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The Dirichlet-to-Neumann operator on rough domains

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ABSTRACT

We consider a bounded connected open set $\Omega \subset \mathbb{R}^d$ whose boundary Γ has a finite $(d-1)$-dimensional Hausdorff measure. Then we define the Dirichlet-to-Neumann operator D_0 on $L_2(\Gamma)$ by form methods. The operator $-D_0$ is self-adjoint and generates a contractive C_0-semigroup $S = (S_t)_{t>0}$ on $L_2(\Gamma)$. We show that the asymptotic behaviour of S_t as $t \to \infty$ is related to properties of the trace of functions in $H^1(\Omega)$ which Ω may or may not have.

1. Introduction

Throughout this paper Ω is a bounded, connected, open set in $\mathbb{R}^d$ with boundary Γ. We consider the $(d-1)$-dimensional Hausdorff measure $\mathcal{H}$ on Γ, where $d \geqslant 2$ and assume throughout that $\mathcal{H}(\Gamma) < \infty$. The purpose of this article is to define the Dirichlet-to-Neumann operator D_0 on $L_2(\Gamma)$ under these mild assumptions on Ω and to study the semigroup $(S_t)_{t>0}$ generated by $-D_0$ on $L_2(\Gamma)$.

For this purpose we define at first the trace in the following way. Given $u \in H^1(\Omega)$, a function $\varphi \in L_2(\Gamma)$ is called a *trace* of u if there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $H^1(\Omega) \cap C(\overline{\Omega})$ such that $\lim_{n\to\infty} u_n = u$ in $H^1(\Omega)$ and $\lim_{n\to\infty} u_n|_\Gamma = \varphi$ in $L_2(\Gamma)$. If $u \in H^1(\Omega)$ then we say that u *has a trace* if and only if there exists a $\varphi \in L_2(\Gamma)$ such that φ is a trace of u. Note that we require that a trace is always in $L_2(\Gamma)$. If u has a trace, then $u \in \widetilde{H}^1(\Omega)$, the closure of $H^1(\Omega) \cap C(\overline{\Omega})$ in $H^1(\Omega)$. In general the space $\widetilde{H}^1(\Omega)$ is a proper subset of $H^1(\Omega)$. An example is

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$$\{(x,y)\in\mathbb{R}^2\colon\ |(x,y)|<1\}\setminus\big([0,1)\times\{0\}\big),$$

the unit disc minus a spoke. If, however, Ω has a continuous boundary (in the sense of graphs, see [11], Definition V.4.1), then $\widetilde{H}^1(\Omega)=H^1(\Omega)$ by the following proposition.

Proposition 1.1. *Suppose Ω has a continuous boundary. Then one has the following.*

(a) *The space $H^1(\Omega)\cap C^\infty(\overline{\Omega})$ is dense in $H^1(\Omega)$. So in particular $\widetilde{H}^1(\Omega)=H^1(\Omega)$.*
(b) *The space $H^1(\Omega)$ is compactly embedded in $L_2(\Omega)$.*

Proof. Statement (a) is in [18], Theorem 1.1.6/2 and statement (b) is in [11], Theorem V.4.17. □

In general not every $u\in\widetilde{H}^1(\Omega)$ has a trace (see Example 9.1).

Alternatively, an element of $H^1(\Omega)$ might have more than one trace (see Example 4.4). This happens if and only if the vector space $\{\varphi\in L_2(\Gamma)\colon\ \varphi$ is a trace of $0\}$ of degenerate traces is non-trivial. By [5], Lemma 4.14 there exists a Borel set $\Gamma_o\subset\Gamma$ such that

$$\{\varphi\in L_2(\Gamma)\colon\ \varphi \text{ is a trace of } 0\}=L_2(\Gamma_o).$$

We say that *the trace on Ω is unique* if the function $\varphi=0\in L_2(\Gamma)$ is the only trace of $u=0\in H^1(\Omega)$. This is equivalent with $\mathcal{H}(\Gamma_o)=0$; i.e. if $L_2(\Gamma_o)=\{0\}$. It is also equivalent with the fact that every element of $H^1(\Omega)$ has at most one trace. Thus the part Γ_o is responsible for the obstruction for the trace to be not unique. We will denote the complement of Γ_o by $\Gamma_r=\Gamma\setminus\Gamma_o$, and we call it the *regular part of the boundary*.

Next we define the (weak) normal derivative via Green's formula. Let $u\in H^1(\Omega)$ be such that $\Delta u\in L_2(\Omega)$ as distribution. We say that u has a *normal derivative in* $L_2(\Gamma)$ if there exists a $\psi\in L_2(\Gamma)$ such that

$$\int_\Omega(\Delta u)v+\int_\Omega\nabla u\cdot\nabla v=\int_\Gamma\psi v\,d\mathcal{H}$$

for all $v\in H^1(\Omega)\cap C(\overline{\Omega})$. In that case ψ is unique. We set $\frac{\partial u}{\partial\nu}:=\psi$ and call it the normal derivative of u. Now we define the Dirichlet-to-Neumann operator D_0 on $L_2(\Gamma)$ as follows. Given $\varphi,\psi\in L_2(\Gamma)$, we say that $\varphi\in D(D_0)$ and $D_0\varphi=\psi$ if there exists a $u\in H^1(\Omega)$ such that $\Delta u=0$ as distribution, φ is a trace of u, the function u has a normal derivative in $L_2(\Gamma)$ and $\frac{\partial u}{\partial\nu}=\psi$. Even though the function u might not have a unique trace, we shall prove in Theorem 3.3 that the operator D_0 is well defined. In fact, D_0 is a self-adjoint operator on $L_2(\Gamma)$ and $-D_0$ generates a positive C_0-semigroup S on $L_2(\Gamma)$ satisfying $S_t\mathbb{1}_\Gamma=\mathbb{1}_\Gamma$ for all $t>0$. This is true without any regularity hypothesis on Ω (besides $\mathcal{H}(\Gamma)<\infty$). One purpose of this paper is to show that diverse properties concerning the asymptotic behaviour of S_t as $t\to\infty$ are related to properties of the trace, which in fact are properties of Ω, which Ω may or may not have.

Here are our main results.

A. Strong convergence of S. Define $P: L_2(\Gamma)\to L_2(\Gamma)$ by $P\varphi=(\frac{1}{\mathcal{H}(\Gamma)}\int_\Gamma\varphi)\mathbb{1}_\Gamma$. So P is the projection from $L_2(\Gamma)$ onto the space of all constant functions.

Theorem 1.2. *The following are equivalent.*

(i) *The trace on Ω is unique.*
(ii) $\mathcal{H}(\Gamma_o)=0$.
(iii) $\dim(\ker D_0)=1$.
(iv) $\lim_{t\to\infty}S_t\varphi=P\varphi$ *for all* $\varphi\in L_2(\Gamma)$.
(v) *S is irreducible.*

The irreducibility of S is surprising since the boundary Γ need not be connected (consider an annulus for example). Thus this result reflects somehow that the operator D_0 is not local.

B. Norm convergence of S. We emphasize that in general not every element in $\widetilde{H}^1(\Omega)$ has a trace and if it has a trace, then it might not be unique. We next characterize when both properties are valid, i.e. every element of $\widetilde{H}^1(\Omega)$ has a trace and this trace is unique. This is true for example if Ω has a Lipschitz boundary.

Theorem 1.3. *The following are equivalent.*

(i) $\lim_{t\to\infty} S_t = P$ *in* $\mathcal{L}(L_2(\Gamma))$.
(ii) *There exists a* $c > 0$ *such that*

$$\int_\Gamma |u|^2 \leqslant c \int_\Omega |\nabla u|^2$$

for all $u \in H^1(\Omega) \cap C(\overline{\Omega})$ *with* $\int_\Gamma u = 0$.
(iii) *There exists a* $c > 0$ *such that*

$$\int_\Gamma |u|^2 \leqslant c\Big(\int_\Omega |\nabla u|^2 + \int_\Omega |u|^2\Big)$$

for all $u \in H^1(\Omega) \cap C(\overline{\Omega})$.
(iv) *Every* $u \in \widetilde{H}^1(\Omega)$ *has a unique trace.*
(v) $0 \notin \sigma_{\rm ess}(D_0)$.

C. Compactness of the resolvent. We shall show that the operator D_0 has compact resolvent if and only if every $u \in \widetilde{H}^1(\Omega)$ has a unique trace $\operatorname{Tr} u$ and the map $\operatorname{Tr} : \widetilde{H}^1(\Omega) \to L_2(\Gamma)$ is compact. This implies that the embedding $\widetilde{H}^1(\Omega) \to L_2(\Omega)$ is also compact. We construct, however, a bounded domain with continuous boundary and with $\mathcal{H}(\Gamma) < \infty$, such that D_0 does not have compact resolvent (even though the embedding $H^1(\Omega) = \widetilde{H}^1(\Omega) \hookrightarrow L_2(\Omega)$ is compact since the boundary is continuous).

The Dirichlet-to-Neumann operator is a well-known object occurring in many applications. In general it is considered on domains of class C^∞, though, see e.g. the monograph of Taylor [21], in particular Section 12C. Then the operator fits into the framework of pseudo-differential operators and also semigroup properties are studied [13,12]. Our point is the very general variational definition which allows an easy approach also for rough domains. On the other hand, the questions concerning trace properties which we investigate here become delicate. They are the main subject of the paper. Some of the trace properties considered here are related to investigations of the Laplace operator with Robin boundary conditions on arbitrary domains as in [9], see also [6].

The paper is organized as follows. In Section 2 we consider the asymptotic behaviour of Markovian semigroups. This section is independent of the Dirichlet-to-Neumann operator. In Section 3 we prove the existence and uniqueness of the Dirichlet-to-Neumann operator on rough domains and show that it is a self-adjoint operator which generates a Markovian semigroup. In Section 4 we prove Theorem 1.2. In addition we give other characterizations of the uniqueness of the trace in terms of the form associated to the Laplacian with Robin boundary conditions and in terms of the relative capacity. In Section 5 we define the trace as a mapping and study its properties. In Section 6 we characterize when every element of $\widetilde{H}^1(\Omega)$ has a trace. Moreover, we prove Theorem 1.3. In Section 7 we characterize when the map $u \mapsto u|_\Gamma$ from $(H^1(\Omega) \cap C(\overline{\Omega}), \|\cdot\|_{H^1(\Omega)})$ into $L_2(\Gamma)$ is compact. Theorem 1.3 and the compactness of the trace can be reformulated in terms of the form associated to the Laplacian with Robin boundary conditions. This is done in Section 8. Finally, in Section 9 we present two striking examples.

Throughout this paper the field is $\mathbb{R}$ and we only consider single valued operators.

A key ingredient in Section 3 is the Maz'ya inequality (5). This remarkable inequality is valid for any open set $\Omega \subset \mathbb{R}^d$ with finite volume, and involves the $(d-1)$-dimensional Hausdorff measure on the boundary Γ of Ω. This is the reason why we choose the $(d-1)$-dimensional Hausdorff measure on Γ. Many statements in this paper are still valid if one has a Borel regular measure on Γ for which inequality (5) is still valid. In order not to clutter this paper we provide Γ with the $(d-1)$-dimensional Hausdorff measure.

The conditions that Ω be bounded and $\mathcal{H}(\Gamma) < \infty$ are convenient in the definition of the Dirichlet-to-Neumann operator. Otherwise one may replace the space $H^1(\Omega) \cap C(\overline{\Omega})$ by the space

$$\left\{u \in H^1(\Omega) \cap C(\overline{\Omega}):\ u|_\Gamma \in L_2(\Gamma)\right\}$$

at many places and require that $\mathcal{H}(K) < \infty$ for every compact $K \subset \Gamma$. For simplicity and easy readability of this paper we assume throughout that Ω is bounded and $\mathcal{H}(\Gamma) < \infty$. Some statements of theorems need otherwise obvious modifications. For the same reason we assume that Ω is connected.

With the restrictions that we use throughout this paper (Ω bounded and connected, $(d-1)$-dimensional Hausdorff measure $\mathcal{H}$ on Γ and $\mathcal{H}(\Gamma) < \infty$) we have many interesting domains. For example, let $C = \bigcap_{n=0}^{\infty} C_n \subset [0,1]$ be the generalized Cantor set with $C_0 = [0,1]$ and for any $n \in \mathbb{N}_0$ construct C_{n+1} by removing the central open interval of length 2^{-2n-1} from any interval of C_n. So $C_1 = [0, \frac{1}{4}] \cup [\frac{3}{4}, 1]$, etc. Then $0 < \mathcal{H}(C \times C) < \infty$. Let $\Omega = ((-1,2) \times (-1,2)) \setminus (C \times C)$. Then Ω is open, bounded, connected and $\mathcal{H}(\partial\Omega) < \infty$. Nevertheless, $\Gamma = \partial\Omega$ is not rectifiable by [2], Example 2.67. A slightly simpler example is the set $((-1,2) \times (-1,2)) \setminus (C \times \{0\})$.

2. Asymptotic behaviour of Markovian semigroups

In this section we put together some asymptotic properties of Markovian semigroups. At first we consider a *self-adjoint semigroup*, i.e. a semigroup consisting of self-adjoint operators.

Proposition 2.1. *Let S be a contractive C_0-semigroup of self-adjoint operators on a Hilbert space H. Then*

$$P_S f = \lim_{t \to \infty} S_t f$$

exists for all $f \in H$ and P_S is the orthogonal projection onto $\ker A$, *where $-A$ denotes the generator of S.*

Proof. By the spectral theorem we may assume that $H = L_2(Y)$, $D(A) = \{f \in L_2(Y):\ mf \in L_2(Y)\}$ and $Af = mf$ for all $f \in D(A)$, where (Y, Σ, μ) is a locally finite measure space and $m : Y \to [0, \infty)$ is a measurable function. Then $\ker A = \{f \in L_2(Y):\ f = 0 \text{ a.e. on } Y \setminus Y_0\}$, where $Y_0 = m^{-1}(\{0\})$. The orthogonal projection P_S onto $\ker A$ is given by $P_S f = \mathbb{1}_{Y_0} f$. Moreover, $S_t f = e^{-tm} f$ for all $t > 0$ and $f \in L_2(Y)$. Now the claim follows from the Lebesgue dominated convergence theorem. □

Next we consider a finite measure space (Γ, Σ, μ). A *Markov operator* T on $L_2(\Gamma)$ is an operator satisfying $T\mathbb{1}_\Gamma = \mathbb{1}_\Gamma$ and $Tf \geqslant 0$ for all $f \in L_2(\Gamma)$ with $f \geqslant 0$. As a consequence $TL_\infty(\Gamma) \subset L_\infty(\Gamma)$ and $T^{(\infty)} := T|_{L_\infty(\Gamma)}$ is contractive. If T is a self-adjoint Markov operator on $L_2(\Gamma)$, then T is contractive for the L_1-norm. Hence for all $p \in [1, \infty]$ there exists a unique $T^{(p)} \in \mathcal{L}(L_p(Y))$ such that $T^{(p)} f = Tf$ for all $f \in L_p(Y) \cap L_2(Y)$. Moreover, $\|T^{(p)}\|_{\mathcal{L}(L_p(Y))} \leqslant 1$. The operator $T^{(\infty)}$ is the adjoint of the operator $T^{(1)}$.

A C_0-semigroup S on $L_2(\Gamma)$ is called *irreducible* if for each $\Gamma_1 \in \Sigma$ with

$$S_t L_2(\Gamma_1) \subset L_2(\Gamma_1)$$

for all $t > 0$ it follows that $\mu(\Gamma_1) = 0$ or $\mu(\Gamma \setminus \Gamma_1) = 0$. Here, and in the sequel, we let $L_2(\Gamma_1) = \{f \in L_2(Y):\ f = 0 \text{ a.e. on } \Gamma \setminus \Gamma_1\}$. A *Markov semigroup* on $L_2(\Gamma)$ is a C_0-semigroup S on $L_2(\Gamma)$ such that S_t is a Markov operator for all $t > 0$. In that case $(S_t^{(p)})_{t>0}$ is a positive contractive C_0-semigroup on $L_p(\Gamma)$ for all $p \in [1,\infty)$. Moreover, $\mathbb{R}\mathbb{1}_\Gamma \subset \ker A$, where $-A$ is the generator of S.

Proposition 2.2. *Let S be a self-adjoint Markov semigroup on $L_2(\Gamma)$. Then S is irreducible if and only if $\ker A = \mathbb{R}\mathbb{1}_\Gamma$, where $-A$ is the generator of S.*

Proof. '$\Rightarrow$'. This follows from [19], Section C-III, Proposition 3.5(c).

'$\Leftarrow$'. Let $\Gamma_1 \in \Sigma$ be such that $S_t L_2(\Gamma_1) \subset L_2(\Gamma_1)$ for all $t > 0$. Set $\Gamma_2 := \Gamma \setminus \Gamma_1$. Then $L_2(\Gamma_2) = L_2(\Gamma_1)^\perp$ and since S_t is self-adjoint, it follows that $S_t L_2(\Gamma_2) \subset L_2(\Gamma_2)$ for all $t > 0$. Now $\mathbb{1}_{\Gamma_1} + \mathbb{1}_{\Gamma_2} = \mathbb{1}_\Gamma = S_t \mathbb{1}_\Gamma = S_t \mathbb{1}_{\Gamma_1} + S_t \mathbb{1}_{\Gamma_2}$ by assumption. Moreover, $S_t \mathbb{1}_{\Gamma_j} \in L_2(\Gamma_j)$ vanishes outside Γ_j for all $j \in \{1,2\}$. Hence $S_t \mathbb{1}_{\Gamma_1} = \mathbb{1}_{\Gamma_1}$ for all $t > 0$. This implies that $\mathbb{1}_{\Gamma_1} \in \ker A$. Since $\ker A = \mathbb{R}\mathbb{1}_\Gamma$ by assumption, it follows that $\mu(\Gamma_1) = 0$ or $\mu(\Gamma_2) = 0$. □

Next we show that a self-adjoint Markov semigroup is irreducible if and only if it converges to an equilibrium. For all $f \in L_1(\Gamma)$ define

$$Pf = \frac{1}{\mu(\Gamma)} \left(\int_\Gamma f \right) \mathbb{1}_\Gamma. \tag{1}$$

Then P defines a positive contractive projection on $L_p(\Gamma)$ for all $p \in [1,\infty]$.

Theorem 2.3. *Let S be a self-adjoint Markov semigroup on $L_2(\Gamma)$. The following are equivalent.*

(i) *S is irreducible.*
(ii) *There exists a $p \in [1,\infty)$ such that $\lim_{t\to\infty} S_t^{(p)} f = Pf$ in $L_p(\Gamma)$ for all $f \in L_p(\Gamma)$.*
(iii) *For all $p \in [1,\infty)$ one has $\lim_{t\to\infty} S_t^{(p)} f = Pf$ in $L_p(\Gamma)$ for all $f \in L_p(\Gamma)$.*

Proof. '(i) $\Rightarrow$ (ii)'. We prove statement (ii) for $p = 2$. If S is irreducible, then $\ker A = \mathbb{R}\mathbb{1}_\Gamma$ by Proposition 2.2, where $-A$ is the generator of S. Then the operator P defined in (1) is the orthogonal projection onto $\ker A$. Then statement (ii) follows from Proposition 2.1.

'(ii) $\Rightarrow$ (iii)'. Let $p \in [1,\infty)$ and suppose that $\lim_{t\to\infty} S_t^{(p)} f = Pf$ in $L_p(\Gamma)$ for all $f \in L_p(\Gamma)$. If $f \in L_p(\Gamma)$ then $\|S_t^{(1)} f - Pf\|_1 \leqslant (\mu(\Gamma))^{\frac{1}{p}-1} \|S_t^{(p)} f - Pf\|_p$ for all $t > 0$. Therefore $\lim_{t\to\infty} S_t^{(1)} f = Pf$ in $L_1(\Gamma)$. Since $L_p(\Gamma)$ is dense in $L_1(\Gamma)$ and $\{P\} \cup \{S_t^{(1)}\colon t > 0\}$ are uniformly bounded in $\mathcal{L}(L_1(\Gamma))$ it follows that $\lim_{t\to\infty} S_t^{(1)} f = Pf$ in $L_1(\Gamma)$ for all $f \in L_1(\Gamma)$.

Finally, let $q \in (1,\infty)$. If $f \in L_\infty(\Gamma)$ then by interpolation

$$\left\|S_t^{(q)} f - Pf\right\|_q \leqslant \left\|S_t^{(1)} f - Pf\right\|_1^\theta \left\|S_t^{(\infty)} f - Pf\right\|_\infty^{1-\theta} \leqslant \left\|S_t^{(1)} f - Pf\right\|_1^\theta \left(2\|f\|_\infty\right)^{1-\theta},$$

where $\theta = \frac{1}{q}$. So $\lim_{t\to\infty} S_t^{(q)} f = Pf$ in $L_q(\Gamma)$. Since $L_\infty(\Gamma)$ is dense in $L_q(\Gamma)$ the claim follows as before.

'(iii) $\Rightarrow$ (i)'. Let $f \in \ker A$. Then $S_t f = f$ for all $t > 0$. Consequently $f = Pf \in \mathbb{R}\mathbb{1}_\Gamma$. We have shown that $\ker A = \mathbb{R}\mathbb{1}_\Gamma$. It follows from Proposition 2.2 that S is irreducible and (i) is valid. □

If A is a self-adjoint operator, then $0 \notin \sigma_{\mathrm{ess}}(A)$ means by definition that 0 is not an accumulation point of $\sigma(A)$ and $\ker A$ is finite dimensional. Thus if S is a self-adjoint irreducible Markov semigroup with generator $-A$ then it follows from Proposition 2.2 that $0 \notin \sigma_{\mathrm{ess}}(A)$ if and only if there exists an $\varepsilon > 0$ such that $\sigma(A) \cap [0,\varepsilon) = \{0\}$. In the next theorem we reformulate this by saying that S_t converges in the operator norm as $t \to \infty$.

Theorem 2.4. *Let S be a self-adjoint irreducible Markov semigroup on $L_2(\Gamma)$ with generator $-A$. The following are equivalent.*

(i) $0 \notin \sigma_{\mathrm{ess}}(A)$.
(ii) $\lim_{t\to\infty} S_t = P$ *in* $\mathcal{L}(L_2(\Gamma))$.
(iii) *There exists an* $\varepsilon > 0$ *such that* $\|S_t - P\|_{\mathcal{L}(L_2(\Gamma))} \leqslant e^{-\varepsilon t}$ *for all* $t > 0$.

In that case one also has $\lim_{t\to\infty} S_t^{(p)} = P$ *in* $\mathcal{L}(L_p(\Gamma))$ *for all* $p \in (1,\infty)$.

Proof. '(i) $\Rightarrow$ (iii)'. We consider the situation in the proof of Proposition 2.1, which was obtained via a unitary transformation. Since S is irreducible one has $\dim\ker A = 1$. Then the hypothesis $0 \notin \sigma_{\mathrm{ess}}(A)$ implies that there exists an $\varepsilon > 0$ such that $\sigma(A) \cap [0,\varepsilon) = \{0\}$. Then $m(y) \geqslant \varepsilon$ for a.e. $y \in Y \setminus Y_0$. Thus

$$\|S_t - P\|_{\mathcal{L}(L_2(Y))} = \|S_t - P\|_{\mathcal{L}(L_2(Y\setminus Y_0))} = \left\|e^{-tm}\right\|_{L_\infty(Y\setminus Y_0)} \leqslant e^{-\varepsilon t} \tag{2}$$

for all $t > 0$.

'(iii) $\Rightarrow$ (ii)'. It is trivial.

'(ii) $\Rightarrow$ (i)'. The space $H_1 = (I-P)(L_2(\Gamma))$ is invariant under S and $\lim_{t\to\infty} \|S_t\|_{\mathcal{L}(H_1)} = 0$. By the spectral theorem, this implies as in (2) that $\lim_{t\to\infty} \|e^{-tm}\|_{L_\infty(Y\setminus Y_0)} = 0$. Hence there exists an $\varepsilon > 0$ such that $m(y) \geqslant \varepsilon$ for a.e. $y \in Y \setminus Y_0$. Again by the spectral theorem this implies (i).

Finally we assume that (ii) is valid. Let $p \in (1,2)$. Let $\theta \in (0,1)$ be such that $\frac{1}{p} = \frac{\theta}{1} + \frac{1-\theta}{2}$. Then

$$\left\|S_t^{(p)} - P\right\|_{\mathcal{L}(L_p(\Gamma))} \leqslant \left\|S_t^{(1)} - P\right\|_{\mathcal{L}(L_1(\Gamma))}^{\theta} \left\|S_t^{(2)} - P\right\|_{\mathcal{L}(L_2(\Gamma))}^{1-\theta} \leqslant 2^\theta \left\|S_t^{(2)} - P\right\|_{\mathcal{L}(L_2(\Gamma))}^{1-\theta}$$

for all $t > 0$ since $S^{(1)}$ is a contraction semigroup. Therefore $\lim_{t\to\infty} S_t^{(p)} = P$ in $\mathcal{L}(L_p(\Gamma))$. The proof for $p \in (2,\infty)$ is similar, or follows by a duality argument. □

The harmonic oscillator on a weighted space (see [10], Theorem 4.3.6) shows that the last assertion is not true, in general, for $p = 1$ even if A has compact resolvent.

3. The Dirichlet-to-Neumann operator on arbitrary domains

In this section we will define the Dirichlet-to-Neumann operator D_0 on $L_2(\Gamma)$ as a self-adjoint operator, and we will show that $-D_0$ generates a Markov semigroup.

Definition 3.1. Let $u \in H^1(\Omega)$ and $\varphi \in L_2(\Gamma)$. We say that φ is a *trace* of u if there exist $u_1, u_2, \ldots \in H^1(\Omega) \cap C(\overline{\Omega})$ such that $\lim_{n\to\infty} u_n = u$ in $H^1(\Omega)$ and $\lim_{n\to\infty} u|_\Gamma = \varphi$ in $L_2(\Gamma)$.

It is well possible that there are different elements of $L_2(\Gamma)$ such that they are both a trace of the same element of $H^1(\Omega)$ (see Section 4). Clearly if $u \in H^1(\Omega)$ has a trace then $u \in \widetilde{H}^1(\Omega)$.

Note that the space $\{v|_\Gamma\colon\ v \in \mathcal{D}(\mathbb{R}^d)\}$ is dense in $C(\Gamma)$ by the Stone–Weierstraß theorem for the uniform norm and therefore it is also in $L_2(\Gamma)$ since $\mathcal{H}$ is Borel regular (see [14], Theorem 2.1.1). Hence $\{v|_\Gamma\colon\ H^1(\Omega) \cap C(\overline{\Omega})\}$ is dense in $L_2(\Gamma)$.

Next we define the normal derivative $\frac{\partial u}{\partial \nu}$ by the Green's formula as follows (cf. [4,3] for the case that Ω has a Lipschitz boundary). If $u \in L_{1,\mathrm{loc}}(\Omega)$, then we denote by $\Delta u \in \mathcal{D}(\Omega)'$ the distributional Laplacian applied to u.

Definition 3.2. Let $u \in H^1(\Omega)$ be such that $\Delta u \in L_2(\Omega)$. We say that u has a normal derivative in $L_2(\Gamma)$ if there exists a $\psi \in L_2(\Gamma)$ such that

$$\int_{\Omega} (\Delta u) v + \int_{\Omega} \nabla u \cdot \nabla v = \int_{\Gamma} \psi v \tag{3}$$

for all $v \in H^1(\Omega) \cap C(\overline{\Omega})$. In that case ψ is uniquely determined by (3), we write $\frac{\partial u}{\partial \nu} := \psi$ and call ψ the *normal derivative* of u.

Now we are able to define the Dirichlet-to-Neumann operator D_0 on $L_2(\Gamma)$. It is part of the following theorem that the operator D_0 is well defined, even though an $H^1(\Omega)$-function might have different functions in $L_2(\Gamma)$ as a trace.

Theorem 3.3. *There exists an operator D_0 on $L_2(\Gamma)$ such that the following holds. Given $\varphi, \psi \in L_2(\Gamma)$ one has $\varphi \in D(D_0)$ and $D_0\varphi = \psi$ if and only if there exists a $u \in H^1(\Omega)$ satisfying*

- $\Delta u = 0$,
- *φ is a trace of u, and,*
- *u has a normal derivative in $L_2(\Gamma)$ and $\frac{\partial u}{\partial \nu} = \psi$.*

Moreover, the operator D_0 is positive and self-adjoint.

Here and in the sequel we always consider the operator Δ in the distributional sense.

For the proof of Theorem 3.3 we will need a generation theorem proved recently in [5] which is valid for arbitrary sectorial forms (without any closability condition). We recall a special case of it.

Theorem 3.4. *Let $D(a)$ be a real vector space and let $a : D(a) \times D(a) \to \mathbb{R}$ be bilinear symmetric such that $a(u) := a(u, u) \geqslant 0$ for all $u \in D(a)$. Let H be a (real) Hilbert space and let $j : D(a) \to H$ be linear with dense image. Then there exists an operator A on H such that for all $\varphi, \psi \in H$ one has $\varphi \in D(A)$ and $A\varphi = \psi$ if and only if there exists a sequence $u_1, u_2, \ldots \in D(a)$ such that*

(a) $\lim_{n,m\to\infty} a(u_n - u_m) = 0$,
(b) $\lim_{n\to\infty} j(u_n) = \varphi$ *in H, and,*
(c) $\lim_{n\to\infty} a(u_n, v) = (\psi, j(v))_H$ *for all $v \in D(a)$.*

Moreover, A is positive and self-adjoint.

Proof. See [5], Theorem 3.2 and Remark 3.5. □

We call A *the operator associated with* (a, j). Note that the operator A in Theorem 3.4 is well defined and it turns out that this will be the reason why the operator D_0 is well defined.

Besides Theorem 3.4, for the proof of Theorem 3.3, we need the following remarkable inequality due to Maz'ya. It was Daners [9] who showed how this inequality can be used efficiently for elliptic and parabolic problems. It follows from Example 3.6.2/1 and Theorem 3.6.3 in [18] and (24) in [6] that there exists a constant $c'_M > 0$ such that

$$\left(\int_{\Omega} |u|^q \right)^{2/q} \leqslant c'_M \left(\int_{\Omega} |\nabla u|^2 + \int_{\Gamma} |u|^2 \right) \tag{4}$$

for all $u \in H^1(\Omega) \cap C(\overline{\Omega})$, where $q = \frac{2d}{d-1}$. Here we use that $\mathcal{H}$ is the $(d-1)$-dimensional Hausdorff measure on Γ and that Ω has finite volume. As a consequence one deduces another Maz'ya inequality: There exists a constant $c_M \geqslant 0$ such that

$$\int_\Omega |u|^2 \leqslant c_M \Bigl(\int_\Omega |\nabla u|^2 + \int_\Gamma |u|^2 \Bigr) \tag{5}$$

for all $u \in H^1(\Omega) \cap C(\overline{\Omega})$.

Inequality (4) implies the following important compactness property (see [18], Corollary 4.11.1/3). It only requires that Ω has finite volume.

Proposition 3.5. *The space $H^1(\Omega) \cap C(\overline{\Omega})$ with norm*

$$\|u\|^2 = \int_\Omega |\nabla u|^2 + \int_\Gamma |u|^2$$

is compactly embedded into $L_2(\Omega)$.

In the proof of Theorem 3.3 we need the form ℓ with form domain $D(\ell) = H^1(\Omega) \cap C(\overline{\Omega})$ given by

$$\ell(u, v) = \int_\Omega \nabla u \cdot \nabla v.$$

The form ℓ is used throughout this paper.

Proof of Theorem 3.3. Let $H = L_2(\Gamma)$. Let $j : D(\ell) \to L_2(\Gamma)$ be defined by $j(u) = u|_\Gamma$. Then clearly j has dense range. Denote by A the operator associated with (ℓ, j) in the sense of Theorem 3.4. We shall show that A has the properties of D_0.

Let $\varphi, \psi \in L_2(\Gamma)$.

Assume that $\varphi \in D(A)$ and $A\varphi = \psi$. Then there exists a sequence $u_1, u_2, \ldots \in D(\ell)$ such that $\lim_{n,m\to\infty} \int_\Omega |\nabla(u_n - u_m)|^2 = 0$, $\lim_{n\to\infty} u_n|_\Gamma = \varphi$ in $L_2(\Gamma)$ and

$$\lim_{n\to\infty} \int_\Omega \nabla u_n \cdot \nabla v = \int_\Gamma \psi v \tag{6}$$

for all $v \in D(\ell)$. It follows from Maz'ya's inequality (5) that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $H^1(\Omega)$. Let $u := \lim_{n\to\infty} u_n$ in $H^1(\Omega)$. Then φ is a trace of u, by definition. Moreover, by (6) we have

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Gamma \psi v$$

for all $v \in D(\ell)$. Taking $v \in C_c^\infty(\Omega)$ we see that $\Delta u = 0$. Consequently,

$$\int_\Omega (\Delta u) v + \int_\Omega \nabla u \cdot \nabla v = \int_\Gamma \psi v$$

for all $v \in D(\ell)$. Therefore u has a normal derivative in $L_2(\Gamma)$ and $\frac{\partial u}{\partial \nu} = \psi$ by Definition 3.2.

Conversely, suppose there exists a $u \in H^1(\Omega)$ such that $\Delta u = 0$, the function φ is a trace of u, the function u has a normal derivative in $L_2(\Gamma)$ and $\frac{\partial u}{\partial \nu} = \psi$. Then there exist $u_1, u_2, \ldots \in D(\ell)$ such that $\lim_{n\to\infty} u_n = u$ in $H^1(\Omega)$ and $\lim_{n\to\infty} u_n|_\Gamma = \varphi$ in $L_2(\Gamma)$. It follows that $\lim_{n,m\to\infty} \ell(u_n - u_m) = 0$ and, since $\Delta u = 0$,

$$\lim_{n\to\infty} \ell(u_n, v) = \lim_{n\to\infty} \int_\Omega \nabla u_n \cdot \nabla v = \int_\Omega \nabla u \cdot \nabla v = \int_\Omega \nabla u \cdot \nabla v + \int_\Omega (\Delta u)v = \int_\Gamma \psi v$$

for all $v \in D(\ell)$ by the definition of $\frac{\partial u}{\partial \nu}$. Hence $\varphi \in D(A)$ and $A\varphi = \psi$.

Therefore the operator with the properties of D_0 is well defined and equals A. In particular D_0 is positive and self-adjoint. This completes the proof of Theorem 3.3. □

In the proof of Theorem 3.3 we also proved the following important fact, which will be used later.

Proposition 3.6. *If $j : D(\ell) \to L_2(\Gamma)$ is defined by $j(u) = u|_\Gamma$, then D_0 is the operator associated with (ℓ, j).*

We now show that the semigroup generated by $-D_0$ is Markovian.

Proposition 3.7. *The C_0-semigroup S on $L_2(\Gamma)$ generated by $-D_0$ is Markovian, i.e. $S_t \geqslant 0$ and $S_t \mathbb{1}_\Gamma = \mathbb{1}_\Gamma$ for all $t > 0$.*

Proof. First we prove that S is positive. Let $L_2(\Gamma)_+ = \{\varphi \in L_2(\Gamma)\colon\ \varphi \geqslant 0\}$ be the positive cone in $L_2(\Gamma)$. The orthogonal projection from $L_2(\Gamma)$ onto $L_2(\Gamma)_+$ is given by $\varphi \mapsto \varphi^+$. Let $u \in H^1(\Omega) \cap C(\overline{\Omega})$. Then $u^+ \in D(\ell)$ and $j(u^+) = (j(u))^+$. Moreover, $\ell(u^+, u - u^+) = -\ell(u^+, u^-) = -\int_\Omega \nabla(u^+) \cdot \nabla(u^-) = 0$ since $\nabla(u^+) = \mathbb{1}_{[u>0]} \nabla u$ and $\nabla(u^-) = -\mathbb{1}_{[u<0]} \nabla u$ by [17], Lemma 7.6. Hence S is positive by Remark 3.12 in [5].

Since $\mathbb{1}_\Gamma \in D(D_0)$ and $D_0 \mathbb{1}_\Gamma = 0$ it follows that $S_t \mathbb{1}_\Gamma = \mathbb{1}_\Gamma$ for all $t > 0$. □

4. Uniqueness of the trace and irreducibility

Recall that

$$\{\varphi \in L_2(\Gamma)\colon\ \varphi \text{ is a trace of } 0\} = L_2(\Gamma_o).$$

Note that if $\mathcal{H}(\Gamma_o) > 0$ then the space Γ_o is non-atomic since $d \geqslant 2$ (see [15], Exercise 264Yg). Hence $\dim L_2(\Gamma_o) = \infty$ if $\mathcal{H}(\Gamma_o) \neq 0$.

If $\varphi \in L_2(\Gamma_o)$, then with the choice $u = 0$ one has $\Delta u = 0$ as distribution, φ is a trace of u and $\frac{\partial u}{\partial \nu} = 0$. Therefore it follows from the definition of the operator D_0 that $\varphi \in \ker D_0$. Thus $L_2(\Gamma_o) \subset \ker D_0$. We next characterize $\ker D_0$. In the proof we use that Ω is connected.

Proposition 4.1. *One has $\ker D_0 = \mathbb{R}\mathbb{1}_\Gamma + L_2(\Gamma_o)$. Hence if $\mathcal{H}(\Gamma_o) = 0$, then $0 \in \sigma_{\mathrm{p}}(D_0)$ with multiplicity 1 and if $\mathcal{H}(\Gamma_o) > 0$, then $0 \in \sigma_{\mathrm{p}}(D_0)$ with infinite multiplicity.*

Proof. Let $\varphi \in \ker D_0$. By Theorem 3.3 there exists a $u \in H^1(\Omega)$ such that $\Delta u = 0$, φ is a trace of u and 0 is the normal derivative of u. Then u has a trace, so $u \in \widetilde{H}^1(\Omega)$. Moreover, $\int_\Omega \nabla u \cdot \nabla v = 0$ for all $v \in H^1(\Omega) \cap C(\overline{\Omega})$. Approximating u by elements in $H^1(\Omega) \cap C(\overline{\Omega})$ gives $\int_\Omega |\nabla u|^2 = 0$. Since Ω is connected, one deduces that u is constant. So $\ker D_0 \subset \mathbb{R}\mathbb{1} + L_2(\Gamma_o)$. The reverse inclusion is clear. □

Proof of Theorem 1.2. Theorem 1.2 is a consequence of Theorem 2.3 and Propositions 2.2 and 4.1. □

If Ω is a Lipschitz domain, then $H^1(\Omega) = \widetilde{H}^1(\Omega)$ and there exists a $c > 0$ such that

$$\int_\Gamma |u|^2 \leqslant c \|u\|^2_{H^1(\Omega)}$$

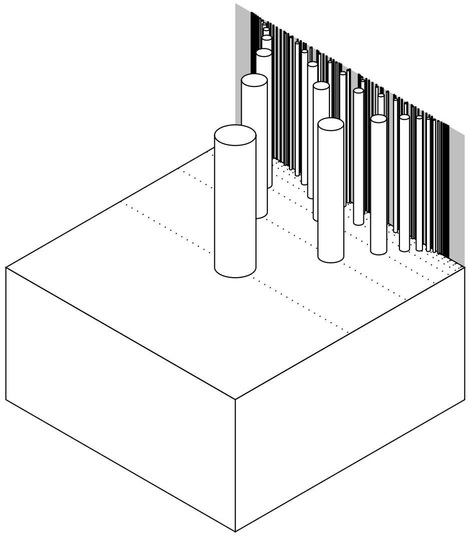

Fig. 1. An example of a domain where $\mathcal{H}(\Gamma_o) > 0$. In fact, the whole gray rectangle belongs to Γ_o.

for all $u \in H^1(\Omega) \cap C(\overline{\Omega})$. This implies in particular that the trace on Ω is unique. For general Ω it follows immediately from this result that the trace on Ω is unique whenever there exists a Borel set $\Lambda \subset \Gamma$ with $\mathcal{H}(\Gamma \setminus \Lambda) = 0$ such that for each point $z \in \Lambda$ there exists an $r > 0$ such that $B(z, r) \cap \Gamma$ is a Lipschitz graph with $B(z, r) \cap \Omega$ on one side.

There is another characterization for the uniqueness of the trace on Ω which involves the relative capacity on Ω. If $A \subset \overline{\Omega}$ is any set, then the *relative capacity* of A with respect to Ω is introduced in [6] by

$$\operatorname{cap}_\Omega A = \inf\bigl\{\|u\|^2_{H^1(\Omega)}\colon u \in \widetilde{H}^1(\Omega) \text{ and there exists an open } V \subset \mathbb{R}^d \text{ such that } A \subset V \text{ and } u \geqslant 1 \text{ a.e. on } \Omega \cap V\bigr\}.$$

We emphasize that the norm $\|\cdot\|_{H^1(\Omega)}$ is used in the definition of relative capacity, not merely the seminorm $u \mapsto \|\nabla u\|_{L_2(\Omega)}$. The usual capacity of the set A is equal to $\operatorname{cap}_{\mathbb{R}^d}(A)$, which is a refinement of the measure of a set. If $\operatorname{cap}_{\mathbb{R}^d}(A) = 0$ then also $|A| = 0$, but the converse is false. For background information on $\operatorname{cap}_{\mathbb{R}^d}$, or more general for capacity associated with Dirichlet forms, we refer to [7], Section I.8. The relative capacity $\operatorname{cap}_\Omega(A)$ takes into account the one-sided effect of Ω if $A \subset \partial\Omega$. Obviously $\operatorname{cap}_\Omega(A) \leqslant \operatorname{cap}_{\mathbb{R}^d}(A)$, but surprisingly it is possible that $\operatorname{cap}_\Omega(A) = 0$ whilst $\operatorname{cap}_{\mathbb{R}^d}(A) > 0$. An example is the grey rectangle in Fig. 1 (see Example 4.4). For more information on relative capacity we refer to [6], where the first example of this kind was constructed.

Now another characterization for the uniqueness of the trace on Ω can be given in terms of the Laplacian on Ω with Robin boundary conditions and also in terms of the relative capacity. Define the form a_R with domain $D(a_R) = H^1(\Omega) \cap C(\overline{\Omega})$ by

$$a_R(u, v) = \int_\Omega \nabla u \cdot \nabla v + \int_\Gamma u v.$$

Then $D(a_R)$ is a pre-Hilbert space with norm $\|u\|^2_{a_R} = a_R(u) + \|u\|^2_{L_2(\Omega)}$. Our second characterization of uniqueness of the trace is as follows.

Proposition 4.2. *The following conditions are equivalent.*

(i) *The trace on Ω is unique.*
(ii) *The form a_R is closable.*
(iii) *For every Borel set $B \subset \Gamma$ with $\operatorname{cap}_\Omega B = 0$ one has $\mathcal{H}(B) = 0$.*

Proof. '(i) ⇒ (ii)'. Let $u_1, u_2, \ldots \in D(a_R)$ be a Cauchy sequence in $D(a_R)$ with $\lim u_n = 0$ in $L_2(\Omega)$. Then $u_1, u_2, \ldots$ is a Cauchy sequence in $H^1(\Omega)$ and $u_1|_\Gamma, u_2|_\Gamma, \ldots$ is a Cauchy sequence in $L_2(\Gamma)$. Hence $u := \lim u_n$ exists in $H^1(\Omega)$ and $\varphi := \lim u_n|_\Gamma$ exists in $L_2(\Gamma)$. Then $u = 0$ since $\lim u_n = 0$ in $L_2(\Omega)$. But the trace on Ω is unique. So $\varphi = 0$ and consequently $\lim a_R(u_n) = 0$. We have shown that a_R is closable.

'(ii) ⇒ (i)'. Let $u_1, u_2, \ldots \in H^1(\Omega) \cap C(\overline{\Omega})$, $\varphi \in L_2(\Gamma)$ and suppose that $\lim u_n = 0$ in $H^1(\Omega)$ and $\lim u_n|_\Gamma = \varphi$ in $L_2(\Gamma)$. Then $u_1, u_2, \ldots$ is a Cauchy sequence in $D(a_R)$. Moreover, $\lim u_n = 0$ in $L_2(\Omega)$ and a_R is closable. Therefore $\lim a_R(u_n) = 0$. This implies that $\lim u_n|_\Gamma = 0$ in $L_2(\Gamma)$ and $\varphi = 0$.

'(ii) ⇔ (iii)'. This is Theorem 3.3 in [6]. □

The regular part of the boundary Γ_r can also be described in a different way. One says that $\mathcal{H}$ is *admissible* if Property (iii) of Proposition 4.2 holds. (In [6] different measures on Γ were considered, not just the $(d-1)$-dimensional Hausdorff measure $\mathcal{H}$ as in this paper. For consistency with [6] we continue to use the phrase '$\mathcal{H}$ is admissible'.) If $\mathcal{H}$ is not necessarily admissible, then there always exists a maximal admissible subset of Γ. More precisely, the following is valid.

Proposition 4.3. *There exists a Borel set $S \subset \Gamma$ such that*

(a) $\operatorname{cap}_\Omega(\Gamma \setminus S) = 0$ *and*
(b) *if $B \subset \Gamma$ is a Borel set with $\operatorname{cap}_\Omega B = 0$, then $\mathcal{H}(B \cap S) = 0$.*

Proof. See Proposition 3.6 in [6]. □

It follows immediately from these two properties that the set S in Proposition 4.3 is $\mathcal{H}$-unique, i.e. if S_1 is another Borel set satisfying (a) and (b), then $\mathcal{H}(S_1 \Delta S) = 0$. If follows from the last paragraph of Section 3 in [6] that the regular part Γ_r equals S up to $\mathcal{H}$-equivalence, i.e. $\mathcal{H}(\Gamma_r \Delta S) = 0$.

In [6], Proposition 5.5 it is shown that always $\mathcal{H}(\Gamma_r) > 0$, without any regularity assumption on the boundary (besides $\mathcal{H}(\Gamma) < \infty$). Moreover, in [6], Example 4.3, an example of a bounded connected open subset $\Omega \subset \mathbb{R}^3$ is given such that $\mathcal{H}(\Gamma) < \infty$ and $\mathcal{H}(\Gamma_o) > 0$. A slightly easier example is as follows, which is a modification of an example at the end of Section 3 in [8]. It also has the property that $\widetilde{H}^1(\Omega) = H^1(\Omega)$.

Example 4.4. For all $(x_0, y_0) \in [0,1] \times [0,1]$ and $r > 0$ let

$$C(x_0, y_0; r) = \left\{(x, y, z) \in \mathbb{R}^3 \colon \left|(x - x_0, y - y_0)\right| \leqslant r \text{ and } z \in [0,1]\right\}$$

be the closed cylinder with axis parallel to the z-axis, radius r, height 1 and standing on $(x_0, y_0, 0)$. Let

$$\Omega = \operatorname{Int}\left(\left([0,1] \times [0,1] \times [-1,0]\right) \cup \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{n-1} C\left(2^{-n}, \frac{k}{n}; 4^{-n}\right)\right).$$

(See Fig. 1.) Then Ω is bounded, connected and $\mathcal{H}(\Gamma) < \infty$. We first show that $\{0\} \times [0,1] \times [0,1] \subset \Gamma_o$, which implies that the trace on Ω is not unique.

For all $m \in \mathbb{N}$ define $u_m \in H^1(\Omega) \cap C(\overline{\Omega})$ by

$$u_m(x, y, z) = \big(0 \vee \big(3^m z\big) \wedge 1\big) \mathbb{1}_{[0, 2^{-m}+4^{-m}]}(x).$$

Then $\|u_m\|^2_{H^1(\Omega)} \leqslant \sum_{n=m}^{\infty} \pi n(4^{-2n} + 3^{2m} 4^{-2n})$ for all $m \in \mathbb{N}$, so $\lim u_m = 0$ in $H^1(\Omega)$. Since $0 \leqslant u_m \leqslant 1$ for all $m \in \mathbb{N}$ it follows from the Lebesgue dominated convergence theorem that $\lim u_m|_\Gamma = \mathbb{1}_{\{0\} \times [0,1] \times [0,1]}$ in $L_2(\Gamma)$. So $\mathbb{1}_{\{0\} \times [0,1] \times [0,1]}$ is a trace of 0 and $\{0\} \times [0,1] \times [0,1] \subset \Gamma_o$, up to $\mathcal{H}$-equivalence.

Finally we show that $H^1(\Omega) = \widetilde{H}^1(\Omega)$. Let $v \in H^1(\Omega) \cap L_\infty(\Omega)$. Define $v_m := v(\mathbb{1} - u_m)$ for all $m \in \mathbb{N}$. Then v_m has a support in a subdomain of Ω with a Lipschitz boundary. So $v_m \in \widetilde{H}^1(\Omega)$. Clearly $\sup_m \|v u_m\|_{H^1(\Omega)} < \infty$. Therefore the sequence $v_1, v_2, \ldots$ has a weakly convergent subsequence in $\widetilde{H}^1(\Omega)$. Moreover, $\lim v u_m = 0$ in $L_2(\Omega)$ and therefore $\lim v_m = v$ weakly in $L_2(\Omega)$. Hence $v \in \widetilde{H}^1(\Omega)$. Since $H^1(\Omega) \cap L_\infty(\Omega)$ is dense in $H^1(\Omega)$ by [16], Theorem 1.4.2(iii), it follows that $H^1(\Omega) \subset \widetilde{H}^1(\Omega)$. Thus $H^1(\Omega) = \widetilde{H}^1(\Omega)$.

In the above example not every element $u \in \widetilde{H}^1(\Omega)$ has a trace. We do not know whether universal existence of a trace implies its uniqueness. More precisely, suppose that every element of $\widetilde{H}^1(\Omega)$ has a trace. Does this imply that the trace on Ω is unique?

5. Mapping properties of the trace

Let $H^1_{\mathcal{H}}(\Omega)$ be the set of all $u \in H^1(\Omega)$ for which there exists a $\varphi \in L_2(\Gamma)$ such that φ is a trace of u. Obviously, $H^1(\Omega) \cap C(\overline{\Omega}) \subset H^1_{\mathcal{H}}(\Omega)$. It follows from the definition of the space $H^1_{\mathcal{H}}(\Omega)$ and the set Γ_r that there exists a unique and well-defined map

$$\mathrm{Tr} : H^1_{\mathcal{H}}(\Omega) \to L_2(\Gamma_r)$$

such that $\mathrm{Tr}\, u$ is a trace of u for all $u \in H^1_{\mathcal{H}}(\Omega)$. Then $\mathrm{Tr}\, u = u|_{\Gamma_r}$ a.e. for all $u \in H^1(\Omega) \cap C(\overline{\Omega})$. We identify $L_2(\Gamma_r)$ in a natural way with the subspace of $L_2(\Gamma)$ of all functions which vanish $\mathcal{H}$-a.e. on Γ_o. Let $c_M > 0$ be the constant as in the Maz'ya inequality (5). Let $u \in H^1_{\mathcal{H}}(\Omega)$. Since $\mathrm{Tr}\, u$ is a trace of u there exists a sequence $(u_n)_{n \in \mathbb{N}}$ in $H^1(\Omega) \cap C(\overline{\Omega})$ such that $\lim u_n = u$ in $H^1(\Omega)$ and $\lim u_n|_\Gamma = \mathrm{Tr}\, u$ in $L_2(\Gamma)$. Applying (5) to u_n and taking the limit $n \to \infty$ gives

$$\int_\Omega |u|^2 \leqslant c_M \left(\int_\Omega |\nabla u|^2 + \int_{\Gamma_r} |\mathrm{Tr}\, u|^2 \right) \tag{7}$$

for all $u \in H^1_{\mathcal{H}}(\Omega)$. Hence one can define the norm $\| \cdot \|_{H^1_{\mathcal{H}}(\Omega)}$ on $H^1_{\mathcal{H}}(\Omega)$ by

$$\|u\|^2_{H^1_{\mathcal{H}}(\Omega)} = \int_\Omega |\nabla u|^2 + \int_{\Gamma_r} |\mathrm{Tr}\, u|^2.$$

Obviously $\mathrm{Tr} : H^1_{\mathcal{H}}(\Omega) \to L_2(\Gamma_r)$ is continuous. On the other hand, we emphasize that in general the map $\mathrm{Tr} : (H^1_{\mathcal{H}}(\Omega), \| \cdot \|_{H^1(\Omega)}) \to L_2(\Gamma_r)$ is not continuous. A counter example is in [9], Remark 3.5(f). It follows from (7) that the norm $\| \cdot \|_{H^1_{\mathcal{H}}(\Omega)}$ is equivalent to the norm

$$u \mapsto \big(\|u\|^2_{H^1(\Omega)} + \|\mathrm{Tr}\, u\|^2_{L_2(\Gamma_r)} \big)^{1/2}.$$

In particular $H^1_{\mathcal{H}}(\Omega)$ is a Hilbert space with inner product

$$(u,v)_{H^1_{\mathcal{H}}(\Omega)} = \int_\Omega \nabla u \cdot \nabla v + \int_{\Gamma_r} \operatorname{Tr} u \operatorname{Tr} v$$

and $H^1_{\mathcal{H}}(\Omega)$ is continuously embedded in $L_2(\Omega)$.

The aim of this section is to study the map Tr. Before doing so, in the following remark, we show how the space $H^1_{\mathcal{H}}(\Omega)$ can be used to give an alternative description of the Dirichlet-to-Neumann operator.

Remark 5.1. The space $D(\ell)$ has the norm

$$u \mapsto \left(\int_\Omega |\nabla u|^2 + \int_\Gamma |u|^2 \right)^{1/2}.$$

First we describe the completion of $D(\ell)$. Define $\Phi : D(\ell) \to H^1_{\mathcal{H}}(\Omega) \oplus L_2(\Gamma_o)$ by $\Phi(u) = (u, u|_{\Gamma_o})$. Then Φ is an isometry with dense range. Therefore the space $H^1_{\mathcal{H}}(\Omega) \oplus L_2(\Gamma_o)$ is 'the' completion of $D(\ell)$ and we identify $D(\ell)$ with $\Phi(D(\ell))$ in the natural manner. Define the form $\tilde{\ell}$ with form domain $D(\tilde{\ell}) = H^1_{\mathcal{H}}(\Omega) \oplus L_2(\Gamma_o)$ by

$$\tilde{\ell}\big((u,\varphi),(v,\psi)\big) = \int_\Omega \nabla u \cdot \nabla v \tag{8}$$

and define the map $\tilde{j} : H^1_{\mathcal{H}}(\Omega) \oplus L_2(\Gamma_o) \to L_2(\Gamma)$ by $\tilde{j}(u,\varphi) = \operatorname{Tr} u + \varphi$. Then $\tilde{\ell}$ and $\tilde{j}$ are the continuous extensions of ℓ and j, where $j : D(\ell) \to L_2(\Gamma)$ is defined by $j(u) = u|_\Gamma$. Therefore D_0 is the operator associated with $(\tilde{\ell}, \tilde{j})$ by [5], Proposition 3.3. Hence if $\varphi, \psi \in L_2(\Gamma)$, then $\varphi \in D(D_0)$ and $D_0\varphi = \psi$ if and only if there exists a $u \in H^1_{\mathcal{H}}(\Omega) \oplus L_2(\Gamma_o)$ such that $\tilde{j}(u) = \varphi$ and

$$\tilde{\ell}(u,v) = \big(\psi, \tilde{j}(v)\big)_{L_2(\Gamma)} \tag{9}$$

for all $v \in H^1_{\mathcal{H}}(\Omega) \oplus L_2(\Gamma_o)$. The latter follows from [5], Theorem 2.1. Then it follows immediately from (9) that the range of D_0 is contained in $L_2(\Gamma_r)$.

We will need the following apparently weaker description of the trace.

Lemma 5.2. *Let $u \in L_2(\Omega)$ and $\varphi \in L_2(\Gamma)$. Suppose there exist $u_1, u_2, \ldots \in H^1(\Omega) \cap C(\overline{\Omega})$ such that $\lim u_n = u$ weakly in $L_2(\Omega)$, $\lim u_n|_{\Gamma_r} = \varphi|_{\Gamma_r}$ weakly in $L_2(\Gamma_r)$ and $\sup \|u_n\|_{H^1(\Omega)} < \infty$. Then $u \in H^1_{\mathcal{H}}(\Omega)$ and φ is a trace of u. In particular, $\operatorname{Tr} u = \varphi \mathbb{1}_{\Gamma_r}$.*

Proof. The sequence $u_1, u_2, \ldots$ is bounded in $H^1(\Omega)$ and the sequence $u_1|_{\Gamma_r}, u_2|_{\Gamma_r}, \ldots$ is bounded in $L_2(\Gamma_r)$. Therefore the sequence $u_1, u_2, \ldots$ is bounded in $H^1_{\mathcal{H}}(\Omega)$. Since the unit ball is weakly compact it follows that, after passing to a subsequence if necessary, the sequence $u_1, u_2, \ldots$ is weakly convergent in $H^1_{\mathcal{H}}(\Omega)$. So $u \in H^1_{\mathcal{H}}(\Omega)$. Since the map Tr is bounded from $H^1_{\mathcal{H}}(\Omega)$ into $L_2(\Gamma_r)$, it is also weakly continuous. Hence $\operatorname{Tr} u = \lim \operatorname{Tr} u_n = \lim u_n|_{\Gamma_r} = \varphi|_{\Gamma_r}$ weakly in $L_2(\Gamma_r)$. So $\varphi \mathbb{1}_{\Gamma_r} = \varphi|_{\Gamma_r}$ is a trace of u. Moreover, $\varphi \mathbb{1}_{\Gamma_o}$ is a trace of 0. Then φ is a trace of u. □

We collect some algebraic properties of the trace.

Proposition 5.3.

(a) *If* $u \in H^1_{\mathcal{H}}(\Omega) \cap L_\infty(\Omega)$*, then* $\operatorname{Tr} u \in L_\infty(\Gamma_r)$ *and* $\|\operatorname{Tr} u\|_\infty \leqslant \|u\|_\infty$*. Moreover, there exist* $u_1, u_2, \ldots \in H^1(\Omega) \cap C(\overline{\Omega})$ *such that* $\|u_n\|_\infty \leqslant \|u\|_\infty$ *for all* $n \in \mathbb{N}$, $\lim u_n = u$ *in* $H^1(\Omega)$ *and* $\lim u_n|_\Gamma = \operatorname{Tr} u$ *in* $L_2(\Gamma)$*.*
(b) *The space* $H^1_{\mathcal{H}}(\Omega) \cap L_\infty(\Omega)$ *is an algebra and* $\operatorname{Tr}(uv) = (\operatorname{Tr} u)(\operatorname{Tr} v)$ *for all* $u, v \in H^1_{\mathcal{H}}(\Omega) \cap L_\infty(\Omega)$*.*

Proof. '(a)'. There exist $u_1, u_2, \ldots \in H^1(\Omega) \cap C(\overline{\Omega})$ such that $\lim u_n = u$ in $H^1(\Omega)$ and $\lim u_n|_\Gamma = \operatorname{Tr} u$ in $L_2(\Gamma)$. For all $n \in \mathbb{N}$ set $v_n = (-M) \vee u_n \wedge M \in H^1(\Omega) \cap C(\overline{\Omega})$, where $M = \|u\|_\infty$. Then $\lim v_n = (-M) \vee u \wedge M = u$ in $H^1(\Omega)$. Moreover, $\lim v_n|_\Gamma = (-M) \vee (\operatorname{Tr} u) \wedge M$ in $L_2(\Gamma)$. So $(-M) \vee (\operatorname{Tr} u) \wedge M$ is a trace of u and $\operatorname{Tr} u = ((-M) \vee (\operatorname{Tr} u) \wedge M)\mathbb{1}_{\Gamma_r} = (-M) \vee (\operatorname{Tr} u) \wedge M$. Then $|\operatorname{Tr} u| \leqslant M$ a.e. Note that $\|v_n\|_\infty \leqslant \|u\|_\infty$ for all $n \in \mathbb{N}$.

'(b)'. Let $u, v \in H^1_{\mathcal{H}}(\Omega) \cap L_\infty(\Omega)$. By statement (a) there exist $u_1, u_2, \ldots, v_1, v_2, \ldots \in H^1(\Omega) \cap C(\overline{\Omega})$ such that $\lim u_n = u$ in $H^1(\Omega)$, $\lim u_n|_\Gamma = \operatorname{Tr} u$ in $L_2(\Gamma)$, $\lim v_n = v$ in $H^1(\Omega)$, $\lim v_n|_\Gamma = \operatorname{Tr} v$ in $L_2(\Gamma)$, and, moreover, $\|u_n\|_\infty \leqslant \|u\|_\infty$ and $\|v_n\|_\infty \leqslant \|v\|_\infty$ for all $n \in \mathbb{N}$. Then $u_n v_n \in H^1(\Omega) \cap C(\overline{\Omega})$ for all $n \in \mathbb{N}$ and

$$\|u_n v_n\|_{H^1(\Omega)} \leqslant \|u_n\|_{H^1(\Omega)} \|v_n\|_\infty + \|u_n\|_\infty \|v_n\|_{H^1(\Omega)} \leqslant \|u_n\|_{H^1(\Omega)} \|v\|_\infty + \|u\|_\infty \|v_n\|_{H^1(\Omega)}$$

for all $n \in \mathbb{N}$. So $\sup \|u_n v_n\|_{H^1(\Omega)} < \infty$. Moreover, $\lim u_n v_n = uv$ in $L_2(\Omega)$ and $\lim(u_n v_n)|_\Gamma = (\operatorname{Tr} u)(\operatorname{Tr} v)$ in $L_2(\Gamma)$. Therefore Lemma 5.2 implies that $uv \in H^1_{\mathcal{H}}(\Omega)$ and $\operatorname{Tr}(uv) = (\operatorname{Tr} u)(\operatorname{Tr} v)$. □

The next lemma is a reformulation of Proposition 3.5.

Lemma 5.4. *The space* $H^1_{\mathcal{H}}(\Omega)$ *is compactly embedded in* $L_2(\Omega)$*.*

Proof. Let $B = \{u \in H^1(\Omega) \cap C(\overline{\Omega}) \colon \int_\Omega |\nabla u|^2 + \int_\Gamma |u|^2 \leqslant 2\}$. By Proposition 3.5 there exists a set $K \subset L_2(\Omega)$ which is compact in $L_2(\Omega)$ such that $B \subset K$. Let $u \in H^1_{\mathcal{H}}(\Omega)$ and suppose that $\|u\|_{H^1_{\mathcal{H}}(\Omega)} \leqslant 1$. There are $u_1, u_2, \ldots \in H^1(\Omega) \cap C(\overline{\Omega})$ such that $\lim u_n = u$ in $H^1(\Omega)$ and $\lim u_n|_\Gamma = \operatorname{Tr} u$ in $L_2(\Gamma)$. Then $u_n \in B \subset K$ for large n and $\lim u_n = u$ in $L_2(\Omega)$. So $u \in K$. □

Clearly $H^1_0(\Omega) \subset \{u \in H^1_{\mathcal{H}}(\Omega) \colon \operatorname{Tr} u = 0\}$. If Ω is a Lipschitz domain, then the converse is valid (see [1], Lemma A 6.10). We next give sufficient conditions for the converse inclusion, which allow Ω to have a cusp.

Proposition 5.5. *Suppose there exists a closed subset* K *of* Γ *such that* $\operatorname{cap}_\Omega K = 0$ *and for all* $z \in \Gamma \setminus K$ *there exists an* $r > 0$ *such that* $B(z, r) \cap \Gamma$ *is a Lipschitz graph with* $B(z, r) \cap \Omega$ *on one side. Then*

$$\{u \in H^1_{\mathcal{H}}(\Omega) \colon \operatorname{Tr} u = 0\} = H^1_0(\Omega).$$

Proof. We may assume that $K \neq \emptyset$. Let $u \in H^1_{\mathcal{H}}(\Omega)$ and suppose that $\operatorname{Tr} u = 0$. We may assume that u is bounded.

Let $\varepsilon > 0$. We first prove that there exists a $\psi \in \widetilde{H}^1(\Omega)$ such that $0 \leqslant \psi \leqslant \mathbb{1}_\Omega$ a.e., $\|\psi\|_{H^1(\Omega)} \leqslant \varepsilon$ and $u(\mathbb{1} - \psi) \in H^1_0(\Omega)$. Define the measure μ on the Borel σ-algebra of $\overline{\Omega}$ by $\mu(A) = |A \cap \Omega|$. Define the form h on $L_2(\overline{\Omega}, \mu)$ with form domain $D(h) = \widetilde{H}^1(\Omega)$ and $h(v, w) = \int_\Omega \nabla v \cdot \nabla w$. Then h is a regular Dirichlet form on $L_2(\overline{\Omega}, \mu)$ and $H^1(\Omega) \cap C(\overline{\Omega})$ is a special standard core for h in the sense of [16]. Moreover, the relative capacity is just the capacity in [16] with respect to the Dirichlet form h on $L_2(\overline{\Omega}, \mu)$. For all $m \in \mathbb{N}$ let

$$K_m = \left\{x \in \overline{\Omega} \colon d(x, K) \leqslant \frac{1}{m}\right\}.$$

Then K_m is compact, $K_1 \supset K_2 \supset \dots$ and $\bigcap_{m=1}^\infty K_m = K$. So by [16], Theorem 2.1.1 there exists an $m \in \mathbb{N}$ such that $\operatorname{cap}_\Omega K_m < \varepsilon$. Next, by [16], Lemma 2.2.7(ii) there exists a $\psi \in H^1(\Omega) \cap C(\overline{\Omega})$ such that $\mathbb{1}_{K_m} \leqslant \psi \leqslant \mathbb{1}$ and $\|\psi\|^2_{H^1(\Omega)} \leqslant \varepsilon$. It is an elementary exercise to see that there exists an open set Ω' in $\mathbb{R}^d$ with Lipschitz boundary such that $\Omega \setminus K_m \subset \Omega' \subset \Omega$. Let $\Gamma' = \partial(\Omega')$. If $x \in \overline{\Omega'}$, then $x \in \overline{\Omega}$. If $x \notin \partial\Omega \cup K_m$ then $x \in \Omega \setminus K_m \subset \Omega'$. So $\Gamma' \subset \Gamma \cup K_m$. By Proposition 5.3(a) there exist $u_1, u_2, \ldots \in H^1(\Omega) \cap C(\overline{\Omega})$ such that $\|u_n\|_\infty \leqslant \|u\|_\infty$ for all $n \in \mathbb{N}$, $\lim u_n = u$ in $H^1(\Omega)$ and $\lim u_n|_\Gamma = \operatorname{Tr} u = 0$ in $L_2(\Gamma)$. For all $n \in \mathbb{N}$ define $v_n = (u_n(\mathbb{1} - \psi))|_{\overline{\Omega'}} \in H^1(\Omega') \cap C(\overline{\Omega'})$ and define $v = (u(\mathbb{1} - \psi))|_{\overline{\Omega'}}$. Then

$$\int_{\Gamma'} |v_n|^2 = \int_{\Gamma'} \bigl|u_n(\mathbb{1} - \psi)\bigr|^2 \leqslant \int_{\Gamma} \bigl|u_n(\mathbb{1} - \psi)\bigr|^2 + \int_{K_m} \bigl|u_n(\mathbb{1} - \psi)\bigr|^2 \leqslant \int_{\Gamma} |u_n|^2.$$

So $\lim v_n|_{\Gamma'} = 0$ in $L_2(\Gamma')$. Moreover, $\lim v_n = v$ in $L_2(\Omega')$ and $\sup \|v_n\|_{H^1(\Omega')} \leqslant \sup \|u_n(\mathbb{1} - \psi)\|_{H^1(\Omega)} < \infty$. So by Lemma 5.2 it follows that $\operatorname{Tr}_{\Omega'} v = 0$. Since Ω' has a Lipschitz boundary it follows that $v \in H^1_0(\Omega') \subset H^1_0(\Omega)$. Then $u(\mathbb{1} - \psi) \in H^1_0(\Omega)$.

Let $n \in \mathbb{N}$. By the above there exists a $\psi_n \in \widetilde{H}^1(\Omega)$ such that $0 \leqslant \psi_n \leqslant 1$ a.e., $\|\psi_n\|_{H^1(\Omega)} \leqslant \frac{1}{n}$ and $u(\mathbb{1} - \psi_n) \in H^1_0(\Omega)$. Then $\sup \|u(\mathbb{1} - \psi_n)\|_{H^1_0(\Omega)} \leqslant \sup \|u(\mathbb{1} - \psi_n)\|_{H^1(\Omega)} < \infty$. So $n \mapsto u(\mathbb{1} - \psi_n)$ has a weakly convergent subsequence in $H^1_0(\Omega)$. Alternatively,

$$\|u\psi_n\|_2 \leqslant \|u\|_\infty \|\psi_n\|_{H^1(\Omega)} \leqslant \frac{1}{n}\|u\|_\infty$$

for all $n \in \mathbb{N}$, so $\lim u(\mathbb{1} - \psi_n) = u$ in $L_2(\Omega)$. Therefore $u \in H^1_0(\Omega)$. $\square$

6. Existence of a trace on $\widetilde{H}^1(\Omega)$

Recall that the trace Tr is defined on the subspace $H^1_{\mathcal{H}}(\Omega)$ of $\widetilde{H}^1(\Omega)$ and that in general the norm on $H^1_{\mathcal{H}}(\Omega)$ is strictly larger than the norm induced from $\widetilde{H}^1(\Omega)$. In this section we characterize whether every element of $\widetilde{H}^1(\Omega)$ has a trace.

We say that Ω *has property* (P) if there exists a $c > 0$ such that

$$\int_\Omega \bigl|u - \langle u\rangle_\Omega\bigr|^2 \leqslant c \int_\Omega |\nabla u|^2$$

for all $u \in H^1(\Omega) \cap C(\overline{\Omega})$, where $\langle u\rangle_\Omega = \frac{1}{|\Omega|}\int_\Omega u$ is the *average* of u on Ω.

Let $\widehat{D_0}$ be the part of the operator D_0 in the space $L_2(\Gamma_r)$. Then $\widehat{D_0}$ is a positive self-adjoint operator on $L_2(\Gamma_r)$.

Theorem 6.1. *The following conditions are equivalent.*

- (i) $H^1_{\mathcal{H}}(\Omega) = \widetilde{H}^1(\Omega)$ *as sets, i.e. every element of* $\widetilde{H}^1(\Omega)$ *has a trace.*
- (ii) *There exists a* $c > 0$ *such that* $\int_{\Gamma_r} |u|^2 \leqslant c \int_\Omega |\nabla u|^2$ *for all* $u \in H^1(\Omega) \cap C(\overline{\Omega})$ *with* $\int_{\Gamma_r} u = 0$.
- (iii) *There exists a* $c > 0$ *such that* $\int_{\Gamma_r} |u|^2 \leqslant c(\int_\Omega |\nabla u|^2 + \int_\Omega |u|^2)$ *for all* $u \in H^1(\Omega) \cap C(\overline{\Omega})$.
- (iv) $0 \notin \sigma_{\mathrm{ess}}(\widehat{D_0})$.

Moreover, if one of these equivalent conditions holds, the space $\widetilde{H}^1(\Omega)$ *is compactly embedded in* $L_2(\Omega)$ *and* Ω *has property* (P).

Proof. '(i) $\Rightarrow$ (iii)'. If (i) is valid, then the norms on the spaces $H^1_{\mathcal{H}}(\Omega)$ and $\widetilde{H}^1(\Omega)$ are equivalent by the closed graph theorem. Since Tr is continuous on $H^1_{\mathcal{H}}(\Omega)$ this implies that (iii) is valid.

'(iii) $\Rightarrow$ (i)'. One always has $H^1_{\mathcal{H}}(\Omega) \subset \widetilde{H}^1(\Omega)$. Therefore the implication follows from Lemma 5.2.

'(ii) $\Rightarrow$ (iii)'. Define $F : (D(\ell), \|\cdot\|_{H^1(\Omega)}) \to \mathbb{R}$ by

$$F(u) = \int_{\Gamma_r} u.$$

We first prove that F is continuous. In order to prove this, it suffices to show that $\ker F$ is closed in $(D(\ell), \|\cdot\|_{H^1(\Omega)})$. Let $u_1, u_2, \ldots \in \ker F$, $u \in D(\ell)$ and suppose that $\lim u_n = u$ in $H^1(\Omega)$. Then for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $\int_\Omega |\nabla(u_n - u_m)|^2 \leqslant \varepsilon$ for all $n, m \geqslant N$. If $c > 0$ is as in (ii), it follows that $\int_{\Gamma_r} |u_n - u_m|^2 \leqslant c\varepsilon$ for all $n, m \geqslant N$, where we used that $\int_{\Gamma_r} (u_n - u_m) = F(u_n - u_m) = 0$ for all $n, m \in \mathbb{N}$. So the sequence $u_1|_{\Gamma_r}, u_2|_{\Gamma_r}, \ldots$ is a Cauchy sequence in $L_2(\Gamma_r)$. Hence $u_1, u_2, \ldots$ is Cauchy sequence in $H^1_{\mathcal{H}}(\Omega)$. Since the space $H^1_{\mathcal{H}}(\Omega)$ is a Hilbert space, the Cauchy sequence converges. Therefore there exists a $\tilde{u} \in H^1_{\mathcal{H}}(\Omega)$ such that $\lim u_n = \tilde{u}$ in $H^1_{\mathcal{H}}(\Omega)$. Then $\lim u_n = \tilde{u}$ in $L_2(\Omega)$, so $u = \tilde{u} \in H^1_{\mathcal{H}}(\Omega)$. But Tr is continuous on $H^1_{\mathcal{H}}(\Gamma)$. So $\lim \operatorname{Tr} u_n = \operatorname{Tr} u$ in $L_2(\Gamma_r)$. Then

$$\int_{\Gamma_r} u = (\operatorname{Tr} u, \mathbb{1}_{\Gamma_r})_{L_2(\Gamma_r)} = \lim (\operatorname{Tr} u_n, \mathbb{1}_{\Gamma_r})_{L_2(\Gamma_r)} = \lim F(u_n) = 0.$$

So $\ker F$ is closed and F is continuous. Hence there is a $c' > 0$ such that $|\langle u|_{\Gamma_r} \rangle_{\Gamma_r}|^2 \leqslant c' \|u\|^2_{H^1(\Omega)}$ for all $u \in H^1(\Omega) \cap C(\overline{\Omega})$, where $\langle \varphi \rangle_{\Gamma_r} = \frac{1}{\mathcal{H}(\Gamma_r)} \int_{\Gamma_r} \varphi$ denote the average of φ for all $\varphi \in L_1(\Gamma_r)$. Finally, let $u \in H^1(\Omega) \cap C(\overline{\Omega})$. Then

$$\begin{aligned}\int_{\Gamma_r} |u|^2 &= \int_{\Gamma_r} \bigl|u - \langle u|_{\Gamma_r} \rangle_{\Gamma_r}\bigr|^2 + \int_{\Gamma_r} \bigl|\langle u|_{\Gamma_r} \rangle_{\Gamma_r}\bigr|^2 \\ &\leqslant c \int_\Omega |\nabla u|^2 + \bigl|\langle u|_{\Gamma_r} \rangle_{\Gamma_r}\bigr|^2 \mathcal{H}(\Gamma_r) \leqslant \bigl(c + c'\mathcal{H}(\Gamma_r)\bigr)\Bigl(\int_\Omega |\nabla u|^2 + \int_\Omega |u|^2\Bigr)\end{aligned}$$

and (iii) is valid.

If (i) is valid, then $\widetilde{H}^1(\Omega)$ is compactly embedded in $L_2(\Omega)$ by Lemma 5.4. Since Ω is connected, it follows that Ω has property (P).

'(iii) $\Rightarrow$ (ii)'. If $c > 0$ is as in (iii), then

$$\int_{\Gamma_r} \bigl|u - \langle u|_{\Gamma_r} \rangle_{\Gamma_r}\bigr|^2 \leqslant \int_{\Gamma_r} \bigl|u - \langle u \rangle_\Omega\bigr|^2 \leqslant c\Bigl(\int_\Omega |\nabla u|^2 + \int_\Omega \bigl|u - \langle u \rangle_\Omega\bigr|^2\Bigr)$$

for all $u \in H^1(\Omega) \cap C(\overline{\Omega})$. Since Ω has property (P), the implication (iii) $\Rightarrow$ (ii) follows.

'(ii) $\Rightarrow$ (iv)'. Suppose (iv) is not valid. Then $0 \in \sigma_{\text{ess}}(\widehat{D_0})$. It follows from Proposition 4.1 that for all $n \in \mathbb{N}$ there exists a $\varphi_n \in D(\widehat{D_0})$ such that $\int_{\Gamma_r} \varphi_n = 0$ and $0 < (\widehat{D_0}\varphi_n, \varphi_n)_{L_2(\Gamma_r)} \leqslant \frac{1}{n} \int_{\Gamma_r} |\varphi_n|^2$. Next there exists a unique $u_n \in H^1_{\mathcal{H}}(\Omega)$ such that $\operatorname{Tr} u_n = \varphi_n$ and $\int_\Omega \nabla u_n \cdot \nabla v = (D_0\varphi_n, \operatorname{Tr} v)_{L_2(\Gamma)} = (\widehat{D_0}\varphi_n, \operatorname{Tr} v)_{L_2(\Gamma_r)}$ for all $v \in H^1_{\mathcal{H}}(\Omega)$. Therefore $\int_\Omega |\nabla u_n|^2 = (\widehat{D_0}\varphi_n, \varphi_n)_{L_2(\Gamma_r)} \leqslant \frac{1}{n} \int_{\Gamma_r} |\operatorname{Tr} u_n|^2$. So (ii) is not valid. Therefore (ii) $\Rightarrow$ (iv).

'(iv) $\Rightarrow$ (ii)'. Let ℓ_c and $\widehat{\ell_c}$ be the closed positive symmetric forms associated with D_0 and $\widehat{D_0}$. Since $0 \notin \sigma_{\text{ess}}(\widehat{D_0})$ it follows from Proposition 4.1 that there is a $\mu > 0$ such that $\widehat{\ell_c}(\varphi) \geqslant \mu \int_{\Gamma_r} |\varphi|^2$ for all $\varphi \in D(\widehat{\ell_c})$ with $\int_{\Gamma_r} \varphi = 0$. So by [5], Theorem 2.5 it follows that $\tilde{\ell}(u) = \ell_c(\operatorname{Tr} u) = \widehat{\ell_c}(\operatorname{Tr} u) \geqslant \mu \int_{\Gamma_r} |\operatorname{Tr} u|^2$

for all $u \in (\ker \mathrm{Tr})^\perp \subset H^1_{\mathcal{H}}(\Omega)$, where the orthoplement is in the Hilbert space $H^1_{\mathcal{H}}(\Omega)$ and $\tilde{\ell}$ is as in (8). Now let $u \in H^1_{\mathcal{H}}(\Omega)$ with $\int_{\Gamma_r} \mathrm{Tr}\, u = 0$. Write $u = u_1 + u_2$ with $u_1 \in (\ker \mathrm{Tr})^\perp$ and $u_2 \in \ker \mathrm{Tr}$. Then $\int_{\Gamma_r} \mathrm{Tr}\, u_1 = 0$. Moreover, $\tilde{\ell}(u_1, u_2) = (u_1, u_2)_{H^1_{\mathcal{H}}(\Omega)} = 0$. Therefore

$$\tilde{\ell}(u) = \tilde{\ell}(u_1) + \tilde{\ell}(u_2) \geqslant \tilde{\ell}(u_1) \geqslant \mu \int_{\Gamma_r} |\mathrm{Tr}\, u_1|^2 = \mu \int_{\Gamma_r} |\mathrm{Tr}\, u|^2.$$

So (ii) is valid. This completes the proof of the theorem. □

Theorem 1.3 characterizes when every element of $\widetilde{H}^1(\Omega)$ has a unique trace.

Proof of Theorem 1.3. If (i) or (v) is valid then the semigroup S is irreducible by Theorem 1.2 and Proposition 4.1. Therefore (i) $\Leftrightarrow$ (v) follows from Theorem 2.4.

'(v) $\Rightarrow$ (ii)'. If (v) is valid, then the trace on Ω is unique. Then (ii) follows from Theorem 6.1.

'(ii) $\Rightarrow$ (iii)'. This is similar to the proof of (ii) $\Rightarrow$ (iii) in the proof of Theorem 6.1.

'(iii) $\Rightarrow$ (iv)'. This is trivial.

'(iv) $\Rightarrow$ (v)'. If (iv) is valid, then the trace on Ω is unique. Then (v) follows from Theorem 6.1. □

7. Compact trace

In the previous section we investigated when the trace is bounded from $\widetilde{H}^1(\Omega)$ into $L_2(\Gamma)$. Now we want to characterize when the trace is compact.

Proposition 7.1. *The following are equivalent.*

(i) *The Dirichlet-to-Neumann operator D_0 has a compact resolvent.*
(ii) *The map $j : D(\ell) \to L_2(\Gamma)$ defined by $j(u) = u|_\Gamma$ is compact, where $D(\ell)$ carries the H^1-norm.*
(iii) *The trace on Ω is unique and the map* Tr *is compact (from $H^1_{\mathcal{H}}(\Omega)$ into $L_2(\Gamma)$).*
(iv) *Every element in $\widetilde{H}^1(\Omega)$ has a unique trace and the map* $\mathrm{Tr} : \widetilde{H}^1(\Omega) \to L_2(\Gamma)$ *is compact.*

Proof. '(i) $\Leftrightarrow$ (ii)'. Let ℓ_c be the closed positive symmetric form on $L_2(\Gamma)$ associated with D_0. Then D_0 has compact resolvent if and only if the embedding from $D(\ell_c)$ into $L_2(\Gamma)$ is compact. Let V be the completion of $D(\ell)$, where $D(\ell)$ has the (usual) norm $u \mapsto (\int_\Omega |\nabla u|^2 + \int_\Gamma |u|^2)^{1/2}$. Let $\tilde{j} : V \to L_2(\Gamma)$ be the continuous extension of the map j. Then $D(\ell_c) = \tilde{j}((\ker \tilde{j})^\perp)$, with the quotient norm of $(\ker \tilde{j})^\perp$ by [5], Theorem 2.5. Therefore the embedding from $D(\ell_c)$ into $L_2(\Gamma)$ is compact if and only if $\tilde{j}|_{(\ker \tilde{j})^\perp} : (\ker \tilde{j})^\perp \to L_2(\Gamma)$ is compact. The latter map is compact if and only if $\tilde{j}$ is compact and clearly that is equivalent with the compactness of the map j.

'(i) $\Rightarrow$ (iv)'. If D_0 has compact resolvent then $0 \notin \sigma_{\mathrm{ess}}(D_0)$. Hence every element of $\widetilde{H}^1(\Omega)$ has a unique trace by Theorem 1.3. Moreover, the norms on $\widetilde{H}^1(\Omega)$ and $H^1_{\mathcal{H}}(\Omega)$ are equivalent. So by (ii) the map $\mathrm{Tr}|_{H^1(\Omega) \cap C(\overline{\Omega})} : (H^1(\Omega) \cap C(\overline{\Omega}), \|\cdot\|_{\widetilde{H}^1(\Omega)}) \to L_2(\Gamma)$ is compact. Then (iii) follows by density.

'(iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii)'. This is trivial. □

Corollary 7.2. *If the Dirichlet-to-Neumann operator D_0 has compact resolvent, then $\widetilde{H}^1(\Omega)$ is compactly embedded in $L_2(\Omega)$.*

We will see in Example 9.4 that the compactness of the embedding of $H^1(\Omega)$ in $L_2(\Omega)$ does not suffice to ensure that the Dirichlet-to-Neumann operator has compact resolvent.

8. Robin boundary conditions for the Laplacian

Finally we wish to consider Robin boundary conditions with a possibly negative measure. For all $\beta \in \mathbb{R}$ define the symmetric densely defined form a_β by

$$a_\beta(u, v) = \int_\Omega \nabla u \cdot \nabla v - \beta \int_\Gamma uv$$

with form domain $D(a_\beta) = H^1(\Omega) \cap C(\overline{\Omega})$.

Proposition 8.1.

(a) *Every element of $\widetilde{H}^1(\Omega)$ has a unique trace if and only if there exists a $\beta > 0$ such that the form a_β is lower bounded.*
(b) *The map* Tr *is compact if and only if for all $\beta > 0$ the form a_β is lower bounded.*

Proof. Statement (a) is easy, by Theorem 1.3(iii) $\Leftrightarrow$ (iv), so it remains to prove statement (b). '$\Rightarrow$'. This is as in [3], Proposition 2.2. '$\Leftarrow$'. Let $u_1, u_2, \ldots \in H^1_{\mathcal{H}}(\Omega)$ and suppose that $\lim u_n = 0$ weakly in $H^1_{\mathcal{H}}(\Omega)$. We shall show that $\lim \operatorname{Tr} u_n = 0$ in $L_2(\Gamma)$. Let $\varepsilon > 0$. There exists an $M \geqslant 0$ such that $\int_\Omega |\nabla u_n|^2 \leqslant M$ for all $n \in \mathbb{N}$. Note that $\widetilde{H}^1(\Omega) = H^1_{\mathcal{H}}(\Omega)$, with equivalent norms, by statement (a) and Theorem 6.1. Choosing $\beta = \frac{M}{\varepsilon}$, it follows from the assumption that there exists a $c > 0$ such that

$$\int_\Gamma |\operatorname{Tr} u|^2 \leqslant \frac{\varepsilon}{M} \int_\Omega |\nabla u|^2 + c \int_\Omega |u|^2$$

first for all $u \in H^1(\Omega) \cap C(\overline{\Omega})$ and then by continuity for all $u \in \widetilde{H}^1(\Omega)$. Then

$$\int_\Gamma |\operatorname{Tr} u_n|^2 \leqslant \varepsilon + c\|u_n\|^2_{L_2(\Omega)} \tag{10}$$

for all $n \in \mathbb{N}$. Since the embedding of the space $H^1_{\mathcal{H}}(\Omega)$ into $L_2(\Omega)$ is compact by Lemma 5.4, one deduces that $\lim u_n = 0$ strongly in $L_2(\Omega)$. Therefore one deduces from (10) that $\limsup \|\operatorname{Tr} u_n\|^2_{L_2(\Gamma)} \leqslant \varepsilon$ and the proposition follows. □

We suppose for the remaining part of this section that every element of $\widetilde{H}^1(\Omega)$ has a unique trace. Let

$$\beta_0 = \sup\{\beta > 0\colon \text{ the form } a_\beta \text{ is lower bounded}\} \in (0, \infty].$$

One has $\beta_0 = \infty$ if Ω is a Lipschitz domain, but in general $\beta_0 < \infty$, see Example 9.4. It follows that a_β is lower bounded for all $\beta \in (-\infty, \beta_0)$. Let $R^{(\beta)}$ be the associated operator.

Proposition 8.2. *Let $\beta \in (-\infty, \beta_0)$ and $u, f \in L_2(\Omega)$. Then $u \in D(R^{(\beta)})$ and $R^{(\beta)} u = f$ if and only if $u \in \widetilde{H}^1(\Omega)$, $-\Delta u = f$, u has a normal derivative in $L_2(\Gamma)$ and $\frac{\partial u}{\partial \nu} = \beta \operatorname{Tr} u$.*

Proof. '$\Rightarrow$'. Let $\beta_1 \in (\beta, \beta_0)$. Then a_{β_1} is lower bounded, so there exists a $\gamma_1 > 0$ such that $a_{\beta_1}(u) \geqslant -\gamma_1 \|u\|^2_{L_2(\Omega)}$ for all $u \in H^1(\Omega) \cap C(\overline{\Omega})$. Then $\beta_1 \int_\Gamma |u|^2 \leqslant \int_\Omega |\nabla u|^2 + \gamma_1 \int_\Omega |u|^2$ and

$$(\beta_1 - \beta) \int_\Gamma |u|^2 \leqslant \int_\Omega |\nabla u|^2 - \beta \int_\Gamma |u|^2 + \gamma_1 \int_\Omega |u|^2 = a_\beta(u) + \gamma_1 \int_\Omega |u|^2 \tag{11}$$

for all $u \in H^1(\Omega) \cap C(\overline{\Omega})$. There exists a Cauchy sequence $u_1, u_2, \ldots$ in $D(a_\beta)$ such that $\lim u_n = u$ in $L_2(\Omega)$ and $\lim a_\beta(u_n, v) = (f, v)_{L_2(\Omega)}$ for all $v \in H^1(\Omega) \cap C(\overline{\Omega})$. Then $\sup a_\beta(u_n) < \infty$ and $\sup \int_\Omega |u_n|^2 < \infty$. So by (11) also $\sup \int_\Gamma |u_n|^2 < \infty$ and subsequently $\sup \int_\Omega |\nabla u_n|^2 < \infty$. So $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\Omega)$ and $(u_n|_\Gamma)_{n\in\mathbb{N}}$ is bounded in $L_2(\Gamma)$. Without loss of generality we may assume that the sequence $(u_n)_{n\in\mathbb{N}}$ is weakly convergent in $H^1(\Omega)$ and $(u_n|_\Gamma)_{n\in\mathbb{N}}$ is weakly convergent in $L_2(\Gamma)$. Since $\lim u_n = u$ in $L_2(\Omega)$ it follows that $u \in H^1(\Omega)$. Then $u \in H^1_{\mathcal{H}}(\Omega)$ by Lemma 5.2. Therefore $u \in \widetilde{H}^1(\Omega)$ by Proposition 8.1(a) and Theorem 6.1. Then

$$\int_\Omega \nabla u \cdot \nabla v - \beta \int_\Gamma (\operatorname{Tr} u) v = \lim_{n\to\infty} a_\beta(u_n, v) = \int_\Omega f v$$

for all $v \in H^1(\Omega) \cap C(\overline{\Omega})$. Therefore $-\Delta u = f$, u has a normal derivative in $L_2(\Gamma)$ and $\frac{\partial u}{\partial \nu} = \beta \operatorname{Tr} u$.

'$\Leftarrow$'. There exist $u_1, u_2, \ldots \in H^1(\Omega) \cap C(\overline{\Omega})$ such that $\lim u_n = u$ in $H^1(\Omega)$ and $\lim u_n|_\Gamma = \operatorname{Tr} u$ in $L_2(\Gamma)$. It follows from the definition of $\frac{\partial u}{\partial \nu}$ that

$$\lim_{n\to\infty} a_\beta(u_n, v) = \int_\Omega \nabla u \cdot \nabla v - \beta \int_\Gamma (\operatorname{Tr} u) v = \int_\Omega f v$$

for all $v \in H^1(\Omega) \cap C(\overline{\Omega})$. Moreover, $\lim u_n = u$ in $L_2(\Omega)$ and $u_1, u_2, \ldots$ is a Cauchy sequence in $D(a_\beta)$. So $u \in D(R^{(\beta)})$ and $R^{(\beta)} u = f$. □

If $\beta \in (-\infty, \beta_0)$ then $R^{(\beta)}$ has compact resolvent by Lemma 5.4 and [5], Lemma 2.7. This has consequences for the Dirichlet-to-Neumann operator.

Proposition 8.3. *If $\beta \in (0, \beta_0)$, then $\dim \ker(D_0 - \beta I) < \infty$ and $\sigma_{\rm p}(D_0) \cap [0, \beta]$ is finite.*

Proof. Let $N \in \mathbb{N}$, $\beta_1, \ldots, \beta_N \in (0, \beta]$ and $\varphi_1, \ldots, \varphi_N$ be an orthonormal system in $L_2(\Gamma)$ such that $D_0 \varphi_n = \beta_n \varphi_n$ for all $n \in \{1, \ldots, N\}$. Then $\varphi_n \in L_2(\Gamma_r)$ since $\beta_n \neq 0$. For all $n \in \{1, \ldots, N\}$ let $u_n \in H^1_{\mathcal{H}}(\Omega)$ be the unique element such that $\operatorname{Tr} u_n = \varphi_n$ and

$$\int_\Omega \nabla u_n \cdot \nabla v = \beta_n \int_\Gamma \varphi_n \operatorname{Tr} v$$

for all $v \in H^1_{\mathcal{H}}(\Omega)$. Then

$$\int_\Omega \nabla u_n \cdot \nabla u_m = \beta_n \int_\Gamma \varphi_n \varphi_m = \beta_n \delta_{nm}$$

and

$$(u_n, u_m)_{H^1_{\mathcal{H}}(\Omega)} = (\beta_n + 1) \delta_{nm}$$

for all $n, m \in \{1, \ldots, N\}$. Therefore $u_1, \ldots, u_N$ is linearly independent in $H^1_{\mathcal{H}}(\Omega)$. Let $\alpha_1, \ldots, \alpha_N \in \mathbb{R}$. Then

$$a_\beta\left(\sum_{n=1}^N \alpha_n u_n\right) = \sum_{n,m=1}^N \alpha_n \alpha_m \left(\int_\Omega \nabla u_n \cdot \nabla u_m - \beta \int_\Gamma \varphi_n \varphi_m\right) = \sum_{n=1}^N |\alpha_n|^2 (\beta_n - \beta) \leqslant 0.$$

Therefore

$$\operatorname{span}\{u_1, \ldots, u_N\} \subset \left\{u \in H^1_{\mathcal{H}}(\Omega)\colon\ a_\beta(u) \leqslant 0\right\}.$$

Since $R^{(\beta)}$ has a compact resolvent, the right hand space is finite dimensional. This proves the proposition. □

9. Examples

In this section we give two striking examples of connected bounded open sets with a continuous boundary such that the Dirichlet-to-Neumann operator does not have compact resolvent. In both examples the trace on Ω is unique. In one example every element of $\widetilde{H}^1(\Omega)$ has a trace, in the other one not. If Ω has a continuous boundary, then the space $H^1(\Omega)$ is compactly embedded in $L_2(\Omega)$ by Proposition 1.1(b). Therefore Ω has the (Neumann type) Poincaré property, which is in this case property (P).

We do not know, however, whether the trace on Ω is unique if Ω has continuous boundary.

In the first example we explicitly give an element of $\widetilde{H}^1(\Omega)$ which does not have a trace.

Example 9.1. Let

$$\Omega = \left\{(x, y) \in \mathbb{R}^2\colon\ 0 < x < 1 \text{ and } -x^4 < y < x^4\right\}.$$

Clearly the set Ω is open, connected and the 1-dimensional Hausdorff measure of the boundary of Ω is finite. Also Ω has a continuous boundary. Therefore $\widetilde{H}^1(\Omega) = H^1(\Omega)$ and $H^1(\Omega)$ is compactly embedded in $L_2(\Omega)$ by Proposition 1.1. Moreover, the trace on Ω is unique since $\Gamma \setminus \{(0,0)\}$ is locally Lipschitz. Define $u : \Omega \to \mathbb{R}$ by $u(x, y) = \frac{1}{x}$. Then $u \in H^1(\Omega)$. Since

$$\int_0^1 \left|u\left(x, x^4\right)\right|^2 \sqrt{1 + \left(4x^3\right)^2}\, dx = \infty,$$

it follows that u does not have a trace. In particular the Dirichlet-to-Neumann operator does not have compact resolvent by Proposition 7.1.

It follows that the semigroup S generated by $-D_0$ is not compact. Therefore S_t does not have a bounded kernel for all $t > 0$. Hence S_t does not map $L_2(\Gamma)$ into $L_\infty(\Gamma)$. Since S is submarkovian, this implies that S is not ultracontractive.

The next estimate is used in Example 9.4, but is also of independent interest.

Lemma 9.2. *Let $e_1, e_2 \in \mathbb{R}^2$ with $\|e_1\| = \|e_2\| = 1$ and $|(e_1, e_2)| \neq 1$. Let $a, b > 0$ and set*

$$\Omega = \left\{s e_1 + t e_2\colon\ s \in (0, a) \text{ and } t \in (0, b)\right\}.$$

Then

$$\int_0^a \left|u(se_1)\right|^2 ds \leqslant \frac{1}{\sqrt{1-|(e_1,e_2)|^2}}\left(\frac{2}{b}\int_\Omega |u|^2 + b\int_\Omega |\nabla u|^2\right)$$

for all $u \in H^1(\Omega) \cap C(\overline{\Omega})$.

Proof. Let $s \in (0,a)$ and $t \in (0,b)$. Then

$$u(se_1) = u(se_1+te_2) - \int_0^t e_2 \cdot (\nabla u)(se_1+re_2)\,dr$$

and therefore

$$\left|u(se_1)\right|^2 \leqslant 2\left|u(se_1+te_2)\right|^2 + 2t\int_0^b \left|(\nabla u)(se_1+re_2)\right|^2 dr.$$

Hence integrating with respect to t over $(0,b)$ and dividing by b yields

$$\left|u(se_1)\right|^2 \leqslant \frac{2}{b}\int_0^b \left|u(se_1+te_2)\right|^2 dt + b\int_0^b \left|(\nabla u)(se_1+re_2)\right|^2 dr$$

and

$$\begin{aligned}\int_0^a \left|u(se_1)\right|^2 ds &\leqslant \frac{2}{b}\int_0^a\int_0^b \left|u(se_1+te_2)\right|^2 dt\,ds + b\int_0^a\int_0^b \left|(\nabla u)(se_1+re_2)\right|^2 dr\,ds \\ &= \frac{1}{\sqrt{1-|(e_1,e_2)|^2}}\left(\frac{2}{b}\int_\Omega |u|^2 + b\int_\Omega |\nabla u|^2\right)\end{aligned}$$

by a change of variables. □

Lemma 9.3. *Let* $a \in (0,1]$. *Define*

$$\Omega = \left\{(x,y) \in \mathbb{R}^2\colon\ 0 < y < a \text{ and } |x| < a^2 - ay\right\}.$$

Let $V = \{u \in H^1(\Omega) \cap C(\overline{\Omega}) : u|_{[-a^2,a^2]\times\{0\}} = 0\}$. *Then*

$$\frac{1}{3} \leqslant \sup\left\{\frac{\|\operatorname{Tr} u\|^2_{L_2(\Gamma)}}{\|u\|^2_{H^1(\Omega)}}\colon\ u \in V \setminus \{0\}\right\} \leqslant 2.$$

Proof. Define $u : \overline{\Omega} \to [0, \infty)$ by $u(x, y) = y$. Then $u \in V$. Moreover, $\int_\Omega |u|^2 = \frac{a^5}{6}$, $\int_\Omega |\nabla u|^2 = a^3$ and $\int_\Gamma |\operatorname{Tr} u|^2 = \frac{2}{3} a^3 \sqrt{1+a^2}$. Therefore $\|\operatorname{Tr} u\|^2_{L_2(\Gamma)} \geqslant \frac{1}{3} \|u\|^2_{H^1(\Omega)}$. This proves the first inequality.

Next let $u \in V$ and $t \in [0, a]$. Then $u(a^2 - at, t) = \int_0^t u_y(a^2 - at, s)\, ds$. So

$$\left|u\left(a^2 - at, t\right)\right|^2 \leqslant t \int_0^t \left|(\nabla u)\left(a^2 - at, s\right)\right|^2 ds \leqslant a \int_0^t \left|(\nabla u)\left(a^2 - at, s\right)\right|^2 ds.$$

Hence

$$\sqrt{1+a^2} \int_0^a \left|u\left(a^2 - at, t\right)\right|^2 dt \leqslant a\sqrt{1+a^2} \int_0^a \int_0^t \left|(\nabla u)\left(a^2 - at, s\right)\right|^2 ds\, dt$$
$$= \sqrt{1+a^2} \int_{\Omega_+} |\nabla u|^2,$$

where $\Omega_+ = \Omega \cap ((0, \infty) \times \mathbb{R})$. So $\int_\Gamma |\operatorname{Tr} u|^2 \leqslant \sqrt{1+a^2} \int_\Omega |\nabla u|^2$, and the lemma follows. □

We next give an example of an open connected bounded set Ω in $\mathbb{R}^2$ with continuous boundary, such that every element of $H^1(\Omega)$ has a unique trace, $H^1(\Omega)$ is compactly embedded in $L_2(\Omega)$, but the Dirichlet-to-Neumann operator does not have compact resolvent.

Example 9.4. Let $\Omega_0 = (-1, 1) \times (-1, 0)$ and for all $n \in \mathbb{N}$ let

$$\Omega_n = \left\{(x, y) \in \mathbb{R}^2 \colon 0 < y < a_n \text{ and } \left|x - 2^{-n}\right| < a_n^2 - a_n y\right\},$$

where $a_n = 4^{-n}$. Let $\Omega = \overset{\circ}{\overline{\bigcup_{n=0}^\infty \Omega_n}}$. (See Fig. 2.) Then Ω is open, connected, the boundary is continuous and $\mathcal{H}(\Gamma) < \infty$.

We show that every element of $\widetilde{H}^1(\Omega)$ has a unique trace by showing that Condition (iii) of Theorem 1.3 is valid. Since the set Ω_0 is Lipschitz, it has the extension property. Therefore there exists a linear map $E : H^1(\Omega_0) \cap C(\overline{\Omega_0}) \to H^1(\mathbb{R}^2) \cap C(\mathbb{R}^2)$ and a constant $c_E > 0$ such that $(Eu)|_{\Omega_0} = u$ and $\|Eu\|^2_{H^1(\mathbb{R}^2)} \leqslant c_E \|u\|^2_{H^1(\Omega_0)}$ for all $u \in H^1(\Omega_0) \cap C(\overline{\Omega_0})$.

Let $u \in H^1(\Omega) \cap C(\overline{\Omega})$. Set $v = u|_{\overline{\Omega_0}}$ and $w = Ev$. Then $w \in H^1(\mathbb{R}^2) \cap C(\mathbb{R}^2)$ and $(u - w)|_{\overline{\Omega_n}} \in H^1(\Omega_n) \cap C(\overline{\Omega_n})$ with $(u - w)|_{[2^{-n} - a_n^2, 2^{-n} + a_n^2] \times \{0\}} = 0$ for all $n \in \mathbb{N}$. Then

$$\int_\Gamma |u|^2 \leqslant \int_{\partial\Omega_0} |u|^2 + \int_{\Gamma \setminus \partial\Omega_0} |u|^2 \leqslant \int_{\partial\Omega_0} |u|^2 + 2 \int_{\Gamma \setminus \partial\Omega_0} |u - w|^2 + 2 \int_{\Gamma \setminus \partial\Omega_0} |w|^2. \tag{12}$$

We estimate the three terms in (12).

First, it follows from Lemma 9.2 that

$$\int_{\partial\Omega_0} |u|^2 \leqslant 8\|u\|^2_{H^1(\Omega_0)} \leqslant 8\|u\|^2_{H^1(\Omega)}.$$

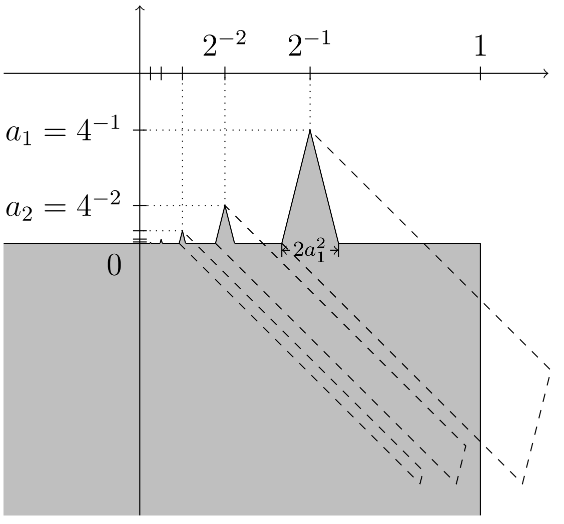

Fig. 2. The domain in Example 9.4.

Secondly, by Lemma 9.3 one deduces that

$$2\int\limits_{\Gamma\setminus\partial\Omega_0} |u-w|^2 \leqslant 2\sum_{n=1}^{\infty}\int\limits_{\partial\Omega_n} |u-w|^2 \leqslant 4\sum_{n=1}^{\infty}\|u-w\|_{H^1(\Omega_n)}^2$$

$$\leqslant 4\|u-w\|_{H^1(\Omega)}^2 \leqslant 8\|u\|_{H^1(\Omega)}^2 + 8\|w\|_{H^1(\Omega)}^2.$$

But

$$\|w\|_{H^1(\Omega)}^2 \leqslant \|w\|_{H^1(\mathbb{R}^2)}^2 \leqslant c_E\|v\|_{H^1(\Omega_0)}^2 \leqslant c_E\|u\|_{H^1(\Omega)}^2.$$

So

$$2\int\limits_{\Gamma\setminus\partial\Omega_0} |u-w|^2 \leqslant 8(1+c_E)\|u\|_{H^1(\Omega)}^2.$$

Therefore it remains to estimate the last term $\int_{\Gamma\setminus\partial\Omega_0}|w|^2$ in (12). Let $n\in\mathbb{N}$ and set

$$\Omega_n' = \left\{\left(2^{-n}-a_n^2, 0\right) + s\frac{1}{\sqrt{1+a_n^2}}(a_n, 1) + t\frac{1}{\sqrt{2}}(1,-1)\colon\ s\in\left(0, a_n\sqrt{1+a_n^2}\right) \text{ and } t\in(0,1)\right\}.$$

Let $\Gamma_n^{(l)} = \partial\Omega_n \cap ((-\infty, 2^{-n})\times(0,\infty))$. Then $\Gamma_n^{(l)} = \partial\Omega_n' \cap ((-\infty, 2^{-n})\times(0,\infty))$ and it follows from Lemma 9.2 that

$$\int\limits_{\Gamma_n^{(l)}} |w|^2 \leqslant 4\|w\|_{H^1(\Omega_n')}^2$$

and therefore, again by disjointness,

$$\sum_{n=1}^{\infty} \int_{\Gamma_n^{(l)}} |w|^2 \leqslant 4\|w\|_{H^1(\mathbb{R}^2)}^2 \leqslant 4c_E\|u\|_{H^1(\Omega)}^2.$$

A similar estimate is valid on the right top boundary of $\partial\Omega_n$. Combining these partial estimates with (12) one deduces that

$$\int_{\Gamma} |u|^2 \leqslant (16 + 24c_E)\|u\|_{H^1(\Omega)}^2.$$

So by Theorem 1.3 every element of $\widetilde{H}^1(\Omega)$ has a unique trace.

For all $n \in \mathbb{N}$ define $u_n \in H^1(\Omega) \cap C(\overline{\Omega})$ by $u_n(x, y) = y\mathbb{1}_{\overline{\Omega_n}}(x, y)$. Then it follows from (the proof of) Lemma 9.3 that $\|\operatorname{Tr} u_n\|_{L_2(\Gamma)}^2 \geqslant \frac{1}{3}\|u_n\|_{H^1(\Omega)}^2$. Since the norms on $H^1(\Omega)$ and $H^1_{\mathcal{H}}(\Omega)$ are equivalent and the functions $u_1, u_2, \ldots$ have disjoint support, it follows that Tr is not compact. Therefore the Dirichlet-to-Neumann operator does not have a compact resolvent. Nevertheless, since Ω has a continuous boundary, the space $H^1(\Omega)$ is compactly embedded in $L_2(\Omega)$ by Proposition 1.1.

Note that as in Example 9.1 the semigroup S is not ultracontractive. Hence there does not exists a $q > 2$ such that $\operatorname{Tr} u \in L_q(\Gamma)$ for all $u \in H^1(\Omega)$. Indeed, otherwise by the closed graph theorem there exists a $c > 0$ such that $\|\operatorname{Tr} u\|_{L_q(\Gamma)} \leqslant c\|u\|_{H^1_{\mathcal{H}}(\Omega)}$ for all $u \in H^1(\Omega) = H^1_{\mathcal{H}}(\Omega)$. Let $t > 0$ and $\varphi \in L_2(\Gamma)$. Then $S_t\varphi \in D(D_0)$, so by Remark 5.1 there exists a $u \in H^1(\Omega)$ such that $\operatorname{Tr} u = S_t\varphi$ and

$$\int_{\Omega} \nabla u \cdot \nabla v = (D_0 S_t\varphi, \operatorname{Tr} v)_{L_2(\Gamma)}$$

for all $v \in H^1(\Omega)$. Then

$$\begin{aligned}
\|S_t\varphi\|_{L_q(\Gamma)}^2 &\leqslant c^2\|u\|_{H^1_{\mathcal{H}}(\Omega)}^2 \\
&= c^2\biggl(\int_{\Omega} |\nabla u|^2 + \|\operatorname{Tr} u\|_{L_2(\Gamma)}^2\biggr) \\
&= c^2\bigl((D_0 S_t\varphi, S_t\varphi)_{L_2(\Gamma)} + \|S_t\varphi\|_{L_2(\Gamma)}^2\bigr) \\
&\leqslant c^2\left(1 + \frac{1}{t}\right)\|\varphi\|_{L_2(\Gamma)}^2.
\end{aligned}$$

But this implies that S is ultracontractive by [20], Lemma 6.1, which is a contradiction. Thus there is no $q > 2$ such that $\operatorname{Tr} u \in L_q(\Gamma)$ for all $u \in H^1(\Omega)$.

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