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Diffusion determines the manifold

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Abstract. We prove under a weak smoothness condition that two Riemannian manifolds are isomorphic if and only if there exists an order isomorphism which intertwines with the Dirichlet type heat semigroups on the manifolds.

1. Introduction

A fundamental problem raised in Kac's famous article [21] 'Can one hear the shape of a drum' is whether two isospectral manifolds are isomorphic. The answer is negative in general. Milnor gave a counter example for compact Riemannian manifolds [22]. In the Euclidean case the first example was given in dimension 4 by Urakawa [29]. Then Gordon–Webb–Wolpert [17] constructed two polygons in $\mathbb{R}^2$ which are isospectral but not isomorphic. Moreover, [16] constructed two isospectral convex open sets in $\mathbb{R}^4$ which are isospectral but not isomorphic. Kac's question in the strict sense, namely whether two isospectral bounded open sets in $\mathbb{R}^2$ with C^∞-boundary are isometric, is still open. But there are recent positive results by Zelditch [32] for open sets in $\mathbb{R}^2$ with analytic boundary verifying some symmetry conditions.

To say that the two manifolds are isospectral means by definition that the corresponding Dirichlet Laplacians have the same eigenvalues counted with multiplicity. This, in turn, can be reformulated by saying that there exists a unitary operator U intertwining the two heat semigroups. The heat semigroups are positive, i.e. positive initial values lead to positive solutions. These positive solutions describe the heat diffusion on the manifold. Thus, if instead of a unitary operator, we consider an order isomorphism U (i.e. U is linear, bijective and $U\varphi \geqq 0$ if and only if $\varphi \geqq 0$) on L_2, then to say that U intertwines the heat semigroups means that U maps the positive solutions to positive solutions. It was shown in [6] that in the Euclidean case, i.e. if we consider open connected sets in $\mathbb{R}^d$, then these sets are necessarily congruent as soon as such an intertwining order isomorphism exists. This may be rephrased by saying that diffusion determines the body. The aim of this paper is to extend this result to manifolds.

There are several notable new features coming into play in the non-Euclidean case. First of all, in [6] a precise regularity condition has been established under which the result is valid. The open sets have to be regular in capacity (this means loosely speaking that they do not have holes of capacity 0). Some effort is made in this paper to extend this notion to manifolds, which is not possible in an immediate way. It turns out that all complete Riemannian manifolds satisfy this regularity condition.

There are other results on Riemannian manifolds where heat flow determines a geometric property. Norris [23] established a Varadhan type [30], [31] equality for a small time limit of the heat kernel in terms of the Riemannian distance. For complete Riemannian manifolds von Renesse–Sturm [25] characterized a lower bound on the Ricci curvature in terms of gradient estimates for the heat semigroup, see also Otto–Villani [24]. In [26], [27] Saloff-Coste proved that two-sided Gaussian bounds for the heat kernel are equivalent to parabolic Harnack inequalities, and are also equivalent to volume doubling together with a scale of Poincaré inequalities.

The problem addressed in this paper is partially motivated by work of Arveson [9], [10], who introduces differential structures in operator algebras. Our results imply uniqueness of these differential structures, the case of compact Riemannian manifolds being of particular interest.

Not all results in the Euclidean case carry over to Riemannian manifolds. We give an example, Example 4.7, of a non-zero lattice homomorphism which intertwines the heat semigroups, but which is not an isomorphism, in contrast to the Euclidean case [5], Theorem 2.1.

Let (M, g) be a Riemannian manifold of dimension d. We always assume that a Riemannian manifold is σ-compact. Then M has a natural Radon measure denoted by $|\cdot|$. Set

$$H^1_{\mathrm{loc}}(M) = \{\varphi \in L_{2,\mathrm{loc}}(M) : \varphi \circ x^{-1} \in H^1_{\mathrm{loc}}(x(V)) \text{ for every chart } (V, x)\}.$$

If $\varphi \in H^1_{\mathrm{loc}}(M)$ and (V, x) is a chart on M then set $\frac{\partial}{\partial x^i}\varphi = (D_i(\varphi \circ x^{-1})) \circ x \in L_{2,\mathrm{loc}}(V)$, where D_i denotes the partial derivative in $\mathbb{R}^d$. Moreover, for all $\varphi, \psi \in H^1_{\mathrm{loc}}(M)$ there exists a unique element $\nabla\varphi \cdot \nabla\psi \in L_{1,\mathrm{loc}}(M)$ such that

$$\nabla\varphi \cdot \nabla\psi|_V = \sum_{i,j=1}^d g^{ij}\left(\frac{\partial}{\partial x^i}\varphi\right)\left(\frac{\partial}{\partial x^j}\psi\right)$$

for every chart (V, x) on M. We let $|\nabla\varphi| = (\nabla\varphi \cdot \nabla\varphi)^{1/2}$. Let $H^1(M)$ be the Hilbert space of all $\varphi \in H^1_{\mathrm{loc}}(M)$ such that both $\varphi, |\nabla\varphi| \in L_2(M)$, with norm $\varphi \mapsto (\|\varphi\|_2^2 + \| \, |\nabla\varphi| \, \|_2^2)^{1/2}$. Moreover, let $H^1_0(M)$ be the closure of $C_c^\infty(M)$ in $H^1(M)$. Define the bilinear form $a : H^1_0(M) \times H^1_0(M) \to \mathbb{R}$ by $a(\psi, \varphi) = \int \nabla\psi \cdot \nabla\varphi$. Then a is closed and positive. The *Dirichlet Laplace–Beltrami operator* Δ on M is the associated self-adjoint operator. If (V, x) is a chart on M then

$$\Delta\varphi = -\sum_{i,j=1}^d \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} g^{ij} \sqrt{g} \frac{\partial}{\partial x^j} \varphi$$

for all $\varphi \in C_c^\infty(V)$.

If (M_1, g_1) and (M_2, g_2) are two Riemannian manifolds then a map $\tau : M_1 \to M_2$ is called an *isometry* if it is a C^∞-diffeomorphism and

$$g_2|_{\tau(p)}\big(\tau_*(v), \tau_*(w)\big) = g_1|_p(v, w)$$

for all $p \in M_1$ and $v, w \in T_p M_1$. A map $\tau : M_1 \to M_2$ is called a *local isometry* if for all $p \in M_1$ there exists an open neighbourhood V of p such that the restriction $\tau|_V : V \to \tau(V)$ is an isometry. The Riemannian manifolds (M_1, g_1) and (M_2, g_2) are called *isomorphic* if there exists an isometry from M_1 onto M_2. If $\tau : M_1 \to M_2$ is an isometry and $p \in [1, \infty)$ then $\varphi \circ \tau \in L_p(M_1)$ and

$$\|\varphi \circ \tau\|_{L_p(M_1)} = \|\varphi\|_{L_p(M_2)} \tag{1}$$

for all $\varphi \in L_p(M_2)$. In particular, the map $\varphi \mapsto \varphi \circ \tau$ is a unitary map from $L_2(M_2)$ onto $L_2(M_1)$ and a unitary map from $H_0^1(M_2)$ onto $H_0^1(M_1)$. Moreover, if $\varphi \in L_2(M_2)$ then $\varphi \in D(\Delta_2)$ if and only if $\varphi \circ \tau \in D(\Delta_1)$ and $\Delta_1(\varphi \circ \tau) = (\Delta_2 \varphi) \circ \tau$, where Δ_j is the Dirichlet Laplace–Beltrami operator on M_j for all $j \in \{1, 2\}$.

A linear operator $U : E \to F$ between two Riesz spaces is said to be a *lattice homomorphism* if

$$U(\varphi \wedge \psi) = (U\varphi) \wedge (U\psi)$$

for all $\varphi, \psi \in E$. For alternative equivalent definitions see [2], Theorem 7.2. Here in this paper in most cases the spaces E and F will be L_p-spaces and then

$$(\varphi \wedge \psi)(x) = \min\{\varphi(x), \psi(x)\} \quad \text{a.e.}$$

Each lattice homomorphism U is positive, i.e. $\varphi \geqq 0$ implies $U\varphi \geqq 0$. An *order isomorphism* $U : E \to F$ is a bijective mapping such that $U\varphi \geqq 0$ if and only if $\varphi \geqq 0$. Equivalently, U is an order isomorphism if and only if U is a bijective lattice homomorphism. Then also U^{-1} is an order isomorphism. Recall that also each positive operator between L_p-spaces is continuous by [2], Theorem 12.3.

The main theorem of this paper is the following. It is valid under some regularity assumptions on the manifolds, namely regularity in capacity, which is optimal for this purpose and which we will explain below.

Theorem 1.1. *Let (M_1, g_1) and (M_2, g_2) be two connected Riemannian manifolds which are both regular in capacity. Let $p \in [1, \infty)$. For all $j \in \{1, 2\}$ let Δ_j be the Dirichlet Laplace–Beltrami operator on M_j and let $S^{(j)}$ be the associated semigroup on $L_p(M_j)$. Then the following two conditions are equivalent*:

(I) *(M_1, g_1) and (M_2, g_2) are isomorphic.*

(II) *There exists a lattice homomorphism $U : L_p(M_1) \to L_p(M_2)$ such that $UL_p(M_1)$ is dense in $L_p(M_2)$ and*

$$US_t^{(1)} = S_t^{(2)} U$$

for all $t > 0$.

Moreover, if U is a lattice homomorphism as in condition (II) *then U is an order isomorphism and there exist $c > 0$ and an isometry $\tau : M_2 \to M_1$ such that $U\varphi = c\varphi \circ \tau$ for all $\varphi \in L_p(M_1)$.*

It turns out that all complete connected Riemannian manifolds, and in particular all compact connected Riemannian manifolds, are regular in capacity. Therefore one immediately has the following corollary.

Corollary 1.2. *Let (M_1, g_1) and (M_2, g_2) be two complete connected Riemannian manifolds. Let $p \in [1, \infty)$. For all $j \in \{1, 2\}$ let Δ_j be the Dirichlet Laplace–Beltrami operator on M_j and let $S^{(j)}$ be the associated semigroup on $L_p(M_j)$. Then the following two conditions are equivalent*:

(I) *(M_1, g_1) and (M_2, g_2) are isomorphic.*

(II) *There exists an order isomorphism $U : L_p(M_1) \to L_p(M_2)$ such that*

$$US_t^{(1)} = S_t^{(2)} U$$

for all $t > 0$.

Now we explain the notion of regularity in capacity for Riemannian manifolds. The *capacity* of a subset A of M is given by

$$\mathrm{cap}_M(A) = \mathrm{cap}(A) = \inf\{\|\varphi\|_{H^1(M)}^2 : \varphi \in H^1(M) \text{ and } \varphi \geqq 1 \text{ on a neighbourhood of } A\}.$$

An open subset Ω of $\mathbb{R}^d$ is called *regular in capacity* [6] if $\mathrm{cap}_{\mathbb{R}^d}\big(B(x; r) \backslash \Omega\big) > 0$ for all $x \in \partial\Omega$ and $r > 0$, where $B(x; r)$ is the Euclidean ball. Biegert and Warma gave several characterizations for regular in capacity. In particular, an open subset Ω of $\mathbb{R}^d$ is regular in capacity if and only if every $\varphi \in H_0^1(\Omega) \cap C(\overline{\Omega})$ is zero everywhere on $\partial\Omega$ ([12], Theorem 3.2). Since $\mathbb{R}^d$ is locally compact it then follows that an open subset Ω of $\mathbb{R}^d$ is regular in capacity if and only if every $\varphi \in H_0^1(\Omega) \cap C_0(\overline{\Omega})$ is zero everywhere on $\partial\Omega$. This characterization allows an extension to general connected Riemannian manifolds. There is a natural distance d_M on a connected Riemannian manifold M. We denote by $B_M(p; r) = B(p; r)$ the associated balls. Let $\tilde{M}$ denote the (metric) completion of M with respect to this distance. Set

$$\partial M = \tilde{M} \backslash M.$$

We say that a connected Riemannian manifold M is *regular in capacity* if $\varphi(p) = 0$ for all $\varphi \in C_0(\tilde{M}) \cap H_0^1(M) = \{\varphi \in C_0(\tilde{M}) : \varphi|_M \in H_0^1(M)\}$ and $p \in \partial M$. Here $C_0(\tilde{M})$ is the closure of the space $C_c(\tilde{M})$ of all continuous functions with compact support, with respect to the supremum norm in the space of all bounded continuous functions on $\tilde{M}$. Clearly every complete connected Riemannian manifold is regular in capacity.

In the Euclidean case, regularity in capacity is a very mild condition on the boundary of an open subset. If $\Omega \subset \mathbb{R}^d$ is open and bounded then it is regular in capacity if it is

Dirichlet regular. The Lebesgue cusp is regular in capacity, but not Dirichlet regular (see [7], Section 7).

If M_1 and M_2 are two isomorphic connected Riemannian manifolds and $\tau : M_1 \to M_2$ is an isometry then τ is *distance preserving*, i.e. $d_{M_2}\big(\tau(p); \tau(q)\big) = d_{M_1}(p; q)$ for all $p, q \in M_1$. Moreover, if M_1 is regular in capacity, then also M_2 is regular in capacity.

Now we can explain why regularity in capacity is the minimal regularity condition in our context. Let M be a connected Riemannian manifold which is complete (or more general, regular in capacity). Let $\emptyset \neq N \subset M$ be a closed subset of capacity zero. Then $\Omega := M \backslash N$ is again a connected Riemannian manifold (see Theorem 2.1) The injection $\tau : \Omega \to M$ defines an isometry which is not surjective. The unitary operator $U : L_2(M) \to L_2(\Omega)$ given by $U\varphi = \varphi \circ \tau$ is an order isomorphism intertwining the two heat semigroups even though Ω and M are not isomorphic. It follows from Theorem 1.1 that Ω is not regular in capacity.

The paper is organized as follows. In the next section we give a sufficient condition to ensure that the distance on a subriemannian manifold equals the induced distance. In Section 3 we show that M_1 and M_2 are isometric if they have sufficiently big isometric open subsets. In Section 4 we prove Theorem 1.1. Finally, in Section 5 we give several characterizations of regularity in capacity.

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2. Distances

If N is a connected open subset of a connected Riemannian manifold M then $d_M(p; q) \leqq d_N(p; q)$ for all $p, q \in N$, where d_M and d_N are the natural distances on M and N. Even if $|M \backslash N| = 0$, then it is easy to construct examples such that the induced distance from d_M on N differs from the distance d_N. We next show that the condition $\operatorname{cap}_M(M \backslash N) = 0$ suffices to have equality.

Theorem 2.1. *Let N be an open subset of a connected Riemannian manifold M and suppose that $\operatorname{cap}_M(M \backslash N) = 0$. Then N is connected and $d_M(p; q) = d_N(p; q)$ for all $p, q \in N$.*

The proof involves an alternative description of the distances and is split in three propositions. The proof of some subresults is present in the literature on non-regular metric spaces in a slightly different setting [3], [4], [14], [18]. First we need L_∞-versions of H^1 and $\nabla\varphi$. Let M be a Riemannian manifold. Set

$$W^{1,\infty}_{\mathrm{loc}}(M) = \big\{\varphi \in L_{\infty,\mathrm{loc}}(M) : \varphi \circ x^{-1} \in W^{1,\infty}_{\mathrm{loc}}\big(x(V)\big) \text{ for every chart } (V, x) \text{ on } M\big\}.$$

For all $\varphi \in W^{1,\infty}_{\rm loc}(M)$ there is a unique $|\nabla\varphi| \in L_{\infty,\rm loc}(M)$ such that

$$|\nabla\varphi|\,|_V = \left(\sum_{i,j=1}^d g^{ij}\Big(\frac{\partial}{\partial x^i}\varphi\Big)\Big(\frac{\partial}{\partial x^j}\varphi\Big)\right)^{1/2}$$

for every chart (V,x) on M, where $\frac{\partial}{\partial x^i}\varphi \in L_{\infty,\rm loc}(M)$ is defined in the natural way. Then define $W^{1,\infty}(M) = \{\varphi \in L_\infty(M) : |\nabla\varphi| \in L_\infty(M)\}$.

The next proposition is folklore and is at the basis of analysis on metric spaces [3], [4]. For the convenience of the reader we include a proof.

Proposition 2.2. *Let M be a connected Riemannian manifold. If $p, q \in M$ then*

$$d_M(p;q) = \sup\{\psi(p) - \psi(q) : \psi \in W^{1,\infty}_{\rm loc}(M), |\nabla\psi| \in L_\infty(M) \text{ and } \|\,|\nabla\psi|\,\|_\infty \leqq 1\}.$$

Proof. '$\leqq$'. If $q \in M$ define $\psi : M \to \mathbb{R}$ by $\psi(p) = d_M(p;q)$. Then $\psi \in W^{1,\infty}_{\rm loc}(M)$, $\|\,|\nabla\psi|\,\|_\infty \leqq 1$ and $d_M(p;q) \leqq \psi(p) - \psi(q)$.

'$\geqq$'. Let $p, q \in M$. Let $\psi \in W^{1,\infty}_{\rm loc}(M)$ with $\|\,|\nabla\psi|\,\|_\infty \leqq 1$. Let $\gamma : [0,1] \to M$ be a C^∞-map with $\gamma(0) = p$ and $\gamma(1) = q$. By regularizing we may assume that ψ is smooth in a neighbourhood of $\gamma([0,1])$. Then

$$\begin{aligned}|\psi(p) - \psi(q)| &\leqq \int_0^1 |\dot\gamma(t)\psi|\,dt \\ &\leqq \int_0^1 g\big(\dot\gamma(t),\dot\gamma(t)\big)^{1/2}|\nabla\psi|\big(\gamma(t)\big)\,dt \leqq \int_0^1 g\big(\dot\gamma(t),\dot\gamma(t)\big)^{1/2}\,dt.\end{aligned}$$

Minimizing over γ gives $|\psi(p) - \psi(q)| \leqq d_M(p;q)$. □

We shall prove that $W^{1,\infty}_{\rm loc}(M) = W^{1,\infty}_{\rm loc}(N)$ if $\mathrm{cap}_M(M\backslash N) = 0$. For all $s \in [0,\infty)$ we denote by $\mathscr{H}^s(A)$ the s-dimensional Hausdorff measure of a subset A of M (see [18], 8.3).

Proposition 2.3. *Let A be a subset of a connected Riemannian manifold M of dimension d. If $\mathrm{cap}(A) = 0$ then $\mathscr{H}^s(A) = 0$ for all $s \in [0,\infty)$ with $s > d-2$.*

Proof. For all $n \in \mathbb{N}$ there exists a $\varphi_n \in H^1(M)$ such that $\varphi_n \geqq 1$ on a neighbourhood of A and $\|\varphi_n\|_{H^1(M)} \leqq 2^{-n}$. Set $\varphi = \sum_{n=1}^\infty \varphi_n \in H^1(M)$. Then for all $m \in \mathbb{N}$ it follows that $\varphi \geqq m$ on a neighbourhood of A. Hence for all $a \in A$ there exists an $\varepsilon > 0$ such that $\langle\varphi\rangle_{a,r} \geqq m$ for all $r \in (0,\varepsilon]$, where $\langle\psi\rangle_{p,r} = |B(p;r)|^{-1}\int_{B(p;r)}\psi$ is the average of ψ over the ball $B(p;r)$ for all $\psi \in L_{1,\rm loc}(M)$, $p \in M$ and $r > 0$.

Let $a \in A$ and suppose that $\limsup_{r\to 0} r^{-s}\int_{B(a;r)}|\nabla\varphi|^2 < \infty$. Then there exist $r_1 \in (0,1]$ and $M \in \mathbb{R}$ such that $\int_{B(a;r)}|\nabla\varphi|^2 \leqq Mr^s$ for all $r \in (0,r_1]$.

It follows from [11], Theorem 5.14, that there exists an $r_2 \in (0, r_1]$ such that $\exp_a$ is a diffeomorphism from $\{v \in T_a M : g_a(v, v) < r_2^2\}$ onto $B(a; r_2)$ and $d(a; \exp_a v) = g_a(v; v)^{1/2}$ for all $v \in T_a M$ with $g_a(v, v) < r_2^2$. Since $T_a M$ is equivalent to $\mathbb{R}^d$, it admits a Poincaré inequality. Hence there exists a $c_1 > 0$ such that

$$\int_{B(a;r)} |\psi - \langle\psi\rangle_{a,r}|^2 \leqq c_1 r^2 \int_{B(a;r)} |\nabla\psi|^2$$

uniformly for all $\psi \in H^1(B(a;r))$ and $r \in (0, r_2]$. Similarly, there exists a $c_2 > 0$ such that $|B(a;r)| \geqq c_2 r^d$ for all $r \in (0, 1]$. Clearly the constants c_1 and c_2 depend on the point a. Then

$$\int_{B(a;r)} |\varphi - \langle\varphi\rangle_{a,r}|^2 \leqq c_1 r^2 \int_{B(a;r)} |\nabla\varphi|^2 \leqq c_1 M r^{s+2}$$

for all $r \in (0, r_2]$. Now the rest of the proof is standard, cf. [14], Theorem 4.7.4. □

Proposition 2.4. *Let N be an open subset of a connected Riemannian manifold M of dimension d and suppose that $\mathscr{H}^{d-1}(M \backslash N) = 0$. Then $W^{1,\infty}_{\mathrm{loc}}(N) = W^{1,\infty}_{\mathrm{loc}}(M)$ and N is connected.*

Proof. Let $\varphi \in W^{1,\infty}_{\mathrm{loc}}(N)$. Using a partition of the unity, normal coordinates and [19], Proposition I.9.10, we may assume that there are $p \in M$ and $r > 0$ such that first φ is compactly supported in the ball $B_M(p;r)$, secondly the restriction Φ of $\exp_p$ to the set $X_{2r} = \{v \in T_p M : g_p(v, v) < (2r)^2\}$ is a diffeomorphism of X_{2r} onto $B_M(p; 2r)$, thirdly $|v| = d_M(p; \exp_p v)$ for all $v \in X_{2r}$ and finally $2^{-1}|v - w| \leqq d_M(\exp_p v; \exp_p w) \leqq 2|v - w|$ for all $v, w \in X_{2r}$, where $|v| = g_p(v, v)^{1/2}$. Then

$$\mathscr{H}^{d-1}\big(X_r \backslash \Phi^{-1}(N \cap B_M(p;r))\big) = \mathscr{H}^{d-1}\big(\Phi^{-1}((M\backslash N) \cap B_M(p;r))\big) = 0$$

where $X_r = \Phi^{-1}(B_M(p;r))$. Moreover, $\varphi \circ \Phi \in W^{1,q}\big(\Phi^{-1}(N \cap B_M(p;r))\big)$ for all $q \in (1, \infty)$ and $\Phi^{-1}(N \cap B_M(p;r))$ is open. Hence it follows from [1], Lemma 9.1.10, that $\varphi \circ \Phi \in W^{1,q}(X_r)$ for all $q \in (1, \infty)$. Then $\varphi \circ \Phi \in W^{1,\infty}(X_r)$ and $\varphi \in W^{1,\infty}(B_M(p;r))$. So $\varphi \in W^{1,\infty}(M)$.

Finally, let $\varphi : N \to \{0, 1\}$ be a continuous function. Then $\varphi \in W^{1,\infty}(N) \subset W^{1,\infty}_{\mathrm{loc}}(M)$. So φ extends to a continuous function on M. But M is connected. Therefore φ is constant and N is connected. □

Proof of Theorem 2.1. This easily follows from the last three propositions. □

3. Quasi isometries are isometries

In this section we prove that two connected Riemannian manifolds, which are regular in capacity, are isomorphic if they have isomorphic open subsets whose complements are polar. Moreover, we give many useful tools to understand and to work with the H^1_0-spaces defined on Riemannian manifolds.

Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds. We say that

$$M_1 \overset{\text{cap}}{\sim} M_2$$

if there exist open sets $M_1' \subset M_1$ and $M_2' \subset M_2$ and an isometry τ from M_2' onto M_1' such that $\text{cap}_{M_1}(M_1 \backslash M_1') = 0 = \text{cap}_{M_2}(M_2 \backslash M_2')$.

The following theorem is the main theorem in this section. It shows that the relation $\overset{\text{cap}}{\sim}$ defined on connected Riemannian manifolds determines the manifold.

Theorem 3.1. *Let (M_1, g_1) and (M_2, g_2) be two connected Riemannian manifolds which are regular in capacity. Then*

$$M_1 \overset{\text{cap}}{\sim} M_2 \Leftrightarrow (M_1, g_1) \text{ and } (M_2, g_2) \text{ are isomorphic.}$$

Explicitly, if M_1' and M_2' are open subsets of M_1 and M_2 such that

$$\text{cap}_{M_1}(M_1 \backslash M_1') = 0 = \text{cap}_{M_2}(M_2 \backslash M_2')$$

and $\tau : M_2' \to M_1'$ is an isometry, then there exists an isometry $\hat{\tau} : M_2 \to M_1$ such that $\hat{\tau}|_{M_2'} = \tau$.

We define the space $H_c^1(M)$ to be the set of all $\varphi \in H^1(M)$ such that there exists a compact subset K of M with $\varphi = 0$ a.e. on $M \backslash K$.

Lemma 3.2. *Let N be an open subset of a Riemannian manifold (M, g) such that $|M \backslash N| = 0$. Then the following are equivalent*:

(I) $\text{cap}_M(M \backslash N) = 0$.

(II) *The restriction map $\psi \mapsto \psi|_N$ from $H^1(M)$ into $H^1(N)$ maps $H_0^1(M)$ onto $H_0^1(N)$.*

(III) $H_0^1(M) = \{\psi|_N : \psi \in H_0^1(M)\} = H_0^1(N)$.

Proof. Clearly (III) is a reformulation of (II).

'(III) $\Rightarrow$ (I)'. Let $K \subset M$ be a compact set. There exists a $\psi \in C_c^\infty(M)$ such that $\psi|_K \geqq 1$. Then $\psi|_N \in H_0^1(N)$ by assumption. Let $\varepsilon > 0$. There exists a $\varphi \in C_c^\infty(N)$ such that $\|\psi|_N - \varphi\|_{H^1(N)}^2 < \varepsilon$. Then $\psi - \varphi \in C_c^\infty(M)$ and $\psi - \varphi \geqq 1$ on $K \backslash N$. So

$$\text{cap}(K \backslash N) \leqq \|\psi - \varphi\|_{H^1(M)}^2 = \|\psi|_N - \varphi\|_{H^1(N)}^2 < \varepsilon,$$

where we used that $|M \backslash N| = 0$ in the equality. Since M is σ-compact one deduces that $\text{cap}(M \backslash N) = 0$.

'(I) $\Rightarrow$ (III)'. Clearly $\{\psi|_N : \psi \in H_0^1(M)\} \supset H_0^1(N)$. Conversely, let $\psi \in C_c^\infty(M)$. Let $\varepsilon > 0$. There exists an open neighbourhood of $M \backslash N$ and a $\chi \in H^1(M)$ such that $0 \leqq \chi \leqq 1$, $\chi|_U = 1$ and $\|\chi\|_{H^1(M)} < \varepsilon$. Then $\big(\psi(\mathbb{1} - \chi)\big)\big|_N \in H_c^1(N) \subset H_0^1(N)$ and

$$\big\|\psi|_N - \big(\psi(\mathbb{1} - \chi)\big)|_N\big\|_{H^1(N)} = \|\psi\chi\|_{H^1(M)} \leqq 3\|\psi\|_{W^{1,\infty}(M)}\|\chi\|_{H^1(M)} \leqq 3\|\psi\|_{W^{1,\infty}(M)}\varepsilon.$$

So $\psi|_N \in H_0^1(N)$. Since $C_c^\infty(M)$ is dense in $H_0^1(M)$ and $\varphi \mapsto \varphi|_N$ is isometric from $H^1(M)$ into $H^1(N)$ the lemma follows. $\square$

If N is an open subset of a Riemannian manifold M and $A \subset N$, then $\mathrm{cap}_N(A) \leqq \mathrm{cap}_M(A)$. The next lemma is instrumental to deduce equality of the two capacities if $\mathrm{cap}_M(M \setminus N) = 0$. It is a kind of L_2-version of Propositions 2.3 and 2.4.

Lemma 3.3. *Let N be an open subset of a manifold M. Suppose that $\mathrm{cap}_M(M \setminus N) = 0$. Then $H^1(N) = H^1(M)$.*

Proof. Let $\varphi \in H^1(N) \cap L_\infty(N)$. We shall prove that $\varphi \in H^1(M)$. Let $n \in \mathbb{N}$. Since $\mathrm{cap}_M(M \setminus N) = 0$ there exists a $\psi_n \in H^1(M)$ such that $\psi_n \geqq 1$ in a neighbourhood of $M \setminus N$ and $\|\psi_n\|_{H_1(M)}^2 \leqq n^{-1}$. We may assume that $0 \leqq \psi_n \leqq 1$. Then $\varphi - \varphi\psi_n \in H^1(N)$. But $\varphi - \varphi\psi_n = \varphi(\mathbb{1} - \psi_n)$ vanishes in a neighbourhood of $M \setminus N$. Therefore we can extend this function by zero to a function $\varphi_n \in H^1(M)$. Then

$$\|\varphi\psi_n\|_{H^1(N)}^2 \leqq 2\|\varphi\|_\infty^2 \|\psi_n\|_{H^1(M)}^2 + 2\|\varphi\|_{H^1(N)}^2 \|\psi_n\|_\infty^2 \leqq 2\|\varphi\|_\infty^2 + 2\|\varphi\|_{H^1(N)}^2$$

for all $n \in \mathbb{N}$. So the sequence $\varphi_1, \varphi_2, \ldots$ is uniformly bounded in $H^1(M)$. Hence it has a weakly convergent subsequence. Passing to a subsequence if necessary, there exists a $\tilde{\varphi} \in H^1(M)$ such that $\lim \varphi_n = \tilde{\varphi}$ weakly in $H^1(M)$. Then $\lim \varphi_n = \tilde{\varphi}$ weakly in $L_2(M)$. But

$$\|\varphi - \varphi_n\|_{L_2(M)} = \|\varphi\psi_n\|_{L_2(M)} \leqq \|\varphi\|_\infty \|\psi_n\|_{L_2(M)} \leqq \|\varphi\|_\infty n^{-1/2}$$

for all $n \in \mathbb{N}$. So $\lim \varphi_n = \varphi$ strongly and therefore also weakly in $L_2(M)$. Hence $\tilde{\varphi} = \varphi$ a.e. and $\varphi \in H^1(M)$.

Thus $H^1(N) \cap L_\infty(N) \subset H^1(M)$. Since $H^1(N) \cap L_\infty(N)$ is dense in $H^1(N)$ the lemma follows. $\square$

Corollary 3.4. *If M is a Riemannian manifold and $N \subset M$ is open with $\mathrm{cap}_M(M \setminus N) = 0$ then $\mathrm{cap}_N(A) = \mathrm{cap}_M(A)$ for all $A \subset N$.*

Corollary 3.5. *Let M_1 and M_2 be two Riemannian manifolds. If $M_1 \overset{\mathrm{cap}}{\sim} M_2$ and M_1', M_2' and τ are as in the definition of $\overset{\mathrm{cap}}{\sim}$ then $\mathrm{cap}_{M_1}\big(\tau(A)\big) = \mathrm{cap}_{M_2}(A)$ for every set $A \subset M_2'$.*

Proof. One deduces from the previous corollary that

$$\mathrm{cap}_{M_1}\big(\tau(A)\big) = \mathrm{cap}_{M_1'}\big(\tau(A)\big) = \mathrm{cap}_{M_2'}(A) = \mathrm{cap}_{M_2}(A) = 0. \quad \square$$

We emphasize that the next proposition does not require the manifolds to be regular in capacity.

Proposition 3.6. *The relation $\overset{\mathrm{cap}}{\sim}$ is an equivalence relation.*

Proof. The reflexivity and symmetry are trivial.

Let M_1, M_2 and M_3 be three Riemannian manifolds and assume that $M_1 \overset{\mathrm{cap}}{\sim} M_2$ and $M_2 \overset{\mathrm{cap}}{\sim} M_3$. Then there exist open $M_1' \subset M_1$, $M_2', M_2'' \subset M_2$ and $M_3'' \subset M_3$ and isometries $\tau : M_2' \to M_1'$ and $\sigma : M_3'' \to M_2''$ such that

$$\mathrm{cap}_{M_1}(M_1 \setminus M_1') = \mathrm{cap}_{M_2}(M_2 \setminus M_2') = \mathrm{cap}_{M_2}(M_2 \setminus M_2'') = \mathrm{cap}_{M_3}(M_3 \setminus M_3'') = 0.$$

Now let $M_2''' = M_2' \cap M_2''$. Then M_2''' is open in M_2 and

$$\mathrm{cap}_{M_2}(M_2 \backslash M_2''') \leqq \mathrm{cap}_{M_2}(M_2 \backslash M_2') + \mathrm{cap}_{M_2}(M_2 \backslash M_2'') = 0.$$

Next set $M_1''' = \tau(M_2''') \subset M_1'$ and $M_3''' = \sigma^{-1}(M_2''') \subset M_3'$. Then M_1''' is open in M_1 and M_3''' is open in M_3. Moreover, $\tau|_{M_2'''} \circ \sigma|_{M_3'''}$ is an isometry from M_3''' onto M_1'''. Since $M_1' \backslash M_1''' = \tau(M_2' \backslash M_2''')$ it follows from Corollary 3.5 that

$$\mathrm{cap}_{M_1}(M_1' \backslash M_1''') = \mathrm{cap}_{M_2}(M_2' \backslash M_2''') = 0.$$

Therefore $\mathrm{cap}_{M_1}(M_1 \backslash M_1''') \leqq \mathrm{cap}_{M_1}(M_1 \backslash M_1') + \mathrm{cap}_{M_1}(M_1' \backslash M_1''') = 0$. So

$$\mathrm{cap}_{M_1}(M_1 \backslash M_1''') = 0.$$

It similarly follows that $\mathrm{cap}_{M_3}(M_3 \backslash M_3''') = 0$. Therefore $M_1 \overset{\mathrm{cap}}{\sim} M_3$. □

Also the next proposition does not assume regular in capacity. But it overshoots the conclusions in Theorem 3.1 since the range of $\tilde{\tau}$ can be bigger than M_1.

Proposition 3.7. *Let M_1 and M_2 be two connected Riemannian manifolds. If $M_1 \overset{\mathrm{cap}}{\sim} M_2$ and if M_1', M_2' and τ are as in the definition of $\overset{\mathrm{cap}}{\sim}$, then there exists a distance preserving isomorphism $\tilde{\tau} : \tilde{M}_2 \to \tilde{M}_1$ such that $\tilde{\tau}|_{M_2'} = \tau$.*

Proof. The function $\tau|_{M_2'} : M_2' \to M_1'$ is distance preserving with respect to the distances $d_{M_2'}$ and $d_{M_1'}$. Then by Theorem 2.1 the map $\tau|_{M_2'}$ is also a distance preserving with respect to the induced distances from M_2 and M_1 on M_2' and M_1'. Since M_2' is dense in M_2 and therefore also in $\tilde{M}_1$ it follows that there exists a unique distance preserving map $\tilde{\tau} : \tilde{M}_2 \to \tilde{M}_1$ such that $\tilde{\tau}|_{M_2'} = \tau$. Similarly there exists a unique distance preserving map $\tilde{\sigma} : \tilde{M}_1 \to \tilde{M}_2$ such that $\tilde{\sigma}|_{M_1'} = \tau^{-1}$. Then $\tilde{\tau} \circ \tilde{\sigma}|_{M_1'}$ is the identity function on M_1'. So by density and continuity $\tilde{\tau} \circ \tilde{\sigma} = I_{\tilde{M}_1}$. Similarly $\tilde{\sigma} \circ \tilde{\tau} = I_{\tilde{M}_2}$ and the proposition follows. □

Now we are able to prove the main theorem of this section.

Proof of Theorem 3.1. The implication $\Leftarrow$ is trivial. Suppose that $M_1 \overset{\mathrm{cap}}{\sim} M_2$ and let M_1', M_2' and τ be as in the definition of $\overset{\mathrm{cap}}{\sim}$. Let $\tilde{\tau}$ be as in Proposition 3.7. If $\varphi \in H_0^1(M_1) \cap C_0(\tilde{M}_1)$ then $\varphi \circ \tilde{\tau} \in C_0(\tilde{M}_2)$. Moreover, $\varphi|_{M_1'} \in H_0^1(M_1')$, so

$$(\varphi \circ \tilde{\tau})|_{M_2'} \in H_0^1(M_2')$$

and therefore $(\varphi \circ \tilde{\tau})|_{M_2} \in H_0^1(M_2)$ by Lemma 3.2. So we can define

$$V : H_0^1(M_1) \cap C_0(\tilde{M}_1) \to H_0^1(M_2) \cap C_0(\tilde{M}_2)$$

by $V\varphi = \varphi \circ \tilde{\tau}$.

Next, let $p \in M_1$. There exists a $\varphi \in C_c^\infty(M_1)$ such that $\varphi(p) = 1$. Then $V\varphi \in H_0^1(M_2) \cap C_0(\tilde{M}_2)$. Moreover, $(V\varphi)\big(\tilde{\tau}^{-1}(p)\big) = \varphi(p) = 1$. Hence $\tilde{\tau}^{-1}(p) \in M_2$ since M_2 is regular in capacity. Similarly $\tilde{\tau}(q) \in M_1$ for all $q \in M_2$ since M_1 is regular in capacity.

Let $\hat{\tau} = \tilde{\tau}|_{M_2}$. Then $\hat{\tau}$ is a topological homeomorphism from M_2 onto M_1. It remains to show that $\hat{\tau}$ and its inverse are smooth and an isometry.

By (1) we can define $U : L_2(M_1) \to L_2(M_2)$ by $U\varphi = \varphi \circ \hat{\tau}$. If $\varphi \in H_0^1(M_1)$ then $\varphi \circ \tau \in H_0^1(M_2') = H_0^1(M_2)$ by isometry, (1) and Lemma 3.2. Therefore U is a bijection from $H_0^1(M_1)$ onto $H_0^1(M_2)$. Let h_1 and h_2 be the forms associated to the Dirichlet Laplace–Beltrami operators on M_1 and M_2, with form domains $H_0^1(M_1)$ and $H_0^1(M_2)$. Then

$$h_1(\varphi) = \int_{M_1} |\nabla\varphi|^2 = \int_{M_1'} |\nabla\varphi|^2$$

$$= \int_{M_2'} |\nabla(\varphi \circ \tau)|^2 = \int_{M_2} |\nabla(\varphi \circ \tau)|^2 = \int_{M_2} |\nabla(\varphi \circ \hat{\tau})|^2 = h_2(U\varphi)$$

for all $\varphi \in C_c^\infty(M_1)$. Since $C_c^\infty(M_1)$ is a core for h_1 it follows that $h_1(\varphi) = h_2(U\varphi)$ for all $\varphi \in H_0^1(M_1)$. Hence if $\varphi \in L_2(M_1)$, then $\varphi \in D(\Delta_1)$ if and only if $U\varphi \in D(\Delta_2)$, and $\Delta_2 U\varphi = U\Delta_1\varphi$ if both conditions are valid. Now let $\varphi \in C_c^\infty(M_1)$. Then

$$U\varphi \in \bigcap_{n=1}^{\infty} D(\Delta_2^n) \subset C^\infty(M_2)$$

by elliptic regularity. (See [15], Theorem 9.11, if $p \neq 1$. If $p = 1$ first apply a Sobolev embedding to embed L_1 into a Sobolev space $W^{s,p}$ with $s < 0$ and $p > 1$, and then apply [15], Theorem 9.11.) So there exists a $\psi \in C^\infty(M_2)$ such that $U\varphi = \psi$ a.e. But $\varphi = \varphi \circ \hat{\tau}$ is continuous. Therefore $\varphi \circ \hat{\tau} = \psi$ pointwise. Thus $\varphi \circ \hat{\tau}$ is smooth for all $\varphi \in C_c^\infty(M_1)$. Therefore $\hat{\tau}$ is a C^∞-map from M_2 onto M_1. Similarly also $\hat{\tau}^{-1}$ is a C^∞-map, so $\hat{\tau}$ is a C^∞-diffeomorphism. Finally, since τ is an isometry and M_2' is dense in M_2 it follows by continuity that also $\hat{\tau}$ is an isometry. This proves Theorem 3.1. □

4. Lattice homomorphisms

In this section, we consider lattice homomorphisms between L_p-spaces on two Riemannian manifolds without the assumption that the manifolds are regular in capacity. The aim is to prove that the associated H_0^1-spaces are equivalent, under the conditions of Theorem 1.1.

The first step is to use elliptic regularity of the Laplace–Beltrami operator to reduce to smooth functions.

Lemma 4.1. *Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds. Let $p \in [1, \infty)$. For all $j \in \{1, 2\}$ let Δ_j be the Dirichlet Laplace–Beltrami operator on M_j and let $S^{(j)}$ be the associated semigroup on $L_p(M_j)$. Let $U : L_p(M_1) \to L_p(M_2)$ be a lattice homomorphism such that*

$$US_t^{(1)} = S_t^{(2)} U \tag{2}$$

for all $t > 0$. Then:

(i) $UC_c^\infty(M_1) \subset C^\infty(M_2)$.

(ii) $U\varphi \geqq 0$ *for all* $\varphi \in C_c^\infty(M_1)$ *with* $\varphi \geqq 0$.

(iii) $(U\varphi)(U\psi) = 0$ *for all* $\varphi, \psi \in C_c^\infty(M_1)$ *with* $\varphi\psi = 0$.

(iv) $\Delta_2 U\varphi = U\Delta_1\varphi$ *for all* $\varphi \in C_c^\infty(M_1)$.

Proof. It follows from (2) that $UD(\Delta_1) \subset D(\Delta_2)$ and $\Delta_2 U\varphi = U\Delta_1\varphi$ for all $\varphi \in D(\Delta_1)$. Hence by iteration $U \bigcap_{n=1}^{\infty} D(\Delta_1^n) \subset \bigcap_{n=1}^{\infty} D(\Delta_2^n)$. But

$$C_c^\infty(M_j) \subset \bigcap_{n=1}^{\infty} D(\Delta_j^n) \subset C^\infty(M_j) \quad \text{for all } j \in \{1,2\}$$

by elliptic regularity. This shows (i) and (iv). Property (ii) follows since U is a lattice homomorphism. Moreover, $|U\varphi| = U|\varphi|$ for all $\varphi \in L_p(M_1)$. Hence if $\varphi, \psi \in C_c(M_1)$ and $\varphi\psi = 0$ then $|\varphi| \wedge |\psi| = 0$ and $|U\varphi| \wedge |U\psi| = U|\varphi| \wedge U|\psi| = U(|\varphi| \wedge |\psi|) = 0$. Therefore

$$|(U\varphi)(U\psi)| = |U\varphi|\,|U\psi| = 0 \quad \text{and} \quad (U\varphi)(U\psi) = 0.$$

This implies property (iii). □

We frequently need the following sufficient condition for point evaluations.

Lemma 4.2. *Let M be a manifold and $F : C_c^\infty(M) \to \mathbb{R}$ a positive linear functional such that*

$$(3) \qquad F(\varphi)F(\psi) = 0 \quad \textit{for all } \varphi, \psi \in C_c^\infty(M) \textit{ with } \varphi\psi = 0.$$

Then there exist $c \in [0, \infty)$ and $p \in M$ such that $F(\varphi) = c\varphi(p)$ for all $\varphi \in C_c^\infty(M)$.

Proof. Arguing as in [14], Corollary 1.8.1, it follows that there exists a unique Radon measure μ on M such that $F(\varphi) = \int \varphi\, d\mu$ for all $\varphi \in C_c^\infty(M)$. Then it follows from (3) that μ is a point measure. Hence there exist $p \in M$ and $c \in [0, \infty)$ such that $F(\varphi) = c\varphi(p)$ for all $\varphi \in C_c^\infty(M)$. □

The next proposition is a manifold version of [2], Theorem 7.22.

Proposition 4.3. *Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds. Suppose there exists a linear map $U : C_c^\infty(M_1) \to C^\infty(M_2)$ such that*

(i) $U\varphi \geqq 0$ *for all* $\varphi \in C_c^\infty(M_1)$ *with* $\varphi \geqq 0$, *and*

(ii) $(U\varphi)(U\psi) = 0$ *for all* $\varphi, \psi \in C_c^\infty(M_1)$ *with* $\varphi\psi = 0$.

Then there exist an open set $M_2' \subset M_2$, a function $\tau : M_2 \to M_1$ and a function $h : M_2 \to [0, \infty)$ such that $M_2' = \{q \in M_2 : h(q) > 0\}$ and $U\varphi = h \cdot (\varphi \circ \tau)$ pointwise for all $\varphi \in C_c^\infty(M_1)$. Moreover, the restrictions $\tau|_{M_2'}$ and $h|_{M_2'}$ are both C^∞.

Proof. Set $M_2' = \{q \in M_2 : (U\varphi)(q) \neq 0 \text{ for some } \varphi \in C_c^\infty(M_1)\}$. Then M_2' is open. Let $q \in M_2$. Then the map $\varphi \mapsto (U\varphi)(q)$ from $C_c^\infty(M_1)$ into $\mathbb{R}$ is linear, positive and satisfies (3). Hence it follows from Lemma 4.2 that there exist $\tau(q) \in M_1$ and $h(q) \in [0, \infty)$ such that

$$(U\varphi)(q) = h(q)\varphi\big(\tau(q)\big)$$

for all $\varphi \in C_c^\infty(M_1)$. So one obtains functions $\tau : M_2 \to M_1$ and $h : M_2 \to [0, \infty)$. Moreover, $M_2' = \{q \in M_2 : h(q) > 0\}$. It remains to show the smoothness of the restrictions of τ and h to the set M_2'.

First we show that the function $\tau|_{M_2'}$ is continuous. Otherwise there are $q, q_1, q_2, \ldots \in M_2'$ and $\varepsilon > 0$ such that $\lim q_n = q$ and $d\big(\tau(q_n); \tau(q)\big) \geqq \varepsilon$ for all $n \in \mathbb{N}$, where d is a distance on M_1. There exists a $\varphi \in C_c^\infty(M_1)$ such that $\varphi\big(\tau(q)\big) = 1$ and $\varphi(p) = 0$ for all $p \in M_1$ with $d\big(p; \tau(q)\big) > \varepsilon$. Then $U\varphi$ is continuous, so

$$h(q) = (U\varphi)(q) = \lim (U\varphi)(q_n) = 0,$$

which is a contradiction.

Secondly, let Ω be a relatively compact open subset of M_2' with $\overline{\Omega} \subset M_2'$. Then $\tau(\overline{\Omega})$ is compact, so there is a $\psi \in C_c^\infty(M_1)$ such that $\psi|_{\tau(\Omega)} = 1$. Then $h|_\Omega = (U\psi)|_\Omega$ is a C^∞-function. Hence $h|_{M_2'}$ is a C^∞-function from M_2' into $(0, \infty)$. Then

$$(\varphi \circ \tau)|_\Omega = (h^{-1} \cdot U\varphi)|_\Omega \in C^\infty(\Omega) \quad \text{for all } \varphi \in C_c^\infty(M_1)$$

and $\tau|_{M_2'}$ is a C^∞-function. □

Using the fact that U intertwines with the Laplace–Beltrami operators implies that τ is almost an isometry.

Proposition 4.4. *Let (M_1, g_1) and (M_2', g_2) be two Riemannian manifolds with Dirichlet Laplace–Beltrami operators Δ_1 and Δ_2. Let $\tau : M_2' \to M_1$ be a C^∞-map and $h : M_2' \to (0, \infty)$ a C^∞-function. Define the map $U : C_c^\infty(M_1) \to C^\infty(M_2')$ by $U\varphi = h \cdot (\varphi \circ \tau)$. Suppose that $\Delta_2 U = U\Delta_1$. Then*

$$g_1|_{\tau(q)}(\alpha, \beta) = g_2|_q\big(\tau^*(\alpha), \tau^*(\beta)\big) \tag{4}$$

for all $q \in M_2'$ and $\alpha, \beta \in T^*_{\tau(q)} M_1$. In particular, $\dim M_2' \geqq \dim M_1$.

If, in addition, $\dim M_1 = \dim M_2'$ *then τ is locally an isometry and h is locally constant.*

Proof. It follows from the identity $\Delta_2 U = U\Delta_1$ that

$$\Delta_2\big(h \cdot (\varphi \circ \tau)\big) = h \cdot \big((\Delta_1\varphi) \circ \tau\big) \tag{5}$$

on M_2' for all $\varphi \in C_c^\infty(M_1)$. Let $q \in M_2'$. There exists a chart (V, x) on M_1 such that $\tau(q) \in V$ and $x\big(\tau(q)\big) = 0$. Let $\Omega \subset M_1$ be a relatively compact subset such that $\tau(q) \in \Omega \subset \overline{\Omega} \subset V$.

Let $d_1 = \dim M_1$ and $d_2 = \dim M_2'$. Let $\lambda_1, \ldots, \lambda_{d_1} \in \mathbb{R}$. For all $t > 0$ there exists a $\varphi_t \in C_c^\infty(M_1)$ such that

$$\varphi_t|_\Omega = e^{t \sum_{k=1}^{d_1} \lambda_k x^k}|_\Omega.$$

Since

$$\Delta_1 = -\sum_{i,j=1}^{d_1} \frac{1}{\sqrt{g_1}} \frac{\partial}{\partial x^i} g_1^{ij} \sqrt{g_1} \frac{\partial}{\partial x^j}$$

on V it follows that

$$\Delta_1 \varphi_t = -\sum_{i,j=1}^{d_1} t^2 g_1^{ij} \lambda_i \lambda_j \varphi_t + t \frac{\lambda_j}{\sqrt{g_1}} \varphi_t \frac{\partial}{\partial x^i} (g_1^{ij} \sqrt{g_1})$$

on Ω. Hence

$$\lim_{t \to \infty} t^{-2} \big(h \cdot ((\Delta_1 \varphi_t) \circ \tau)\big)(q) = -h(q) \sum_{i,j=1}^{d_1} g_1^{ij}|_{\tau(q)} \lambda_i \lambda_j.$$

Next, let (W, y) be a chart on M_2' such that $q \in W$. Then it follows similarly that

$$\begin{aligned}\lim_{t \to \infty} t^{-2} \big(\Delta_2 \big(h \cdot (\varphi_t \circ \tau)\big)\big)(q) &= -\sum_{k,l=1}^{d_2} h(q) g_2^{kl}|_q \left(\frac{\partial}{\partial y^k} \sum_{i=1}^{d_1} \lambda_i x^i \circ \tau \right)(q) \left(\frac{\partial}{\partial y^l} \sum_{j=1}^{d_1} \lambda_j x^j \circ \tau \right)(q) \\ &= -\sum_{i,j=1}^{d_1} h(q) g_2|_q \big(\tau^*(dx^i), \tau^*(dx^j)\big) \lambda_i \lambda_j.\end{aligned}$$

But then (5) gives

$$\sum_{i,j=1}^{d_1} g_1^{ij}|_{\tau(q)} \lambda_i \lambda_j = \sum_{i,j=1}^{d_1} g_2|_q \big(\tau^*(dx^i), \tau^*(dx^j)\big) \lambda_i \lambda_j$$

for all $\lambda_1, \ldots, \lambda_{d_1} \in \mathbb{R}$ and $g_1^{ij}|_{\tau(q)} = g_2|_q\big(\tau^*(dx^i), \tau^*(dx^j)\big)$ for all $i, j \in \{1, \ldots, d_1\}$. Hence $g_1|_{\tau(q)}(\alpha, \beta) = g_2|_q\big(\tau^*(\alpha), \tau^*(\beta)\big)$ for all $\alpha, \beta \in T^*_{\tau(q)} M_1$. In particular, τ^* is injective and $d_2 \geqq d_1$.

Finally suppose that $d_1 = d_2$. Since $\tau^*|_{\tau(q)}$ is injective for all $q \in M_2'$ it follows that $\tau^*|_{\tau(q)}$ is bijective for all $q \in M_2'$. Hence τ is locally a C^∞-diffeomorphism. Moreover, $\tau^*|_{\tau(q)}$ is an orthogonal map and therefore also $\tau_*|_q$ is an orthogonal map for all $q \in M_2'$. In particular, τ is locally an isometry.

It then also follows that $\Delta_2(\varphi \circ \tau) = (\Delta_1 \varphi) \circ \tau$ on M_2' for all $\varphi \in C_c^\infty(M_1)$. If $q \in M_2'$ and Ω is a relatively compact open subset of M_2' with $q \in \Omega \subset \overline{\Omega} \subset M_2'$, and if one chooses $\varphi \in C_c^\infty(M_1)$ such that $\varphi|_{\tau(\Omega)} = 1$, then it follows from (5) that $(\Delta_2 h)(q) = 0$. Then for all $\varphi \in C_c^\infty(M_1)$ one deduces from (5) that

$$\begin{aligned} h \cdot \big((\Delta_1 \varphi) \circ \tau\big) &= \Delta_2\big(h \cdot (\varphi \circ \tau)\big) \\ &= (\Delta_2 h) \cdot (\varphi \circ \tau) + 2(\nabla_2 h) \cdot \big(\nabla_2(\varphi \circ \tau)\big) + h \cdot \Delta_2(\varphi \circ \tau) \\ &= 2(\nabla_2 h) \cdot \big(\nabla_2(\varphi \circ \tau)\big) + h \cdot \big((\Delta_1 \varphi) \circ \tau\big) \end{aligned}$$

on M_2'. So $(\nabla_2 h) \cdot \big(\nabla_2(\varphi \circ \tau)\big) = 0$ on M_2'. Since τ is locally a diffeomorphism it follows that $\nabla_2 h = 0$ and h is locally constant. This completes the proof of Proposition 4.4. □

In the next lemmas we consider injectivity and density of the range of U.

Lemma 4.5. *Assume the hypothesis of Lemma* 4.1. *Moreover, assume* $U \neq 0$, *the manifold* M_1 *is connected and the manifolds have equal dimension.*

Then there exist open sets $M_1' \subset M_1$ *and* $M_2' \subset M_2$, *a map* $\tau : M_2 \to M_1$ *and a bounded function* $h : M_2 \to [0, \infty)$ *such that* $M_2' = \{q \in M_2 : h(q) > 0\}$, $M_1' = \tau(M_2')$, $\tau|_{M_2'}$ *is a local isometry,* $h|_{M_2'}$ *is a* C^∞*-function and*

$$U\varphi = h \cdot (\varphi \circ \tau)$$

pointwise for all $\varphi \in C_c^\infty(M_1)$ *and a.e. for all* $\varphi \in L_p(M_1)$. *Moreover,* U *is injective and* $|M_1 \backslash M_1'| = 0$.

Proof. Since $C_c^\infty(M_1)$ is dense in $L_p(M_1)$ and $U \neq 0$ it follows that the restriction of U to $C_c^\infty(M_1)$ does not vanish. So we can apply Lemma 4.1 and Proposition 4.3. We use the notation of Proposition 4.3. Set $M_1' = \tau(M_2')$. It follows from the inverse function theorem that M_1' is open in M_1. Moreover, since $M_2' \neq \emptyset$ it follows that $M_1' \neq \emptyset$ and $|M_1'| \neq 0$.

Now let $\varphi \in L_p(M_1)$. Since $C_c^\infty(M_1)$ is dense in $L_p(M_1)$ there exists a sequence $\varphi_1, \varphi_2, \ldots \in C_c^\infty(M_1)$ such that $\lim \varphi_n = \varphi$ in $L_p(M_1)$. Since U is continuous it follows that $\lim U\varphi_n = U\varphi$ in $L_p(M_2)$. Passing to subsequences, if necessary, we may assume that $\lim \varphi_n = \varphi$ a.e. and $\lim U\varphi_n = U\varphi$ a.e. But since M_1' is σ-compact and $\tau|_{M_2'}$ is locally an isometry it follows that $\tau^{-1}(N)$ is a null-set in M_2' for every null-set N in M_1'. Therefore $\lim \varphi_n \circ \tau = \varphi \circ \tau$ a.e. on M_2'. Then

$$U\varphi = \lim U\varphi_n = \lim h \cdot (\varphi_n \circ \tau) = h \cdot (\varphi \circ \tau)$$

a.e. on M_2'. In addition one has $U\varphi = \lim U\varphi_n = 0$ a.e. on $M_2 \backslash M_2'$. So $U\varphi = h \cdot (\varphi \circ \tau)$ a.e. on M_2.

Next we show that U is injective. Let $\varphi \in L_2(M_1)$ and suppose that $U\varphi = 0$. Then $U|\varphi| = |U\varphi| = 0$, so we may assume that $\varphi \geqq 0$. Fix $t > 0$. Then

$$0 = S_t^{(2)} U\varphi = US_t^{(1)}\varphi = h \cdot (S_t^{(1)}\varphi) \circ \tau$$

a.e. on M_2'. Since also τ maps M_2'-null-sets into M_1'-null-sets and $h > 0$ pointwise, one deduces that $S_t^{(1)}\varphi = 0$ a.e. on M_1'. But M_1 is connected and if $\varphi \neq 0$ then $(S_t^{(1)}\varphi)(p) > 0$ for all $p \in M_1$ and in particular for all $p \in M_1'$. So $\varphi = 0$ and U is injective. It is obvious that this implies that $|M_1 \backslash M_1'| = 0$.

Finally we show that h is bounded by $\|U\|$. Let $q \in M_2'$ and $\varepsilon > 0$. Since $\tau|_{M_2'}$ is locally an isometry and $h|_{M_2'}$ is continuous there exists an open neighbourhood Ω of q in M_2' such that $\tau|_\Omega : \Omega \to \tau(\Omega)$ is an isometry and $h|_\Omega \geqq (1-\varepsilon)h(q)$. Fix $\varphi \in C_c^\infty(\tau(\Omega))$ with $\varphi \neq 0$. Then

$$(1-\varepsilon)h(q)\|\varphi \circ \tau\|_{L_p(\Omega)} \leqq \|h \cdot (\varphi \circ \tau)\|_{L_p(\Omega)} \leqq \|U\varphi\|_{L_p(M_2)}$$
$$\leqq \|U\|\,\|\varphi\|_{L_p(M_1)} \leqq \|U\|\,\|\varphi\|_{L_p(\tau(\Omega))} = \|U\|\,\|\varphi \circ \tau\|_{L_p(\Omega)}$$

where we used (1) in the last step. So $h(q) \leqq \|U\|$. □

Lemma 4.6. *Assume the hypothesis of Lemma* 4.1. *Let M_2' be an open subset of M_2, let $\tau : M_2 \to M_1$ and $h : M_2 \to [0,\infty)$ be two functions such that $M_2' = \{q \in M_2 : h(q) > 0\}$ and the restrictions $\tau|_{M_2'}$ and $h|_{M_2'}$ are both C^∞-maps. Suppose that $U\varphi = h \cdot (\varphi \circ \tau)$ for all $\varphi \in C_c^\infty(M_1)$.*

Then the following are equivalent:

(I) *$UL_p(M_1)$ is dense in $L_p(M_2)$.*

(II) *$UC_c^\infty(M_1)$ is dense in $L_p(M_2)$.*

(III) *For every pair of disjoint measurable subsets A_1 and A_2 in M_2 with $0 < |A_1|, |A_2| < \infty$ the functionals*

$$\varphi \mapsto \int_{A_1} U\varphi \quad \textit{and} \quad \varphi \mapsto \int_{A_2} U\varphi$$

from $C_c^\infty(M_1)$ into $\mathbb{R}$ are linearly independent.

(IV) *The map $\tau|_{M_2'}$ is injective and $|M_2 \backslash M_2'| = 0$.*

Moreover, these conditions imply that the dimensions of M_1 and M_2 are equal.

Proof. Clearly (I) ⇔ (II) ⇒ (III).

Next we show that (III) or (IV) implies that $\dim M_1 = \dim M_2$. Obviously $U \neq 0$, so $M_2' \neq 0$ and $d_2 \geqq d_1$ by Proposition 4.4, where $d_1 = \dim M_1$ and $d_2 = \dim M_2$. Fix $q \in M_2'$ and set $p = \tau(q)$. Let (V, x) be a chart on M_1 with $p \in V$ and (W, y) a chart on M_2' with $q \in W$. Let $V' = x(V)$, $W' = y(W)$, $p' = x(p)$ and $q' = y(q)$. Define the C^∞-map $f : W' \to V'$ by $f = x \circ \tau \circ y^{-1}$. Then it follows from (4) that $(Df)(q')$ has maximal rank. Suppose that $k = d_2 - d_1 > 0$. Then it follows from the inverse function theorem that there exist open $W'' \subset \mathbb{R}^{d_2}$, $\delta > 0$ and a C^∞-diffeomorphism $F : W'' \to B(p'; 3\delta) \times (-3\delta, 3\delta)^k$ such that $q \in W'' \subset W'$, $B(p'; 3\delta) \subset V'$, $F(q') = (p', 0)$ and $f(G(u,v)) = u$ for all $(u,v) \in B(p'; 3\delta) \times (-3\delta, 3\delta)^k$, where $G = F^{-1}$.

In particular, f is not injective. This contradicts the injectivity of τ in (IV).

In order to obtain a contradiction with condition (III) we proceed as follows. Define the C^∞-function $\Phi : B(p';3\delta) \times (-3\delta, 3\delta)^k \to (0,\infty)$ by

$$\Phi(u,v) = \big((h\sqrt{g_2}) \circ y^{-1} \circ G\big)(u,v)|(JG)(u,v)|,$$

where JG denotes the Jacobian determinant of G. If $A \subset y^{-1}(W'')$ is measurable then

$$\int_A U\varphi = \int_{(F\circ y)(A)} \Phi(u,v)(\varphi \circ x^{-1})(u)\, d(u,v) \tag{6}$$

for all $\varphi \in C_c(M_1)$. By compactness, there are $m, M > 0$ such that $m \leqq \Phi(u,v) \leqq M$ for all $(u,v) \in \overline{B(p';2\delta)} \times [-2\delta, 2\delta]^k$. Let $u \in \overline{B(p';2\delta)}$. Then

$$\int_{[-2\delta,2\delta]^{k-1}\times[-\delta m M^{-1},0)} \Phi(u,v)\, dv \leqq (4\delta)^{k-1}\delta m \leqq \int_{[-2\delta,2\delta]^{k-1}\times(0,\delta]} \Phi(u,v)\, dv.$$

So there exists a $\beta(u) \in (0,\delta]$ such that

$$\int_{[-2\delta,2\delta]^{k-1}\times[-\delta m M^{-1},0)} \Phi(u,v)\, dv = \int_{[-2\delta,2\delta]^{k-1}\times(0,\beta(u)]} \Phi(u,v)\, dv.$$

Now choose

$$A_1 = (y^{-1} \circ F^{-1})\big(\overline{B(p';2\delta)} \times [-2\delta,2\delta]^{k-1} \times [-\delta m M^{-1}, 0)\big)$$

and

$$A_2 = \big\{(y^{-1} \circ F^{-1})(u,v) : u \in \overline{B(p';2\delta)} \text{ and } v \in [-2\delta,2\delta]^{k-1} \times \big(0,\beta(u)\big]\big\}.$$

Then $A_1 \cap A_2 = \emptyset$, $0 < |A_1|, |A_2| < \infty$ and $\int_{A_1} U\varphi = \int_{A_1} U\varphi$ for all $\varphi \in C_c^\infty(M_1)$ by (6) and the choice of $\beta(u)$. This contradicts (III). So $d_1 = d_2$. Thus all four conditions imply that the dimensions of M_1 and M_2 are equal.

'(III) $\Rightarrow$ (IV)'. Clearly (III) implies that $|M_2 \backslash M_2'| = 0$. Suppose (III) and τ is not injective. Then there are $q_1, q_2 \in M_2'$ such that $\tau(q_1) = \tau(q_2)$ and $q_1 \neq q_2$. Since $\dim M_1 = \dim M_2$ it follows from Proposition 4.4 that $\tau|_{M_2'}$ is locally an isomorphism. There are open connected relative compact $\Omega_1, \Omega_2 \subset M_2'$ such that $\Omega_1 \cap \Omega_2 = \emptyset$ and for all $j \in \{1,2\}$ one has $q_j \in \Omega_j$ and $\tau_j = \tau|_{\Omega_j} : \Omega_j \to \tau(\Omega_j)$ is an isometry. Since Ω_j is connected there is a $c_j \in (0,\infty)$ such that $h|_{\Omega_j} = c_j$. Without loss of generality we may assume that $\tau_1(\Omega_1) = \tau_2(\Omega_2)$. Then for all $\varphi \in C_c^\infty(M_1)$ one has $(U\varphi)(q) = h(q)\varphi\big(\tau_1(q)\big) = c_1\varphi\big(\tau_1(q)\big)$ for all $q \in \Omega_1$. So

$$\int_{\Omega_1} U\varphi = c_1 \int_{\Omega_1} \varphi \circ \tau_1 = c_1 \int_{\tau_1(\Omega_1)} \varphi$$

where we used (1). Similarly

$$\int_{\Omega_1} U\varphi = c_2 \int_{\tau_2(\Omega_2)} \varphi$$

for all $\varphi \in C_c^\infty(M_1)$. But $\tau_1(\Omega_1) = \tau_2(\Omega_2)$. This contradicts the independence of the functionals.

'(IV) $\Rightarrow$ (II)'. Since $|M_2 \backslash M_2'| = 0$ the space $C_c^\infty(M_2')$ is dense in $L_p(M_2)$. Therefore it suffices to show that $C_c^\infty(M_2') \subset U C_c^\infty(M_1')$.

Using again that $\dim M_1 = \dim M_2$ it follows from Proposition 4.4 that $\tau|_{M_2'}$ is locally an isomorphism and $h|_{M_2'}$ is locally constant. If $\psi \in C_c^\infty(M_2')$ and there exists an open connected set Ω in M_2' such that $\operatorname{supp} \psi \subset \Omega$, then h is constant on Ω, say c, and $\varphi = c^{-1} \psi \circ (\tau|_{M_2'})^{-1} \in C_c^\infty(M_1') \subset C_c^\infty(M_1)$ satisfies $U\varphi = \psi$. Then the general case follows by a partition of the unity. $\square$

For open subsets in $\mathbb{R}^d$ the surjectivity of U follows from the fact that $U \neq 0$ (see [5], Theorem 2.1). In general the condition $U \neq 0$ is not sufficient to establish the surjectivity of U.

Example 4.7. Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. Let g_1 be the Riemannian metric on $M_1 = S^1$ such that $(g_1)|_{e^{i\theta}} \left(\frac{\partial}{\partial x^1}\Big|_{e^{i\theta}}, \frac{\partial}{\partial x^1}\Big|_{e^{i\theta}} \right) = 1$ for each $\theta \in \mathbb{R}$, where (V, x) is a chart on S^1 such that V is an open neighbourhood of $e^{i\theta}$, $\theta \in x(V)$ and $x^{-1}(\xi) = e^{i\xi}$ for all $\xi \in x(V)$. Set $M_2 = S_1$ and choose the Riemannian metric g_2 on M_2 by $g_2 = 4g_1$.

Define $U : L_2(M_1) \to L_2(M_2)$ by

$$(U\varphi)(z) = \varphi(z^2).$$

Then U is a lattice homomorphism, $U \neq 0$ and $U S_t^{(1)} = S_t^{(2)} U$ for all $t > 0$, where $S^{(j)}$ is the semigroup on $L_2(M_j)$ generated by the Dirichlet Laplace–Beltrami operator on M_j for all $j \in \{1, 2\}$. Moreover, M_1 and M_2 are regular in capacity. But the Riemannian manifolds (M_1, g_1) and (M_2, g_2) are not isomorphic.

We combine the previous results.

Proposition 4.8. *Let (M_1, g_1) and (M_2, g_2) be two connected Riemannian manifolds. Let $p \in [1, \infty)$. For all $j \in \{1, 2\}$ let Δ_j be the Dirichlet Laplace–Beltrami operator on M_j and let $S^{(j)}$ be the associated semigroup on $L_p(M_j)$. Let $U : L_p(M_1) \to L_p(M_2)$ be a lattice homomorphism such that $U L_p(M_1)$ is dense in $L_p(M_2)$ and*

$$U S_t^{(1)} = S_t^{(2)} U \tag{7}$$

for all $t > 0$.

Then U is an order isomorphism and there exist open connected sets $M_1' \subset M_1$ and $M_2' \subset M_2$, a map $\tau : M_2 \to M_1$ and a constant $c > 0$ such that $M_1' = \tau(M_2')$, $\tau|_{M_2'} : M_2' \to M_1'$ is an isometry, and

$$U\varphi = c \mathbb{1}_{M_2'} \cdot (\varphi \circ \tau)$$

pointwise for all $\varphi \in C_c^\infty(M_1)$ and a.e. for all $\varphi \in L_p(M_1)$. Moreover,

$$\operatorname{cap}(M_1 \backslash M_1') = \operatorname{cap}(M_2 \backslash M_2') = 0$$

and for all $\tilde{p} \in [1, \infty)$ there exists an order isomorphism $\tilde{U}$ such that $\tilde{U}\varphi = U\varphi$ for all $\varphi \in L_p(M_1) \cap L_{\tilde{p}}(M_1)$. Finally, if $p = 2$ then U maps $H_0^1(M_1)$ continuously into $H_0^1(M_2)$.

Proof. We use the notation as above. Let $\tilde{p} \in [1, \infty)$. Then the map

$$\tilde{U} : L_{\tilde{p}}(M_1) \to L_{\tilde{p}}(M_2)$$

defined by $\tilde{U}\varphi = h \cdot (\varphi \circ \tau)$ (a.e.) for all $\varphi \in L_{\tilde{p}}(M_1)$ is well defined since h is bounded and $\tau^{-1}(N)$ is a null-set in M_2' for every null-set N in M_1'. It is a lattice homomorphism and is consistent with U. Moreover, $UL_{\tilde{p}}(M_1)$ is dense in $L_{\tilde{p}}(M_2)$ by Proposition 4.6 (IV) $\Rightarrow$ (I).

Therefore, for the remainder of the proof we may assume that $p = 2$. Then one deduces from (7) that $(I + \Delta_2)^{-1/2} U = U(I + \Delta_1)^{-1/2}$. Hence

$$\begin{aligned} UH_0^1(M_1) &= U(I + \Delta_1)^{-1/2} L_2(M_1) \\ &= (I + \Delta_2)^{-1/2} UL_2(M_1) \subset (I + \Delta_2)^{-1/2} L_2(M_2) = H_0^1(M_2). \end{aligned}$$

Then by the closed graph theorem the restriction of U to $H_0^1(M_1)$ is a continuous map from $H_0^1(M_1)$ into $H_0^1(M_2)$. Next $UL_2(M_1)$ is dense in $L_2(M_2)$ and $(I + \Delta_2)^{-1/2}$ is continuous from $L_2(M_2)$ onto $H_0^1(M_2)$. So $UH_0^1(M_1)$ is dense in $H_0^1(M_2)$. Therefore $UC_c^\infty(M_1)$ is dense in $H_0^1(M_2)$.

Now suppose $\operatorname{cap}(M_2 \backslash M_2') > 0$. There exist compact subsets $K_1 \subset K_2 \subset \cdots$ of M_2 such that $M_2 = \bigcup_{n=1}^{\infty} K_n$. Then $\operatorname{cap}(M_2 \backslash M_2') = \lim_{n \to \infty} \operatorname{cap}(K_n \backslash M_2')$ by [13], Proposition 8.1.3c. Hence there exists an $n \in \mathbb{N}$ such that $\operatorname{cap}(K_n \backslash M_2') > 0$. Let $\psi \in C_c^\infty(M_2)$ be such that $\psi|_{K_n} = 1$. Since $UC_c^\infty(M_1)$ is dense in $H_0^1(M_2)$ there exists a $\varphi \in C_c^\infty(M_1)$ such that $\|\psi - U\varphi\|_{H^1(M_2)}^2 < \operatorname{cap}(K_n \backslash M_2')$. Then $\psi - U\varphi = 1$ on $K_n \backslash M_2'$ by definition of M_2'. Moreover, $\psi - U\varphi$ is continuous and $\psi - U\varphi \in H_0^1(M_2) \subset H^1(M_2)$. Hence

$$\operatorname{cap}(K_n \backslash M_2') \leqq \|\psi - U\varphi\|_{H^1(M_2)}^2.$$

This is a contradiction. Hence $\operatorname{cap}(M_2 \backslash M_2') = 0$.

This allows to apply Theorem 2.1 to deduce that M_2' is connected. Then $h|_{M_2'}$ is constant, say $c > 0$. It follows from (1) that (for any $p \in [1, \infty)$) the map U is an isometry between L_p-spaces and since the range is dense, it is surjective.

Finally, $H_0^1(M_2') = \{\psi|_{M_2'} : \psi \in H_0^1(M_2)\}$ by Lemma 3.2, since $\operatorname{cap}(M_2 \backslash M_2') = 0$. But $H_0^1(M_2) = UH_0^1(M_1)$. In addition, $UH_0^1(M_1') = H_0^1(M_2')$ since $\tau|_{M_2'} : M_2' \to M_1'$ is an isometry. Therefore

$$UH_0^1(M_1') = \{(U\varphi)|_{M_2'} : \varphi \in H_0^1(M_1)\} = \{U(\varphi|_{M_1'}) : \varphi \in H_0^1(M_1)\}.$$

Hence $H_0^1(M_1') = \{\varphi|_{M_1'} : \varphi \in H_0^1(M_1)\}$. Using Lemma 3.2 again one deduces that $\operatorname{cap}(M_1 \backslash M_1') = 0$. $\square$

Now we are able to prove the main theorem of this paper.

Proof of Theorem 1.1. The implication (I) $\Rightarrow$ (II) is trivial. But if condition (II) is valid then Proposition 4.8 implies that $M_1 \overset{\text{cap}}{\sim} M_2$. Hence the Riemannian manifolds (M_1, g_1) and (M_2, g_2) are isomorphic by Theorem 3.1. Moreover, there exist $c > 0$ and an isometry $\tau : M_2 \to M_1$ such that $U\varphi = c\varphi \circ \tau$ for all $\varphi \in L_p(M_1)$. $\square$

In fact, it follows from the proof that under condition (II) it follows that $M_2' = M_2$ and $M_1' = M_1$. Therefore $U\varphi = c\varphi \circ \tau$ for all $\varphi \in L_p(M_1)$.

5. Regularity in capacity

The purpose of this section is to characterize the notion of regularity in capacity by various other properties. Among those several are functional analytic in nature. Of special interest is a characterization via relative capacity. Recall that $\tilde{M}$ is the completion of M with respect to the natural distance and $\partial M = \tilde{M} \backslash M$. The relative capacity is defined on subsets of $\tilde{M}$ instead of M. It had been introduced in [8] for an open subset Ω in $\mathbb{R}^d$. Since it depends on the set Ω in [8] it is called relative capacity. The following definition on manifolds is similar to the Euclidean one.

Let μ be the trivial extension to $\tilde{M}$ of the natural Radon measure $|\cdot|$ on M, that is, for a Borel set $B \subset \tilde{M}$ we let $\mu(B) = |B \cap M|$. For a subset $A \subset \tilde{M}$ the *relative capacity* of A (with respect to M) is given by

$$\operatorname{rcap}(A) = \inf\{\|\varphi\|_{H^1(M)}^2 : \varphi \in \tilde{H}^1(M) \text{ and } \varphi \geqq 1 \ \mu\text{-a.e. on a neighbourhood of } A\}$$

where $\tilde{H}^1(M)$ is defined to be the closure of the space $H^1(M) \cap C_c(\tilde{M})$ in $H^1(M)$.

Note that the relative capacity is the usual capacity as defined in [13], Section I.8, on the space $\tilde{M}$ with respect to the Dirichlet form $(\psi, \varphi) \mapsto \int_{\tilde{M}} \nabla\psi \cdot \nabla\varphi$ and form domain $\tilde{H}^1(M)$. We consider $\tilde{H}^1(M)$ instead of $H^1(M)$ in order to fulfill condition (D) in [13], Subsection I.8.2, and therefore to use the notion of relative quasi-continuity and relative quasi-everywhere (r.q.e.). We do not need that $\tilde{M}$ is locally compact, although it is a consequence of the embedding theorem of Nash. In general, however, the completion of a locally compact metric space is not locally compact. We are grateful to Robin Nitka for showing us a counter example.

The following characterization of regularity of capacity is our main result in this section. Note that condition (V) is formulated completely in terms of relative capacity of the boundary $\partial\Omega$.

Theorem 5.1. *Let M be a connected Riemannian manifold. Then the following conditions are equivalent*:

(I) *M is regular in capacity.*

(II) *The space $C_c^\infty(M)$ is dense in $H_0^1(M) \cap C_0(\tilde{M})$.*

(III) *For every lattice homomorphism* $F : H_0^1(M) \cap C_0(\tilde{M}) \to \mathbb{R}$ *there exist* $c \in \mathbb{R}$ *and* $p \in M$ *such that* $F(\varphi) = c\varphi(p)$ *for all* $\varphi \in H_0^1(M) \cap C_0(\tilde{M})$.

(IV) *For every multiplicative functional* τ *on the Banach algebra* $H_0^1(M) \cap C_0(\tilde{M})$ *there exists a* $p \in M$ *such that* $\tau(\varphi) = \varphi(p)$ *for all* $\varphi \in H_0^1(M) \cap C_0(\tilde{M})$.

(V) *For every* $p \in \partial M$ *and* $r > 0$ *one has* $\operatorname{rcap}(\partial M \cap B_{\tilde{M}}(p,r)) > 0$.

For the proof of Theorem 5.1, we need a characterization of the space $H_0^1(M)$ in terms of the relative capacity. This result is also of independent interest.

Theorem 5.2. *Let M be a connected Riemannian manifold. Then*

$$(8) \qquad H_0^1(M) = \{\varphi \in \tilde{H}^1(M) : \tilde{\varphi} = 0 \text{ r.q.e. on } \partial M\}$$

where $\tilde{\varphi}$ *denotes the relative quasi-continuous version of* φ.

Proof. '$\subset$'. Since $C_c^\infty(M) \subset H^1(M) \cap C_c(\tilde{M})$ one deduces by closure in $H^1(M)$ that $H_0^1(M) \subset \tilde{H}^1(M)$. Let $\varphi \in H_0^1(M)$. Then it follows from the proof of Proposition 8.2.1 in [13] that there exists a sequence $\varphi_1, \varphi_2, \ldots \in C_c^\infty(M)$ such that $\lim_{n\to\infty} \varphi_n = \tilde{\varphi}$ r.q.e. on $\tilde{M}$. So $\tilde{\varphi} = 0$ r.q.e. on ∂M.

'$\supset$'. Let $D_0^1(M)$ denote the right-hand side of (8). Let $\varphi \in D_0^1(M) \cap L^\infty(\tilde{M})$. We may assume that $\varphi \geqq 0$. Then $\varphi \in \tilde{H}^1(M)$. It follows from the definition of $\tilde{H}^1(M)$ and the proof of Proposition 8.2.1 in [13] that there exist $\varphi_1, \varphi_2, \ldots \in H^1(M) \cap C_c(\tilde{M})$ such that $\lim_{n\to\infty} \varphi_n = \tilde{\varphi}$ in $H^1(M)$ and for all $\varepsilon > 0$ there exists an open $U \subset \tilde{M}$ such that $\operatorname{rcap}(U) < \varepsilon$ and $\lim_{n\to\infty} \varphi_n = \tilde{\varphi}$ uniformly on $\tilde{M} \backslash U$. We may assume that $0 \leqq \varphi_n \leqq \|\varphi\|_\infty$ and $\|\varphi_n\|_{H^1(M)} \leqq 2\|\varphi\|_{H^1(M)}$ for all $n \in \mathbb{N}$.

Let $\varepsilon \in (0,1]$. Then there exist $n \in \mathbb{N}$ and an open $U \subset \tilde{M}$ such that $\|\varphi_n - \varphi\|_{H^1(M)} \leqq \varepsilon$, $\operatorname{rcap}(U) < \varepsilon$ and $|\varphi_n - \tilde{\varphi}| \leqq \varepsilon$ uniformly on $\tilde{M} \backslash U$. Since $\tilde{\varphi} = 0$ r.q.e. on ∂M there exists an open $V \subset \tilde{M}$ such that $\{x \in \partial M : \tilde{\varphi}(x) \neq 0\} \subset V$ and $\operatorname{rcap}(V) < \varepsilon$. Consequently, $\varphi_n \leqq \varepsilon$ uniformly on $(\partial M) \backslash W$ where $W = U \cup V$, and $\operatorname{rcap}(W) \leqq \operatorname{rcap}(U) + \operatorname{rcap}(V) \leqq 2\varepsilon$. Let $\chi \in \tilde{H}(M)$ be such that $\chi \geqq 1$ on W and $\|\chi\|_{H^1(M)}^2 < 3\varepsilon$. We may assume that $\chi = 1$ pointwise on W and $0 \leqq \chi \leqq 1$ on M. Let $\sigma = (\varphi_n - 2\varepsilon)^+$ and $\tau = \sigma(\mathbb{1} - \chi)$. Then $\|\sigma\|_{H^1(M)} \leqq 2\|\varphi\|_{H^1(M)}$ and $\|\tau\|_{H^1(M)} \leqq 4\|\varphi\|_{H^1(M)} + 2\|\varphi\|_\infty$. Moreover,

$$\|\sigma - \tau\|_2 = \|\sigma\chi\|_2 \leqq \|\varphi\|_\infty \|\chi\|_2 \leqq 2\varepsilon^{1/2} \|\varphi\|_\infty.$$

Then $\operatorname{supp} \tau \subset \operatorname{supp} \sigma \cap W^c$, which is a compact subset of M. So $\tau \in H_c^1(M) \subset H_0^1(M)$.

It follows from the above that for all $m \in \mathbb{N}$ there exist $\varphi_m, \sigma_m \in H^1(M) \cap L_\infty$ and $\tau_m \in H_0^1(M) \cap L_\infty$ such that $\|\varphi - \varphi_m\|_{H^1(M)} \leqq \frac{1}{m}$, $\|\sigma_m - \tau_m\|_2 \leqq \frac{1}{m}$ and $0 \leqq \varphi_m - \sigma_m \leqq \frac{1}{m}$ for all $m \in \mathbb{N}$, and the sequences $\sigma_1, \sigma_2, \ldots$ and $\tau_1, \tau_2, \ldots$ are bounded in $H^1(M)$. We next show that $\tau_1, \tau_2, \ldots$ has a subsequence which converges to φ weakly in $H^1(M)$.

Clearly $\lim \varphi_m = \varphi$ strongly and hence weakly in $H^1(M)$. The sequences $\sigma_1, \sigma_2, \ldots$ and $\tau_1, \tau_2, \ldots$ are bounded in $H^1(M)$. Hence, by passing to a subsequence if necessary, these sequences are weakly convergent in $H^1(M)$. Since $0 \leqq \varphi_m - \sigma_m \leqq \frac{1}{m}$ for all $m \in \mathbb{N}$ it follows from the Lebesgue dominated convergence theorem that $\lim \varphi_m - \sigma_m = 0$ in $L_{2,\mathrm{loc}}$. Therefore it follows by the uniqueness of the weak limit that $\lim \varphi_m - \sigma_m = 0$ weakly in $H^1(M)$. Because $\lim \sigma_m - \tau_m = 0$ in L_2 it follows that $\lim \sigma_m - \tau_m = 0$ weakly in $H^1(M)$. Then $\lim \tau_m = \varphi$ weakly in $H^1(M)$. So $\varphi \in H_0^1(M)$ and $D_0^1(M) \cap L_\infty(\tilde{M}) \subset H_0^1(M)$.

Finally, if $\varphi \in D_0^1(M)$ then $(-n) \vee \varphi \wedge n \in D_0^1(M) \cap L_\infty(\tilde{M}) \subset H_0^1(M)$ for all $n \in \mathbb{N}$ and $\lim (-n) \vee \varphi \wedge n = \varphi$ in $H^1(M)$. So $\varphi \in H_0^1(M)$. □

Finally we prove the characterizations of regular in capacity.

Proof of Theorem 5.1. '(I) ⇒ (II)'. Let $\varphi \in H_0^1(M) \cap C_0(\tilde{M})$ and $\varepsilon > 0$. We may assume that $\varphi \geqq 0$. Since $\varphi \in C_0(\tilde{M})$ there exists a compact $K \subset \tilde{M}$ such that $\varphi(q) < \varepsilon$ for all $q \in \tilde{M} \backslash K$. Moreover, $\varphi(q) = 0$ for all $q \in \partial M$ since M is regular in capacity. Let $U = \{q \in \tilde{M} : \varphi(q) < \varepsilon\}$. Then U is open and $\tilde{M} \backslash M \subset U$. Moreover,

$$\operatorname{supp}(\varphi - \varepsilon)^+ \subset (\tilde{M} \backslash U) \cap K$$

and hence compact. But $(\tilde{M} \backslash U) \cap K \subset M$. Therefore $(\varphi - \varepsilon)^+ \in H_0^1(M) \cap C_c(M)$. Using a partition of the unity one deduces that $C_c^\infty(M)$ is dense in $H_0^1(M) \cap C_c(M)$. Finally, $\lim_{\varepsilon \downarrow 0} (\varphi - \varepsilon)^+ = \varphi$ in $H_0^1(M) \cap C_0(\tilde{M})$. So $C_c^\infty(M)$ is dense in $H_0^1(M) \cap C_0(\tilde{M})$.

'(II) ⇒ (I)'. Suppose M is not regular in capacity. Then there are

$$\varphi \in H_0^1(M) \cap C_0(\tilde{M})$$

and $p \in \partial M$ such that $\varphi(p) \neq 0$. Then $\|\varphi - \psi\|_{C_0(\tilde{M})} \geqq |\varphi(p)|$ for all $\psi \in C_c^\infty(M)$, so $C_c^\infty(M)$ is not dense in $H_0^1(M) \cap C_0(\tilde{M})$.

'(II) ⇒ (III)'. Let $F : H_0^1(M) \cap C_0(\tilde{M}) \to \mathbb{R}$ be a lattice homomorphism. Then F is continuous by [28], Theorem V.5.5(ii). Arguing as at the end of the proof of Lemma 4.1 it follows from Lemma 4.2 that there are $c \in \mathbb{R}$ and $p \in M$ such that $F(\varphi) = c\varphi(p)$ for all $\varphi \in C_c^\infty(M)$. Since F is continuous and $C_c^\infty(M)$ is dense in $H_0^1(M) \cap C_0(\tilde{M})$ it follows that $F(\varphi) = c\varphi(p)$ for all $\varphi \in H_0^1(M) \cap C_0(\tilde{M})$.

'(III) ⇒ (I)'. Suppose M is not regular in capacity. Then there are

$$\psi \in H_0^1(M) \cap C_0(\tilde{M})$$

and $p \in \partial M$ such that $\psi(p) \neq 0$. Define $F : H_0^1(M) \cap C_0(\tilde{M}) \to \mathbb{R}$ by $F(\varphi) = \varphi(p)$. Then F is a continuous lattice homomorphism. So by assumption there are $q \in M$ and $c \in \mathbb{R}$ such that $F(\varphi) = c\varphi(q)$ for all $\varphi \in H_0^1(M) \cap C_0(\tilde{M})$. Let $\chi \in C_c^\infty(M)$ be such that $\chi(q) = 1$. Then $\psi(\mathbb{1} - \chi) = \psi - \psi\chi \in H_0^1(M) \cap C_0(\tilde{M})$. Therefore

$$0 \neq \psi(p) = \big(\psi(\mathbb{1} - \chi)\big)(p) = F\big(\psi(\mathbb{1} - \chi)\big) = c\big(\psi(\mathbb{1} - \chi)\big)(q) = 0.$$

This is a contradiction.

'(II) $\Rightarrow$ (IV)'. Let $\tau : H_0^1(M) \cap C_0(\tilde{M}) \to \mathbb{C}$ be a (non-zero) multiplicative functional. Then τ is continuous by [20], Theorem C.21. Therefore it follows by condition (II) that $\tau|_{C_c^\infty(M)} : C_c^\infty(M) \to \mathbb{C}$ is a (non-zero) multiplicative functional. Let $p, q \in \operatorname{supp} \tau|_{C_c^\infty(M)}$ with $p \neq q$ and let U and V be two disjoint open neighbourhoods of p and q respectively. Then there exist $\varphi \in C_c^\infty(U)$ and $\psi \in C_c^\infty(V)$ such that $\tau(\varphi) \neq 0$ and $\tau(\psi) \neq 0$. But then $\varphi\psi = 0$ and

$$0 = \tau(\varphi\psi) = \tau(\varphi)\tau(\psi) \neq 0.$$

This is a contradiction. So there exists a $p \in M$ such that $\operatorname{supp} \tau|_{C_c^\infty(M)} = \{p\}$.

Next we show that τ is positive. Let $\varphi \in C_c^\infty(M)$ and suppose that $\varphi \geqq 0$. If $\varphi(p) > 0$ then there exist $\psi \in C_c^\infty(M)$ and a neighbourhood V of p such that $\varphi|_V = \psi^2|_V$. Then $\tau(\varphi) = \tau(\psi^2) = \tau(\psi)^2 \geqq 0$. Alternatively, suppose that $\varphi(p) = 0$. Let V be a relative compact neighbourhood of p. Then by continuity there exists a $c > 0$ such that $|\tau(\psi)| \leqq c\|\psi\|_{W^{1,\infty}(V)}$ for all $\psi \in C_c^\infty(V)$. We may assume that $\operatorname{supp} \varphi \subset V$. Then $\lim_{\varepsilon \downarrow 0} (\varphi - \varepsilon)^+ = \varphi$ in $W^{1,\infty}(V)$, so by regularizing it follows that there are $\varphi_1, \varphi_2, \ldots \in C_c^\infty(V)$ such that $\lim \varphi_k = \varphi$ in $W^{1,\infty}(V)$ and φ_k vanishes in a neighbourhood of p for all $k \in \mathbb{N}$. Then $\tau(\varphi) = \lim \tau(\varphi_k) = 0$.

Now it follows from Lemma 4.2 that there are $c \in [0, \infty)$ and $p \in M$ such that $\tau(\varphi) = c\tau(\varphi)$ for all $\varphi \in C_c^\infty(M)$. Then $c^2 = 1$ and since $\tau \neq 0$ it follows that $c = 1$. Since $C_c^\infty(M)$ is dense in $H_0^1(M) \cap C_0(\tilde{M})$ one establishes that $\tau(\varphi) = \varphi(p)$ for all $\varphi \in H_0^1(M) \cap C_0(\tilde{M})$.

'(IV) $\Rightarrow$ (I)'. This proof is similar to the proof (III) $\Rightarrow$ (I).

'(I) $\Rightarrow$ (V)'. Assume that there exist $p \in \partial M$ and $r > 0$ such that

$$\operatorname{rcap}\big(B_{\tilde{M}}(p; r) \cap \partial M\big) = 0.$$

Then there exist a $\tilde{M}$-open neighbourhood U of $B_{\tilde{M}}(p; r) \cap \partial M$ and a function $\chi \in \tilde{H}^1(M)$ such that $\chi \geqq 1$ a.e. on $U \cap M$. Let $\rho \in (0, r)$ be such that $M \backslash B_{\tilde{M}}(p; \rho) \neq \emptyset$. Define $\psi : \tilde{M} \to \mathbb{R}$ by $\psi(q) = d_{\tilde{M}}\big(q; \tilde{M} \backslash B(p; \rho)\big)$. Then $\psi \in C(\tilde{M})$ and $\psi|_M \in W^{1,\infty}(M)$. Set $\varphi = \chi\psi$. Then $\varphi \in C(\tilde{M})$ and by an elementary argument one deduces that $\varphi|_M \in \tilde{H}^1(M)$. Moreover, $\varphi = 0$ r.q.e. on ∂M. By Theorem 5.2 it follows that $\varphi|_M \in H_0^1(M)$. So $\varphi \in H_0^1(M) \cap C_0(\tilde{M})$. Since $\varphi(p) \neq 0$ it follows from the definition that the manifold M is not regular in capacity.

'(V) $\Rightarrow$ (I)'. If M is not regular in capacity then there exist $\varphi \in H_0^1(M) \cap C_0(\tilde{M})$ and $p \in \partial M$ such that $\varphi(p) \neq 0$. Without loss of generality we may assume that $\varphi(p) = 2$.

Let $r \in (0,1)$ be such that $\varphi \geqq 1$ on $B_{\tilde{M}}(p;r)$. Since $\varphi \in H_0^1(M)$ one deduces from Theorem 5.2 that $\varphi = 0$ r.q.e. on ∂M. Then $\mathrm{rcap}\big(B_{\tilde{M}}(p;r) \cap \partial M\big) \leqq \mathrm{rcap}\big(\{q \in \partial M : \varphi(q) \neq 0\}\big) = 0$. □

References

[1] *D. R. Adams* and *L. I. Hedberg*, Function spaces and potential theory, Grundl. Math. Wiss. **314**, Springer-Verlag, Berlin 1996.

[2] *C. D. Aliprantis* and *O. Burkinshaw*, Positive operators, Pure Appl. Math. **119**, Academic Press, Orlando 1985.

[3] *L. Ambrosio*, *N. Gigli* and *G. Savaré*, Gradient flows in metric spaces and in the space of probability measures, Lectures Math. ETH Zürich, Birkhäuser Verlag, Basel 2005.

[4] *L. Ambrosio* and *P. Tilli*, Topics on analysis in metric spaces, Oxford Lect. Ser. Math. Appl. **25**, Oxford University Press, New York 2004.

[5] *W. Arendt*, Different domains induce different heat semigroups on $C_0(\Omega)$, in: G. Lumer and L. Weis, eds., Evolution equations and their applications in physical and life sciences, Lect. Notes Pure Appl. Math. **215**, Marcel Dekker, New York (2001), 1–14.

[6] *W. Arendt*, Does diffusion determine the body?, J. reine angew. Math. **550** (2002), 97–123.

[7] *W. Arendt* and *D. Daners*, Varying domains: Stability of the Dirichlet and the Poisson problem, Discrete Contin. Dyn. Syst. (A) **21** (2008), 21–39.

[8] *W. Arendt* and *M. Warma*, The Laplacian with Robin boundary conditions on arbitrary domains, Potential Anal. **19** (2003), 341–363.

[9] *W. Arveson*, Dynamical invariants for noncommutative flows, in: Operator algebras and quantum field theory (Rome 1996), International Press, Cambridge, MA, 1997.

[10] *W. Arveson*, Noncommutative dynamics and E-semigroups, Springer Monogr. Math., Springer, New York 2003.

[11] *T. Aubin*, A course in differential geometry, Grad. Stud. Math. **27**, Amer. Math. Soc., Providence, RI, 2001.

[12] *M. Biegert* and *M. Warma*, Regularity in capacity and the Dirichlet Laplacian, Potential Anal. **25** (2006), 289–305.

[13] *N. Bouleau* and *F. Hirsch*, Dirichlet forms and analysis on Wiener space, de Gruyter Stud. Math. **14**, Walter de Gruyter & Co., Berlin 1991.

[14] *L. C. Evans* and *R. F. Gariepy*, Measure theory and fine properties of functions, Stud. Adv. Math., CRC Press, Boca Raton 1992.

[15] *D. Gilbarg* and *N. S. Trudinger*, Elliptic partial differential equations of second order, 2nd edn., Grundl. Math. Wiss. **224**, Springer-Verlag, Berlin 1983.

[16] *C. S. Gordon* and *D. L. Webb*, Isospectral convex domains in Euclidean space, Math. Res. Lett. **1** (1994), 539–545.

[17] *C. Gordon*, *D. L. Webb* and *S. Wolpert*, Isospectral plane domains and surfaces via Riemannian orbifolds, Invent. Math. **110** (1992), 1–22.

[18] *J. Heinonen*, Lectures on analysis and metric spaces, Universitext, Springer, New York 2001.

[19] *S. Helgason*, Differential geometry, Lie groups, and symmetric spaces, Pure Appl. Math. **80**, Academic Press, New York 1978.

[20] *E. Hewitt* and *K. A. Ross*, Abstract harmonic analysis I, 2nd edn., Grundl. Math. Wiss. **115**, Springer-Verlag, Berlin 1979.

[21] *M. Kac*, Can one hear the shape of a drum?, Amer. Math. Monthly **73** (1966), 1–23.

[22] *J. Milnor*, Eigenvalues of the Laplace operator on certain manifolds, Proc. Nat. Acad. Sci. U.S.A. **51** (1964), 542.

[23] *J. R. Norris*, Heat kernel asymptotics and the distance function in Lipschitz Riemannian manifolds, Acta Math. **179** (1997), 79–103.

[24] *F. Otto* and *C. Villani*, Generalization of an inequality by Talagrand and links with the logarithmic Sobolev inequality, J. Funct. Anal. **173** (2000), 361–400.

[25] *M.-K. von Renesse* and *K.-T. Sturm*, Transport inequalities, gradient estimates, entropy, and Ricci curvature, Comm. Pure Appl. Math. **58** (2005), 923–940.

[26] *L. Saloff-Coste*, A note on Poincaré, Sobolev, and Harnack inequalities, Int. Math. Res. Not. **1992** (1992), no. 2, 27–38.

[27] *L. Saloff-Coste*, Parabolic Harnack inequality for divergence-form second-order differential operators, Potential Anal. **4** (1995), 429–467.
[28] *H. H. Schaefer*, Topological vector spaces, Grad. Texts Math. **3**, Springer-Verlag, New York 1966.
[29] *H. Urakawa*, Bounded domains which are isospectral but not congruent, Ann. Sci. Éc. Norm. Sup. (4) **15** (1982), 441–456.
[30] *S. R. S. Varadhan*, Diffusion processes in a small time interval, Comm. Pure Appl. Math. **20** (1967), 659–685.
[31] *S. R. S. Varadhan*, On the behavior of the fundamental solution of the heat equation with variable coefficients, Comm. Pure Appl. Math. **20** (1967), 431–455.
[32] *S. Zelditch*, Spectral determination of analytic bi-axisymmetric plane domains, Geom. Funct. Anal. **10** (2000), 628–677.

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