphi)(q)$ from $C^\infty(M_1)$ into $\mathbf{R}$ is linear, positive and non-zero. So by Lemma 2.2 there exist $\tau(q) \in M_1$ and $h(q) \in (0, \infty)$ such that

$$(U\varphi)(q) = h(q) \, \varphi(\tau(q)) \tag{4}$$

for all $\varphi \in C^\infty(M_1)$. So one obtains functions $\tau \colon M_2 \to M_1$ and $h \colon M_2 \to (0, \infty)$. Similarly, there exist $\widetilde{\tau} \colon M_1 \to M_2$ and $\widetilde{h} \colon M_1 \to (0, \infty)$ such that $(U^{-1}\psi)(p) = \widetilde{h}(p) \, \psi(\widetilde{\tau}(p))$ for all $\psi \in C^\infty(M_2)$ and $p \in M_1$. Then $\varphi(p) = \widetilde{h}(p) \, h(\widetilde{\tau}(p)) \, \varphi(\tau(\widetilde{\tau}(p)))$ for all $\varphi \in C^\infty(M_1)$ and $p \in M_1$. Choosing $\varphi = \mathbb{1}$ gives $\widetilde{h}(p) \, h(\widetilde{\tau}(p)) = 1$. Hence $\varphi = \varphi \circ \tau \circ \widetilde{\tau}$ for all $\varphi \in C^\infty(M_1)$ and $\tau \circ \widetilde{\tau} = I$. Similarly $\widetilde{\tau} \circ \tau = I$ and τ is a bijection.

Choosing again $\varphi = \mathbb{1}$ in (4) gives $h = U\mathbb{1} \in C^\infty(M_2)$. Hence $\varphi \circ \tau = h^{-1} U\varphi \in C^\infty(M_2)$ for all $\varphi \in C^\infty(M_1)$ and τ is a C^∞-function. Thus τ is a C^∞-diffeomorphism and the two manifolds have the same dimension. Let $d = \dim M_1 = \dim M_2$.

It follows from Property (iii) that

$$\Delta_2(h \cdot (\varphi \circ \tau)) = h \cdot ((\Delta_1 \varphi) \circ \tau) \tag{5}$$

for all $\varphi \in C^\infty(M_1)$. Let $q \in M_2$. There exists a chart (V, x) on M_1 such that $\tau(q) \in V$ and $x(\tau(q)) = 0$. Let $\Omega \subset M_1$ be open such that $\tau(q) \in \Omega \subset \overline{\Omega} \subset V$. Let $\lambda_1, \ldots, \lambda_d \in \mathbf{R}$. For all $t > 0$ there exists a $\varphi_t \in C^\infty(M_1)$ such that

$$\varphi_t|_\Omega = e^{t \sum_{k=1}^d \lambda_k x^k}|_\Omega .$$

Since

$$\Delta_1 = \sum_{i,j=1}^d \frac{1}{\sqrt{g_1}} \, \frac{\partial}{\partial x^i} \, g_1^{ij} \, \sqrt{g_1} \, \frac{\partial}{\partial x^j}$$

on V it follows that

$$\Delta_1 \varphi_t = \sum_{i,j=1}^d t^2 g_1^{ij} \, \lambda_i \, \lambda_j \, \varphi_t - t \, \frac{\lambda_j}{\sqrt{g_1}} \, \varphi_t \, \frac{\partial}{\partial x^i} (g_1^{ij} \, \sqrt{g_1})$$

on Ω. Hence

$$\lim_{t\to\infty} t^{-2}\Big(h \cdot ((\Delta_1\varphi_t)\circ\tau)\Big)(q) = h(q)\sum_{i,j=1}^{d} g_1^{ij}(\tau(q))\,\lambda_i\,\lambda_j.$$

Next, $(\tau^{-1}(V), y)$ is a chart on M_2, where $y = x\circ\tau$. Then it follows similarly that

$$\begin{aligned}\lim_{t\to\infty} t^{-2}\Big(\Delta_2(h\cdot(\varphi_t\circ\tau))\Big)(q) &= \sum_{i,j=1}^{d} h(q)\,g_2^{ij}(q)\Big(\frac{\partial}{\partial y_i}\sum_{k=1}^{d}\lambda_k x^k\circ\tau\Big)(q)\Big(\frac{\partial}{\partial y_j}\sum_{l=1}^{d}\lambda_l x^l\circ\tau\Big)(q)\\ &= \sum_{i,j=1}^{d} h(q)\,g_2^{ij}(q)\Big(\frac{\partial}{\partial y_i}\sum_{k=1}^{d}\lambda_k y^k\Big)(q)\Big(\frac{\partial}{\partial y_j}\sum_{l=1}^{d}\lambda_l y^l\Big)(q)\\ &= \sum_{i,j=1}^{d} h(q)\,g_2^{ij}(q)\,\lambda_i\,\lambda_j.\end{aligned}$$

But then (5) gives

$$\sum_{i,j=1}^{d} g_1^{ij}(\tau(q))\,\lambda_i\,\lambda_j = \sum_{i,j=1}^{d} g_2^{ij}(q)\,\lambda_i\,\lambda_j$$

for all $\lambda_1,\ldots,\lambda_d \in \mathbf{R}$ and $(g_1^{ij}\circ\tau)(q) = g_2^{ij}(q)$ for all $i,j\in\{1,\ldots,d\}$. Hence $g_{1\,ij}|_{\tau(q)} = g_{2\,ij}|_q$. In particular,

$$g_1|_{\tau(q)}(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}) = g_2|_q(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}) = g_2|_q(\tau_*\frac{\partial}{\partial x^i},\tau_*\frac{\partial}{\partial x^j})$$

for all $i,j\in\{1,\ldots,d\}$. Hence τ is an isomorphism from (M_2,g_2) onto (M_1,g_1). □

Now the implications II⇒I and III⇒I in Theorem 1.1 follow easily from Lemma 2.1 and Proposition 2.3. Substituting $\varphi = \mathbb{1}$ in (5) gives $\Delta_2 h = 0$ in the proof of Proposition 2.3. If M_2 is connected this implies that h is constant. Then the last part in Theorem 1.1 is obvious.

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