COMMUNICATIONS ON
PURE AND APPLIED ANALYSIS
Volume **11**, Number **6**, November **2012**

doi:10.3934/cpaa.2012.11.

pp. –

FRIEDLANDER'S EIGENVALUE INEQUALITIES AND THE DIRICHLET-TO-NEUMANN SEMIGROUP

Wolfgang Arendt

Institute of Applied Analysis
University of Ulm, D - 89069 Ulm, Germany

Rafe Mazzeo

Department of Mathematics
Stanford University, Stanford, CA 94305, USA

Abstract. If Ω is any compact Lipschitz domain, possibly in a Riemannian manifold, with boundary $\Gamma = \partial\Omega$, the Dirichlet-to-Neumann operator $\mathcal{D}_\lambda$ is defined on $L^2(\Gamma)$ for any real λ. We prove a close relationship between the eigenvalues of $\mathcal{D}_\lambda$ and those of the Robin Laplacian Δ_μ, i.e. the Laplacian with Robin boundary conditions $\partial_\nu u = \mu u$. This is used to give another proof of the Friedlander inequalities between Neumann and Dirichlet eigenvalues, $\lambda^N_{k+1} \le \lambda^D_k$, $k \in \mathbb{N}$, and to sharpen the inequality to be strict, whenever Ω is a Lipschitz domain in $\mathbb{R}^d$. We give new counterexamples to these inequalities in the general Riemannian setting. Finally, we prove that the semigroup generated by $-\mathcal{D}_\lambda$, for λ sufficiently small or negative, is irreducible.

1. **Introduction.** Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with $\partial\Omega = \Gamma$. Let $\lambda^D_1 < \lambda^D_2 \le \lambda^D_3 \le \cdots$ and $\lambda^N_1 < \lambda^N_2 \le \lambda^N_3 \le \cdots$ be the eigenvalues of the Dirichlet and Neumann Laplacians on Ω, respectively. There is a beautiful set of inequalities discovered by Friedlander [9] which compares the elements of these two lists, namely

$$\lambda^N_{k+1} \le \lambda^D_k \quad \text{for all } k. \tag{1.1}$$

The fundamental tool in his proof is the Dirichlet-to-Neumann operator associated to $\Delta - \lambda$; his methods require that $\partial\Omega$ be at least $\mathcal{C}^1$. Friedlander's inequalities have attracted substantial attention since then, starting from a geometric recasting of his argument by the second author [19]. More recently, Filonov [8] discovered a substantially simpler proof of (1.1) based on the minimax characterization of eigenvalues, assuming only that Ω has finite measure and that the inclusion $H^1(\Omega) \subset L^2(\Omega)$ be compact. An extension of Filonov's ideas by Gesztesy and Mitrea [10] provides a comparison between generalized Robin and Dirichlet eigenvalues, while Safarov [24] showed how to describe all of this in a purely abstract setting involving only quadratic forms on Hilbert spaces.

The present paper is a substantially shortened version of the preprint [3], which apparently provided some motivation for [10], and hence should be placed before that paper in the chronology. We have decided to revise it for publication since we believe that the point of view espoused here is still of interest and should lead to further progress on some of the questions we consider. We return to the use

2000 *Mathematics Subject Classification.* Primary: 35P15; Secondary: 47D06.

Key words and phrases. Eigenvalue inequalities, Dirichlet-to-Neumann operator.

of the Dirichlet-to-Neumann operator, formulated weakly so that our argument applies on Lipschitz domains. (This is still less general than the domains considered by Filonov.) Our starting point is the folklore observation that if λ and μ are real numbers, then μ is an eigenvalue of the Dirichlet-to-Neumann operator $\mathcal{D}_\lambda$ associated to $\Delta - \lambda$ if and only if λ is an eigenvalue of the Robin Laplacian Δ_μ, i.e. the operator Δ on Ω with boundary condition $\partial_\nu u = \mu u$. We prove that λ depends strictly monotonically on μ, and vice versa. This has been rediscovered several times before our proof of it in [3]; it is equivalent to the monotonicity for $\mathcal{D}_\lambda$ used by Friedlander [9], see also [19], but traces back at least as far as the paper of Grégoire, Nédélec and Planchard [11] in the mid '70's, though they in turn attribute the idea to earlier unpublished work of Caseau. This relationship and monotonicity was known to S.T. Yau in the '70's as well. In any case, this is a lovely set of ideas which deserves to be more widely appreciated and utilized. We show here that it leads directly to yet another proof of (1.1). We also show that (1.1) need not be true for general manifolds with boundary. This was already discussed in [19], and the counterexample given there is any spherical cap larger than a hemisphere. We prove here that (1.1) also fails if Ω is the complement of a sufficiently small set in any closed manifold M.

Our second goal in this paper is to present some facts about the semigroup associated to the Dirichlet-to-Neumann operator $\mathcal{D}_\lambda$ (for any $\lambda < \lambda_1^D$). Specifically, we prove that it is positive and irreducible. While this is somewhat disjoint from the question of eigenvalue inequalities, the proof is yet another illustration of the close link between the Robin Laplacian and $\mathcal{D}_\lambda$. A consequence of this is that the first eigenvalue of $\mathcal{D}_\lambda$ is simple and has a strictly positive eigenfunction. Note that this irreducibility of T requires only that Ω be connected, though its boundary may have several components. This reflects the non-local nature of $\mathcal{D}_\lambda$.

We mention also the recent paper [4] which considers a number of issues related to the ones here. For general information about eigenvalue problems we refer to [14] and [15].

We shall be brief since various of the papers cited above contain good expositions of all the background material needed here, as well as the history of eigenvalue inequalities preceding (1.1). The next section contains a short review of the correspondence between coercive symmetric forms and self-adjoint operators and the weak formulation of normal derivatives on Lipschitz domains, and then records the quadratic forms underlying the various operators we study in this paper. §3 describes the eigenvalue monotonicity and its application to the proof of the eigenvalue inequalities. The Dirichlet-to-Neumann semigroup is the subject of §4.

2. **The Robin Laplacian and Dirichlet-to-Neumann operator.** Let H be an infinite dimensional separable Hilbert space and V another Hilbert space which is embedded as a dense subspace in H, so that $V \subset H \subset V^*$. Suppose that a is a closed, symmetric, real-valued, coercive quadratic form, i.e.

$$a(u) + \omega \|u\|_H^2 \geq \alpha \|u\|_V^2 \quad \text{for all } u \in V$$

for some $\omega \in \mathbb{R}$ and $\alpha > 0$. Associated to a is a bounded operator $A_1 : V \to V^*$. Also associated to a is an unbounded self-adjoint operator A_2 on H with domain $\mathcal{D}(A_2) \subset V \subset H$. Thus $x \in \mathcal{D}(A_1)$ and $A_1 x = y \in V^*$ if and only if $a(x, v) = \langle y, v\rangle$ for all $v \in V$. The operator A_2 is the part of A_1 in $\mathcal{D}(A_2)$, and hence we simply write either operator as A and drop the subscript. The form a is **accretive** (i.e. $a(u) \geq 0$ for all $u \in V$) if and only if A is nonnegative (i.e. $\langle Au, u\rangle_H \geq 0$ for all

$u \in \mathcal{D}(A)$). Furthermore, A has compact resolvent, and hence discrete spectrum, if and only if the inclusion $\mathcal{D}(A) \hookrightarrow H$ is compact, which is certainly the case if $V \hookrightarrow H$ is compact. Assuming that this is so, then we denote by $\{e_n, \lambda_n\}$ the eigendata for A, so the e_n are an orthonormal basis for H, $Ae_n = \lambda_n e_n$ for all n, and $\lambda_1 \leq \lambda_2 \leq \cdots \nearrow \infty$. The standard max-min characterization of the eigenvalues is

$$\lambda_n = \sup_{V_{n-1} \in \mathcal{G}_{n-1}(V)} \inf\{a(u) : u \in V_{n-1},\ ||u|| = 1\}. \tag{2.1}$$

where $\mathcal{G}_{n-1}(V)$ denotes the set of all subspaces of V of codimension $n-1$.

Let (Ω, g) be a compact Riemannian manifold with Lipschitz boundary. In other words, we assume that Ω is a connected, compact subset in a larger smooth manifold M, that the metric g on Ω is the restriction of a smooth metric on M, and that $\Gamma = \partial\Omega$ is locally a Lipschitz graph such that Ω lies locally on one side of Γ. (The results below extend in a straightforward manner if we only assume that M has a $\mathcal{C}^{1,1}$ structure and that the metric g is Lipschitz.) We refer to [12], [13], [17], [18] for more about the (straightforward) generalizations of the analytic facts used in this paper from the setting of Lipschitz domains in $\mathbb{R}^d$ to domains in manifolds.

The volume form and gradient for g lead naturally to the Hilbert spaces $L^2(\Omega)$ and $H^1(\Omega)$, as well as the space $L^2(\Gamma)$. As usual, $H^1_0(\Omega)$ is the closure of $\mathcal{C}^\infty_0(\Omega)$ in $H^1(\Omega)$. The boundary restriction map $u \mapsto u|_\Gamma := \operatorname{Tr} u$ is well-defined for any $u \in H^1(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$, and this map extends to a bounded operator $\operatorname{Tr} : H^1(\Omega) \to L^2(\Gamma)$, with nullspace $H^1_0(\Omega)$. We write $u_{|\Gamma}$ or $\operatorname{Tr} u$ interchangeably.

We next recall the weak formulations of well-known operators and identities.

a) If $u \in H^1(\Omega)$, we say that $\Delta u \in L^2(\Omega)$ if there exists $f \in L^2(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \overline{\nabla v}\, dV_g = \int_\Omega f \overline{v}\, dV_g \quad \text{for all } v \in H^1_0(\Omega).$$

b) Suppose that $u \in H^1(\Omega)$ and $\Delta u \in L^2(\Omega)$. We say that $\partial_\nu u \in L^2(\Gamma)$ if there exists $b \in L^2(\Gamma)$ such that

$$\int_\Omega \left(\nabla u \cdot \overline{\nabla v} - \Delta u\, \overline{v}\right) dV_g = \int_\Gamma b \overline{v}\, d\sigma_g \quad \text{for all } v \in H^1(\Omega),$$

and we then write $\partial_\nu u = b$.

To be explicit, our conventions are that $\Delta = -\operatorname{div} \nabla$ and ν is the outer unit normal; also, dV_g and $d\sigma_g$ are the volume forms on Ω and Γ associated to g. Here and later we often omit the trace signs under the integral, e.g. simply write $\int_\Gamma bv = \int_\Gamma bv_{|\Gamma}$. These definitions are set so that Green's formula still holds:

$$\int_\Omega \left(\nabla u \cdot \overline{\nabla v} - \Delta u \overline{v}\right) dV_g = \int_\Gamma \partial_\nu u\, \overline{v}\, d\sigma_g$$

for all $v \in H^1(\Omega)$ whenever $u \in H^1(\Omega)$, $\Delta u \in L^2(\Omega)$ and $\partial_\nu u \in L^2(\Gamma)$.

Consider the form, for any $\mu \in \mathbb{R}$,

$$b_\mu(u, v) = \int_\Omega \nabla u \cdot \overline{\nabla v}\, dV_g - \mu \int_\Gamma u \overline{v}\, d\sigma_g, \tag{2.2}$$

for $u, v \in H^1(\Omega)$. It is not hard to show that b_μ is coercive, and hence determines an operator Δ_μ. Letting $v \in H^1_0(\Omega)$ shows that Δ_μ is just the standard Laplacian in the interior, and we then deduce that $u \in \mathcal{D}(\Delta_\mu)$ implies $\partial_\nu u = \mu u$, at least in the weak sense. Thus, altogether,

$$\mathcal{D}(\Delta_\mu) = \left\{u \in L^2(\Omega) : \Delta u \in L^2(\Omega), \partial_\nu u \text{ exists and } \partial_\nu u = \mu u_{|\Gamma}\right\}.$$

The special case $\mu = 0$ corresponds to the Neumann Laplacian Δ^N.

We next consider the form

$$b_{-\infty}(u,v) = \int_\Omega \nabla u \cdot \overline{\nabla v}\, dV_g, \tag{2.3}$$

for $u, v \in H_0^1(\Omega)$. The discussion in the next section motivates why the moniker $b_{-\infty}$ is reasonable. The coercivity of this form is obvious, and its corresponding operator is the Dirichlet Laplacian Δ^D.

Since $H^1(\Omega)$ is compactly included in $L^2(\Omega)$, each of these operators has discrete spectrum. We write

$$\sigma(\Delta_\mu) = \{\lambda_j(\mu)\}_{j=1}^\infty,\ \sigma(\Delta^D) = \{\lambda_j^D\}_{j=1}^\infty, \text{ and } \sigma(\Delta^N) = \{\lambda_j^N\}_{j=1}^\infty.$$

Thus, $\lambda_j(0) = \lambda_j^N$, whereas $\lim_{\mu\to-\infty} \lambda_j(\mu) = \lambda_j^D$ (see Proposition 3.2 below). Hence the Robin eigenvalues interpolate between the Dirichlet and Neumann eigenvalues.

We now define, for each $\lambda \in \mathbb{R}$, the Dirichlet-to-Neumann operator $\mathcal{D}_\lambda$. If $\lambda \in \mathbb{R} \setminus \sigma(\Delta^D)$, then the classical definition is that if g is a (sufficiently smooth) function on Γ and u is the unique function on Ω such that $(\Delta - \lambda)u = 0$, $\operatorname{Tr} u = g$, then $\mathcal{D}_\lambda g = \partial_\nu u_{|\Gamma}$. Note that u is indeed uniquely defined if and only if $\lambda \notin \sigma(\Delta^D)$. There are several equivalent ways to circumvent this apparent need to avoid the Dirichlet eigenvalues. The first and most classical is simply to consider the Cauchy data subspace, sometimes also called the Calderon subspace, which is defined for any $\lambda \in \mathbb{R}$ by

$$\mathcal{C}_\lambda = \{(g,h) \in L^2(\Gamma) \times L^2(\Gamma) : \exists\, u \in H^1(\Omega) \text{ such that } \Delta u = \lambda u, u_{|\Gamma} = g\ ,\ \partial_\nu u = h\}\ .$$

It follows from Proposition 1 below that $\mathcal{C}_\lambda$ is a closed subspace of $L^2(\Gamma) \times L^2(\Gamma)$. If $\lambda \notin \sigma(\Delta^D)$, then $\mathcal{C}_\lambda$ intersects $\{0\} \times L^2(\Gamma)$ only at the origin, and hence there is a densely defined closed operator $\mathcal{D}_\lambda$ on $L^2(\Gamma)$ for which $\mathcal{C}_\lambda$ is the graph.

We may also consider $\mathcal{C}_\lambda$ as a multi-valued selfadjoint operator when $\lambda \in \sigma(\Delta^D)$. In order to avoid this, we define $\mathcal{D}_\lambda$ as follows. Let $\lambda \in \sigma(\Delta^D)$ and define $K(\lambda) := \{h \in L^2(\Gamma) : (0,h) \in \mathcal{C}_\lambda\}$; clearly

$$K(\lambda) = \{\partial_\nu w : w \in \ker(\lambda - \Delta^D),\ \partial_\nu w \in L^2(\Gamma)\}.$$

Let $L^2_\lambda(\Gamma) := K(\lambda)^\perp$ (the orthogonal taken in $L^2(\Gamma)$). Since $\dim K(\lambda) < \infty$, $L^2_\lambda(\Gamma)$ is an infinite-dimensional closed subspace of $L^2(\Gamma)$. We now let $\mathcal{D}_\lambda$ be the unique operator on $L^2_\lambda(\Gamma)$ whose graph is $\mathcal{C}_\lambda \cap (L^2_\lambda(\Gamma) \times L^2_\lambda(\Gamma))$. In this way, the operator $\mathcal{D}_\lambda$ is defined for all $\lambda \in \mathbb{R}$. It follows from our definition of the normal derivative that $\mathcal{D}_\lambda$ is symmetric. In order to show that $\mathcal{D}_\lambda$ is self-adjoint (i.e., that $(is - \mathcal{D}_\lambda)$ is invertible for $s \in \mathbb{R} \setminus \{0\}$), we use the following result by Grégoire, Nédélec and Planchard [11, Proposition 1].

Proposition 1. *Fix $\lambda \in \mathbb{R}$ and $s \in \mathbb{R} \setminus \{0\}$. Then given any $h \in L^2(\Gamma)$, there exists a unique $u \in H^1(\Omega)$ which satisfies*

$$\begin{aligned} \Delta u &= \lambda u \\ is u_{|\Gamma} - \partial_\nu u &= h\ . \end{aligned}$$

This solution u is uniquely determined by the condition that

$$\int_\Omega \nabla u \cdot \overline{\nabla v} - \lambda \int_\Omega u \overline{v} = \int_\Gamma \left((is u_{|\Gamma} - h\right) \overline{v}_{|\Gamma}$$

for all $v \in H^1(\Omega)$.

For $h \in L^2(\Gamma)$, let $R(is)h = u_{|_\Gamma}$ where u is the solution above. By the Closed Graph Theorem, $L^2(\Gamma) \ni h \mapsto u \in H^1(\Omega)$ is bounded, so by the compactness of $\mathrm{Tr} : H^1(\Omega) \to L^2(\Gamma)$, we see that that $R(is)$ is compact on $L^2(\Gamma)$. We let $L^2_\lambda(\Gamma) = L^2(\Gamma)$ if $\lambda \notin \sigma(\Delta^D)$. The relationship with $\mathcal{D}_\lambda$ is as follows.

Proposition 2. *The operator* $\mathcal{D}_\lambda$ *is selfadjoint for every* $\lambda \in \mathbb{R}$. *In fact, for* $s \in \mathbb{R} \setminus \{0\}$ *the resolvent is given by*

$$(is - \mathcal{D}_\lambda)^{-1}h = R(is)h \quad (h \in L^2_\lambda(\Gamma)).$$

In particular, $\mathcal{D}_\lambda$ *has compact resolvent.*

Proof. Let $h \in L^2_\lambda(\Gamma)$, $R(is)h = g$, and define the corresponding $u \in H^1(\Omega)$ as in Proposition 1 with $u_{|_\Gamma} = g$. We claim that $g \in K(\lambda)^\perp$. In fact, let $\partial_\nu w \in K(\lambda)$, where $w \in \ker(\lambda - \Delta^D)$. Since $w \in H^1_0(\Omega)$, it follows from the last identity in Proposition 1 for $v := w$

$$\begin{aligned} 0 &= \int_\Omega \nabla u \cdot \overline{\nabla w} - \lambda \int_\Omega u\,\overline{w} = \int_\Omega \nabla u \cdot \overline{\nabla w} - \int_\Omega u\,\overline{\Delta w} \\ &= \int_\Gamma u_{|_\Gamma}\,\overline{\partial_\nu w} = \langle g, \partial_\nu w \rangle_{L^2(\Gamma)}\,. \end{aligned}$$

Thus $g \in K(\lambda)^\perp = L^2_\lambda(\Gamma)$.

Since $isg - \partial_\nu u = h \in L^2_\lambda(\Gamma)$, it follows that $\partial_\nu u \in L^2_\lambda(\Gamma)$ as well. Moreover, since $\partial_\nu u = isg - h$ one has $(g, isg - h) \in \mathcal{C}_\lambda \cap (L^2_\lambda(\Gamma) \times L^2_\lambda(\Gamma))$. Thus $g \in \mathcal{D}(\mathcal{D}_\lambda)$ and $\mathcal{D}_\lambda g = isg - h$. □

Lemma 2.1. *Let* $(g,h) \in \mathcal{C}_\lambda$. *Then* $g \in \mathcal{D}(\mathcal{D}_\lambda)$. *If* $\langle g,h\rangle = 0$, *then* $\langle \mathcal{D}_\lambda g, g\rangle = 0$.

Proof. Let $(g,h) \in \mathcal{C}_\lambda$. We show that $g \in L^2_\lambda(\Gamma)$. By definition, there exists $u \in H^1(\Omega)$ such that $\Delta u = \lambda u, u_{|_\Gamma} = g, \partial_\nu u = h$. Let $k \in K(\lambda)$. We have to show that $\langle k, g\rangle = 0$. There exists $w \in \ker(\lambda - \Delta^D)$ such that $k = \partial_\nu w$. Thus $\langle k, g\rangle = \int_\Gamma \partial_\nu w \bar{u} = -\int_\Omega \Delta w \bar{u} + \int_\Omega \nabla w \overline{\nabla u} = -\int_\Omega \lambda w \bar{u} + \int_\Omega w \overline{\Delta u} = 0$ since $w \in H^1_0(\Omega)$. This proves the claim. Since $\mathcal{C}_\lambda$ is closed, also $K(\lambda)$ is a closed subspace of $L^2(\Gamma)$. Thus we can write $h = h_0 + h_1$ with $h_0 \in L^2_\lambda(\Gamma), h_1 \in K(\lambda)$. Hence $g \in \mathcal{D}(\mathcal{D}_\lambda)$ and $\mathcal{D}_\lambda g = h_0$. Now assume that $\langle g,h\rangle = 0$. Since $g \in L^2_\lambda(\Gamma)$ one has $\langle g, h_1\rangle = 0$. Consequently, $\langle g, \mathcal{D}_\lambda g\rangle = \langle g, h_0\rangle = \langle g, h - h_1\rangle = 0$. □

We will see later, in Theorem 3.1, that the operator $\mathcal{D}_\lambda$ is bounded below. Thus its spectrum consists of eigenvalues $\alpha_k(\lambda)$, $k = 1, 2, \ldots$, which we arrange in increasing order repeated according to multiplicity.

Remark 1. Using Proposition 1 Grégoire et al. [11] define the unitary operator $B(s)$ on $L^2(\Gamma)$:

$$B(s) = (is - \mathcal{C}_\lambda)(is + \mathcal{C}_\lambda)^{-1}$$

for $\lambda = s^2$ (where $\mathcal{C}_\lambda$ is considered as a multi-valued operator, which is such that its resolvent is single-valued). Hence as λ increases, the poles of $\mathcal{D}_\lambda$ as λ crosses a Dirichlet eigenvalue transform to a more innocuous spectral flow across the value 1.

We conclude this discussion by an alternative form definition of $\mathcal{D}_\lambda$ in the case where $\lambda \notin \sigma(\Delta^D)$.

Lemma 2.2. *For any $\lambda \in \mathbb{R} \setminus \sigma(\Delta^D)$, define $H^1(\lambda) = \{u \in H^1(\Omega) : \Delta u = \lambda u\}$. Then*

$$H^1(\Omega) = H^1_0(\Omega) \oplus H^1(\lambda).$$

Proof. If $\lambda \notin \sigma(\Delta^D)$, then $\Delta^D - \lambda : H^1_0(\Omega) \to (H^1_0(\Omega))^*$ is an isomorphism. Let $u \in H^1(\Omega)$ and consider the element $F \in H^1_0(\Omega)^*$ given by $F(v) = \int_\Omega (\nabla u \cdot \nabla v - \lambda u v)$. Since $\lambda \notin \sigma(\Delta^D)$, there exists $u_0 \in H^1_0(\Omega)$ such that $(\Delta^D - \lambda)u_0 = F$. Thus $u_1 := u - u_0 \in H^1(\lambda)$, and hence $u = u_0 + u_1 \in H^1_0(\Omega) + H^1(\lambda)$. We have now shown that $H^1(\Omega) = H^1_0(\Omega) + H^1(\lambda)$. Since $\lambda \notin \sigma(\Delta^D)$ one has $H^1_0(\Omega) \cap H^1(\lambda) = \{0\}$. The fact that $H^1(\Omega)$ is the topological direct sum of these two spaces follows from the open mapping theorem. ☐

Since Tr $: H^1(\lambda) \to L^2(\Gamma)$ is injective and $H^1(\lambda) \hookrightarrow L^2(\Omega)$ is compact, it is not difficult to show that there exist $\alpha > 0, \omega \geq 0$ such that

$$\int_\Omega |\nabla u|^2 - \lambda \int_\Omega |u|^2 + \omega \int_\Gamma |u|^2 \geq \alpha \|u\|^2_{H^1}$$

for every $u \in H^1(\lambda)$. Define

$$a_\lambda(u_{|\Gamma}, v_{|\Gamma}) = \int_\Omega \nabla u \cdot \overline{\nabla v} - \lambda \int_\Omega u\overline{v}.$$

Then a_λ is a closed, symmetric form on $L^2(\Gamma)$ and $\mathcal{D}_\lambda$ is the associated self-adjoint operator. We refer to [3] for more details.

3. **Eigenvalue comparison.** We now recall the relationship between the eigenvalues $\lambda_k(\mu)$ of Δ_μ and $\alpha_k(\lambda)$ of $\mathcal{D}_\lambda$.

Theorem 3.1. *Let $\lambda, \mu \in \mathbb{R}$. Then*

a) $\mu \in \sigma(\mathcal{D}_\lambda) \Leftrightarrow \lambda \in \sigma(\Delta_\mu)$;
b) $\dim \ker(\mu - \mathcal{D}_\lambda) = \dim \ker(\lambda - \Delta_\mu)$.

Proof. Both assertions follow from the fact that the mapping

$$S : \ker(\Delta_\mu - \lambda) \longrightarrow \ker(\mathcal{D}_\lambda - \mu), \qquad u \mapsto \operatorname{Tr} u$$

is an isomorphism.

To prove this, let $u \in \ker(\Delta_\mu - \lambda)$. Then $b_\mu(u, v) = \lambda \int_\Omega u\overline{v}$ for all $v \in H^1(\Omega)$, i.e.

$$\int_\Omega \left(\nabla u \cdot \overline{\nabla v} - \lambda u\overline{v}\right) = \mu \int_\Gamma u\overline{v} \tag{3.1}$$

for all $v \in H^1(\Omega)$. This implies that $(\operatorname{Tr} u, \mu \operatorname{Tr} u) \in \mathcal{C}_\lambda$. Let $\partial_\nu w \in K(\lambda)$ where $w \in \ker(\lambda - \Delta^D)$. Then

$$\begin{aligned} \int_\Gamma \partial_\nu w \, \overline{\operatorname{Tr} u} &= \int_\Omega \nabla w \cdot \overline{\nabla u} - \int_\Omega \Delta w \, \overline{u} \\ &= \int_\Omega \nabla w \cdot \overline{\nabla u} - \lambda \int_\Omega w \, \overline{u} \\ &= \int_\Omega w \, \overline{\Delta u} - \lambda \int_\Omega w \, \overline{u} = 0 \end{aligned}$$

since $w \in H^1_0(\Omega)$ and $\Delta u = \lambda u$. Thus $\operatorname{Tr} u \in K(\lambda)^\perp = L^2_\lambda(\Gamma)$. Hence $\operatorname{Tr} u \in \mathcal{D}(\mathcal{D}_\lambda)$ and $\mathcal{D}_\lambda \operatorname{Tr} u = \mu \operatorname{Tr} u$. Thus S is well-defined.

Next, S is injective, since if $u \in \ker(\Delta_\mu - \lambda)$ is such that $\operatorname{Tr} u = 0$, then $u = 0$ by Proposition 1. To show surjectivity, let $\varphi \in \ker(\mathcal{D}_\lambda - \mu)$. Then there exists

$u \in H^1(\Omega)$ such that $\Delta u = \lambda u$, $\partial_\nu u = \mu \operatorname{Tr} u$, $\varphi = \operatorname{Tr} u$. Thus $u \in \mathcal{D}(\Delta_\mu)$ and $\Delta_\mu u = \lambda u$. □

We now describe how the Robin eigenvalues $\lambda_k(\mu)$ vary with μ.

Proposition 3. *For each k, the function $\lambda_k(\mu)$ is strictly decreasing and satisfies*

$$\lim_{\mu\to-\infty} \lambda_k(\mu) = \lambda_k^D, \quad \lim_{\mu\to\infty} \lambda_k(\mu) = -\infty.$$

Proof. It follows from the definition of b_μ and the max-min definition of eigenvalues that λ_k is at least nonincreasing. To see that it decreases strictly, suppose that $\lambda_k(\mu_1) = \lambda_k(\mu_2)$ for some $\mu_1 < \mu_2$. Then setting $\lambda := \lambda_k(\mu_1)$, it follows from Theorem 3.1 that $\mu \in \sigma(\mathcal{D}_\lambda)$ for all $\mu \in [\mu_1, \mu_2]$. But this is impossible since $\sigma(\mathcal{D}_\lambda)$ is discrete.

Standard eigenvalue perturbation theory shows that each λ_k is continuous in μ, and is even analytic if one follows the eigenvalue branches correctly across their crossings, see [16]). We refer to [3, Theorem 2.4] for the proof that $\lim_{\mu\to-\infty} \lambda_k(\mu) = \lambda_k^D$. On the other hand, if there exist $k \in \mathbb{N}$ and $\lambda \in \mathbb{R}$ such that $\lambda_k(\mu) > \lambda > -\infty$ for all $\mu \in \mathbb{R}$, then by Theorem 3.1, for that value of λ, $\sigma(\mathcal{D}_\lambda) \subset \{\mu \in \mathbb{R}, \lambda_j(\mu) = \lambda, j = 1, \dots k-1\}$, which is a finite set. This is impossible. □

We are now almost in a position to reprove the Friedlander eigenvalue inequalities.

Lemma 3.2. *If $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain, then $\sigma(\mathcal{D}_\lambda) \cap (-\infty, 0) \neq \emptyset$ for any $\lambda > 0$.*

Proof. Fix $\lambda > 0$ and define $W := \{\omega \in \mathbb{R}^d : |\omega|^2 = \lambda\}$. For $\omega \in W$, set $u_\omega(x) = e^{i\omega x}$. Then $u_\omega \in H^1(\Omega)$, $\Delta u_\omega = \lambda u_\omega$ and $(\partial_\nu u_\omega)(x) = i\langle \omega, \nu(x)\rangle e^{i\omega x}$ on Γ. Thus $(u_{w_{|\Gamma}}, \partial_\nu u_w) \in \mathcal{C}_\lambda$. It follows from the divergence theorem that

$$\int_\Gamma g_\omega \overline{\partial_\nu u_\omega} = -i \int_\Gamma \langle \omega, \nu(z)\rangle = 0 \ .$$

Now it follows from Lemma 2.1 that $g_w := u_{w_{|\Gamma}} \in \mathcal{D}(\mathcal{D}_\lambda)$ and

$$\langle \mathcal{D}_\lambda g_w, g_w \rangle = 0$$

for all $w \in W$. Suppose that $\mathcal{D}_\lambda$ is a nonnegative operator. Then for any $h \in \mathcal{D}(\mathcal{D}_\lambda)$,

$$\langle \mathcal{D}_\lambda g_\omega, h\rangle \le \langle \mathcal{D}_\lambda g_\omega, g_\omega\rangle^{\frac{1}{2}} \langle \mathcal{D}_\lambda h, h\rangle^{\frac{1}{2}} = 0,$$

which implies that $\mathcal{D}_\lambda g_\omega = 0$ for every $\omega \in W$. This is a contradiction since it would mean that $\ker \mathcal{D}_\lambda$ is infinite-dimensional. □

This same set of test functions was used in [9] and later in [8] for the same purpose.

Using this Lemma and Theorem 3.1, we now obtain the strict Friedlander inequalities.

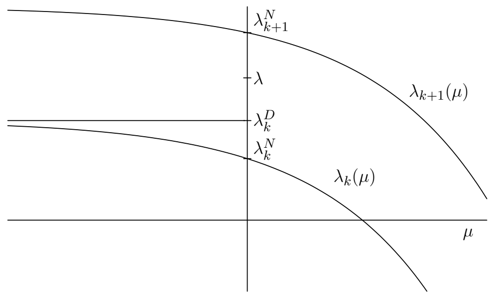
Theorem 3.3. *If $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain, then*

$$\lambda_{k+1}^N < \lambda_k^D \quad \textit{for all} \quad k \in \mathbb{N} \ .$$

Proof. Suppose that $\lambda_{k+1}^N \ge \lambda_k^D$ for some k. Choose any $\lambda \in [\lambda_k^D, \lambda_{k+1}^N]$; then for any $\mu < 0$,

$$\begin{aligned} j \le k \quad &\Rightarrow \quad \lambda_j(\mu) \le \lambda_k(\mu) < \lambda_k^D \le \lambda, \\ j \ge k+1 \quad &\Rightarrow \quad \lambda_j(\mu) \ge \lambda_{k+1}(\mu) > \lambda_{k+1}(0) = \lambda_{k+1}^N \ge \lambda. \end{aligned}$$

Hence $\lambda_j(\mu) \neq \lambda$ for any $j \in \mathbb{N}$. It follows from Theorem 3.1 that $\mu \notin \sigma(\mathcal{D}_\lambda)$, which contradicts Lemma 3.2. □



A quantitative version of this inequality which appears in [9] for $\lambda \notin \sigma(\Delta^D)$ can be proved by similar considerations.

Proposition 4. *For any $\lambda \in \mathbb{R}$, define*

$$\begin{aligned} N^N(\lambda) &= \operatorname{card}\{k \in \mathbb{N} : \lambda_k^N \leq \lambda\}, \\ N^D(\lambda) &= \operatorname{card}\{k \in \mathbb{N} : \lambda_k^D \leq \lambda\}. \end{aligned}$$

Then $N^N(\lambda) - N^D(\lambda)$ is the number of all eigenvalues of $\mathcal{D}_\lambda$ which are ≤ 0.

Proof. Let $\lambda \in \mathbb{R}$. Then

$$\left\{k \in \mathbb{N} : \lambda_k^N \leq \lambda < \lambda_k^D\right\} = \left\{k \in \mathbb{N} : \exists \mu \leq 0 \text{ such that } \lambda_k(\mu) = \lambda\right\}.$$

Since $N^N(\lambda) - N^D(\lambda) = \operatorname{card}\left\{k \in \mathbb{N} : \lambda_k^N \leq \lambda < \lambda_k^D\right\}$ the claim follows from Theorem 3.1. □

We now consider the functions $\mu_k : (-\infty, \lambda_k^D) \to \mathbb{R}$ which are the inverses of the $\lambda_k(\mu)$, $k = 1, 2, \ldots$. These are well-defined by the strict monotonicity of the λ_k, of course, and each μ_k is continuous (see [3]), strictly decreasing and satisfies

$$\lim_{\lambda \to -\infty} \mu_k(\lambda) = \infty, \qquad \lim_{\lambda \to \lambda_k^D} \mu_k(\lambda) = -\infty.$$

Theorem 3.1 now gives the following description of the spectrum of $\mathcal{D}_\lambda$, where we use the convention $\lambda_0^D = -\infty$.

Proposition 5. *For any $\lambda \in \mathbb{R}$, choose $n \in \mathbb{N}$ such that $\lambda_{n-1}^D \leq \lambda < \lambda_n^D$. Then*

$$\sigma(\mathcal{D}_\lambda) = \{\mu_j(\lambda) : j \geq n\} .$$

We conclude this section with a broader class of counterexamples of (1.1) when Ω is no longer Euclidean than those presented in [19]. We first quote an old result by Rauch and Taylor which is a special case of Theorem 2.3 in [23]:

Lemma 3.4. *Let (M^d, g) be a compact Riemannian manifold, $d \geq 2$. Let $K_1 \supset K_2 \supset \ldots \supset \{a\}$ be a decreasing sequence of closed subsets with Lipschitz boundary which decrease to a point $a \in M$. Set $\Omega_n = M \setminus K_n$, and denote by Δ_M the Laplace operator on all of M. Then for all $k \in \mathbb{N}$,*

$$\lim_{n\to\infty} \lambda_k^D(\Omega_n) = \lim_{n\to\infty} \lambda_k^N(\Omega_n) = \lambda_k(M)$$

Remark 2. The precise criterion in [23] is that the capacities of the sets K_n tend to 0.

Proposition 6. *Choose any $k \in \mathbb{N}$ such that $\lambda_k(M) < \lambda_{k+1}(M)$. Then for n sufficiently large, $\lambda_k^D(\Omega_n) < \lambda_{k+1}^N(\Omega_n)$.*

Proof. This is immediate from

$$\lim_{n\to\infty} \lambda_{k+1}^N(\Omega_n) = \lambda_{k+1}(M) > \lambda_k(M) = \lim_{n\to\infty} \lambda_k^D(\Omega_n).$$

□

On the other hand, a straightforward perturbation result using the variational characterization of the eigenvalues also proves the following.

Proposition 7. *Let (M^d, g) be any compact Riemannian manifold and let $k \in \mathbb{N}$. Then for any $\lambda > 0$ there exists an r_0 which depends on λ and g such that*

$$\lambda_{k+1}^N(\Omega) < \lambda_k^D(\Omega)$$

for any Lipschitz domain Ω in M which is contained in a geodesic ball $B_{r_0}(p)$, and for all k such that $\lambda_k^D(\Omega) \leq \lambda$.

Note that from this sort of perturbation argument, it is impossible to discern whether these inequalities hold for all k independent of the size of Ω.

4. **Positivity.** We now turn to a study of the semigroup generated by $-D_\lambda$ on $L^2(\Gamma)$. Some of the facts established here appear also in the paper [7].

A $\mathcal{C}_0$-semigroup $T = (T(t))_{t\geq 0}$ on a space L^p is called **positive** if $T(t)f \geq 0$ for all $t \geq 0$ whenever $0 \leq f \in L^p$.

Theorem 4.1. *If $\lambda < \lambda_1^D$, then the semigroup generated by $-D_\lambda$ on $L^2(\Gamma)$ is positive.*

Proof. If w is any function, then we recall the standard notation $w^+ = \max\{w, 0\}$ and $w^- = -\min\{w, 0\}$. If $u \in H^1(\Omega)$, then both $u^\pm \in H^1(\Omega)$ as well. Let $\varphi = \operatorname{Tr} u$, which is an element of the space V consisting of all boundary traces of elements of $H^1(\Omega)$. Then the terms $\varphi^\pm$ in its decomposition are precisely the boundary traces $\operatorname{Tr} u^\pm$, and in particular both $\varphi^\pm \in V$ as well.

By the Beurling-Deny criterion (see [6] or [22, Theorem 2.6]), the semigroup $T(t)$ is positive if and only if $a_\lambda(\varphi^+, \varphi^-) \leq 0$ for all $\varphi \in V$. Now suppose that $u \in H^1(\lambda)$, and write $u^\pm = u_0^\pm + u_1^\pm \in H_0^1 \oplus H^1(\lambda)$. We use this short notation here even though u_0^- is not the positive part u_0 (and similarly for u_0^-, u_1^+, u_1^-). Since $u = (u_0^+ - u_0^-) + (u_1^+ - u_1^-) \in H^1(\lambda)$, i.e. u has no component in $H_0^1(\Omega)$, it follows that $u_0^+ = u_0^-$. We then compute

$$\begin{aligned} a_\lambda(\varphi^+, \varphi^-) &= \int_\Omega \left(\nabla u_1^+ \cdot \nabla u_1^- - \lambda u_1^+ u_1^-\right) \\ &= \int_\Omega \nabla(u_1^+ + u_0)\nabla(u_1^- + u_0) \\ &\quad - \int_\Omega \left(\nabla u_1^+ \nabla u_0 + \nabla u_0 \nabla u_1^- + |\nabla u_0|^2\right) \\ &\quad -\lambda \int_\Omega \left((u_1^+ + u_0)(u_1^- + u_0) - u_1^+ u_0 - u_0 u_1^- - (u_0)^2\right) \\ &= \int_\Omega \nabla u^+ \nabla u^- - \lambda \int_\Omega u^+ u^- - \int_\Omega |\nabla u_0|^2 + \lambda \int_\Omega (u_0)^2 \leq 0, \end{aligned}$$

by the Poincaré inequality. In the last identity we used that fact that

$$\int_\Omega \nabla u_1^\pm \nabla u_0 = \lambda \int_\Omega u_1^\pm u_0$$

since $u_1^\pm \in H^1(\lambda)$. □

Let (Y, Σ, ν) be a measure space endowed with a positive $\mathcal{C}_0$-semigroup T acting on $L^p(Y)$ for some $1 \le p < \infty$. A subspace $\mathcal{J} \subset L^p(Y)$ is called a **closed ideal** if and only if there exists $S \in \Sigma$ such that $\mathcal{J} = \{f \in L^p(Y) : f = 0$ a.e. on $Y \setminus S\} =: L^p(S)$. A closed ideal $\mathcal{J}$ is said to be **invariant** (with respect to T) if $T(t)\mathcal{J} \subset \mathcal{J}$ for all $t > 0$. The semigroup T is called **irreducible** if the only invariant closed ideals are $\mathcal{J} = \{0\}$ and $\mathcal{J} = L^p(Y)$. For any $f \in L^p(Y)$ we write $f > 0$ if $f(y) \ge 0$ a.e. and $\nu(\{y \in Y : f(y) > 0\}) > 0$, while $f \gg 0$ if $f(y) > 0$ a.e. If T is holomorphic then irreducibility implies that $T(t)f \gg 0$ for all $t > 0$ and $f > 0$ (see [20, C-III.Theorem 3.2.(b)]).

Irreducible semigroups have interesting spectral properties. Assume that T is positive and irreducible (and hence that its generator $-B$ has compact resolvent). Denote by $\lambda_1(B)$ the first eigenvalue of B. Then the eigenspace $\ker(\lambda_1(B) - B)$ has dimension 1. Moreover, there exists a strictly positive eigenvector u; i.e. $u \in D(B), Bu = \lambda_1(B)u$ and $u \gg 0$. This actually characterizes the first eigenvalue: whenever $\lambda \in \mathbb{R}$ is an eigenvalue with positive eigenvector, then $\lambda = \lambda_1(B)$. This set of results is frequently referred to as the Krein-Rutman Theorem, see [20] for more information. The following comparison result will be used below. Let $\tilde{T}$ be another $\mathcal{C}_0$-semigroup on $L^p(Y)$ whose generator $\tilde{B}$ has compact resolvent. If

$$T(t)f \le \tilde{T}(t)f$$

for all $t \ge 0$ and $f \ge 0$ then $\lambda_1(B) \le \lambda_1(\tilde{B})$. Moreover,

$$\lambda_1(B) = \lambda_1(\tilde{B}) \text{ if and only if } B = \tilde{B} \tag{4.1}$$

(see [1, Theorem 1.3]).

An example of this is the semigroup generated by the Robin Laplacian, or slightly more generally, the Laplacian Δ_β with boundary conditions $\partial_\nu u = \beta u_{|_\Gamma}$ for some fixed $\beta \in L^\infty(\Gamma)$. To make this precise, let Ω be a compact manifold with Lipschitz boundary Γ, as before, and fix $\beta \in L^\infty(\Gamma)$. Then define the form

$$a_\beta(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dV_g - \int_\Gamma \beta u v \, d\sigma_g \tag{4.2}$$

with domain $H^1(\Omega)$. The associated self-adjoint operator is Δ_β, and has domain

$$\mathcal{D}(\Delta_\beta) = \{u \in H^1(\Omega) : \Delta u \in L^2(\Omega),\ \partial_\nu u \in L^2(\Gamma),\ \partial_\nu u = \beta u\}.$$

Moreover, $-\Delta_\beta$ generates a positive irreducible $\mathcal{C}_0$-semigroup T_β on $L^2(\Omega)$ and if $\tilde{\beta} \le \beta$ then

$$0 \le T_\beta(t) \le T_{\tilde{\beta}}(t).$$

We refer to [5] for this and further information. The Krein-Rutman Theorem shows that if $\tilde{\beta} \le \beta$, then

$$\lambda_1(\Delta_\beta) = \lambda_1(\Delta_{\tilde{\beta}}) \text{ if and only if } \Delta_\beta = \Delta_{\tilde{\beta}}\ . \tag{4.3}$$

Let us return to our primary goal, which is to prove the

Theorem 4.2. *Suppose that Ω is connected, and let $\lambda < \lambda_1^D$. Then the semigroup T generated by $-\mathcal{D}_\lambda$ on $L^2(\Gamma)$ is irreducible.*

Remark 3. It is somewhat surprising that this holds only assuming that Ω, but not necessarily Γ, is connected. There is an elegant criterion by Ouhabaz [22] which shows that the semigroup generated by Δ_β is irreducible, and we shall use this last result to deduce the irreducibity of T. It does not seem to be easy to prove the irreducibility more directly using the usual criteria such as the one in [22].

Proof. Let $\Gamma_1 \subset \Gamma$ be a Borel set and assume that the closed ideal $L^2(\Gamma_1) := \{b \in L^2(\Gamma) : b_{|\Gamma_2} = 0 \text{ a.e.}\}$ is invariant under T, where we have set $\Gamma_2 = \Gamma \setminus \Gamma_1$. Then $T_1(t) := T(t)_{|_{L^2(\Gamma_1)}}$ is a positive $\mathcal{C}_0$-semigroup on $L^2(\Gamma_1)$ and $T_1(t)$ is compact for all $t > 0$. Consequently its generator $-A_1$ has compact resolvent. Let μ be the first eigenvalue of A_1. By the Krein-Rutman Theorem there exists $0 < b \in L^2(\Gamma_1)$ such that $T_1(t)b = e^{-\mu t}b$ $(t > 0)$. Consequently, $T(t)b = e^{-\mu t}b$ for all $t > 0$, and hence $b \in \mathcal{D}(\mathcal{D}_\lambda)$ and $\mathcal{D}_\lambda b = \mu b$. By the definition of $\mathcal{D}_\lambda$ there exists $u \in H^1(\Omega)$ such that $\operatorname{Tr} u = b$ and $\partial_\nu u = \mu b$. We show that $u \geq 0$. In fact,

$$\int_\Omega (\nabla u \cdot \nabla v - \lambda uv) = \mu \int_\Gamma uv \tag{4.4}$$

for all $v \in H^1(\Omega)$. Since $\operatorname{Tr} u \geq 0$, one has $u^- \in H^1_0(\Omega)$. Thus inserting $v = u^-$ into this equation gives

$$-\int_\Omega |\nabla u^-|^2 + \lambda \int_\Omega |u^-|^2 = 0 \ .$$

Combined with the Poincaré inequality, $\int_\Omega |\nabla u^-|^2 \geq \lambda_1^D \int_\Omega |u^-|^2$, we obtain $\lambda \int_\Omega |u^-|^2 \geq \lambda_1^D \int_\Omega |u^-|^2$. Since $\lambda < \lambda_1^D$ we deduce that $\int_\Omega |u^-|^2 = 0$, and hence $u^- = 0$, i.e. $u \geq 0$. It follows that $u \in \mathcal{D}(\Delta_\mu)$ and $\Delta_\mu u = \lambda u$. Since $u \geq 0$ but $u \not\equiv 0$, the Krein-Rutman Theorem implies that $\lambda = \lambda_1(\mu)$ is the first eigenvalue of Δ_μ. Now define $\beta \in L^\infty(\Gamma)$ by

$$\beta(z) = \begin{cases} \mu & z \in \Gamma_1 \\ \mu_1 & z \in \Gamma_2 \ , \end{cases}$$

where $\mu_1 \neq 0$ is chosen so that $\mu < \mu_1$. Since $\operatorname{Tr} u = 0$ on Γ_2, it follows from (4.4) that

$$a_\beta(u, v) = \int_\Omega \nabla u \nabla v - \int_\Gamma \beta uv = \int_\Omega \nabla u \nabla v - \mu \int_\Gamma uv = \lambda \int_\Omega uv$$

for all $v \in H^1(\Omega)$. This implies that $u \in \mathcal{D}(\Delta_\beta)$ and $\Delta_\beta u = \lambda u$. Since $u \geq 0$ (and u is nontrivial), applying the Krein-Rutman Theorem once again gives that $\lambda = \lambda_1(\Delta_\beta)$. We have shown that $\lambda_1(\Delta_\beta) = \lambda_1(\Delta_\mu)$. Since $\mu \leq \beta$, the semigroup T_β generated by $-\Delta_\beta$ satisfies $0 \leq T_\beta(t) \leq T_\mu(t)$. Now it follows from (4.3) that $\Delta_\mu = \Delta_\beta$. This implies that $a_\beta = a_\mu$. In particular

$$\int_{\Gamma_1} \mu u^2 + \int_{\Gamma_2} (\mu_1) u^2 = \int_\Gamma \mu u^2$$

for all $u \in H^1(\Omega)$. Hence $\int_{\Gamma_2} u^2 \, d\sigma = 0$ for all $u \in H^1(\Omega)$. In particular $\int_\Gamma u^2 1_{\Gamma_2} \, d\sigma = 0$ for all $u \in \mathcal{D}(\mathbb{R}^d)$. Since $\{\operatorname{Tr} u : u \in \mathcal{C}_0^\infty(\mathbb{R}^d)\}$ is dense in $\mathcal{C}(\Gamma)$, we see that $\int_\Gamma \varphi^2 1_{\Gamma_2} \, d\sigma = 0$ for all $\varphi \in \mathcal{C}(\Gamma)$. This implies that the Borel measure $1_{\Gamma_2} \, d\sigma$ is 0. Hence $\sigma(\Gamma_2) = 0$. □

Acknowledgement. The first author is most grateful to the Department of Mathematics of Stanford University for its hospitality and inspiring atmosphere during the time this work was carried out. The second author is grateful to Jean-Claude Nédélec for pointing out the paper [11]; he was supported by the NSF grant DMS-0805529.

REFERENCES

[1] W. Arendt and C. Batty, *Domination and ergodicity for positive semigroups,* Proc. Amer. Math. Soc., **114** (1992), 743–747.

[2] W. Arendt and C. Batty, *Exponential stability of diffusion equations with absorption,* Diff. Int. Equ., **6** (1993), 1009–1024.

[3] W. Arendt and R. Mazzeo, *Spectral properties of the Dirichlet-to-Neumann operator on Lipschitz domains*, Ulmer Seminare, **Heft 12** (2007), 28–38.

[4] W. Arendt and T. ter Elst, *The Dirichlet to Neumann operator on rough domains*, J. Differential Equations, **251** (2011), 2100–2124. arXiv:1005.0875

[5] W. Arendt and M. Warma, *Dirichlet and Neumann boundary conditions: What is between?* J. Evolution Equ., **3** (2003), 119–136.

[6] E. B. Davies, "Heat Kernels and Spectral Theory," Cambridge University Press 1990.

[7] H. Emamirad and I. Laadnani, *An approximating family for the Dirichlet-to-Neumann semigroup*, Adv. Diff. Equ., **11** (2006), 241–257.

[8] N. Filinov, *On an inequality between Dirichlet and Neumann eigenvalues for the Laplace operator*, St. Petersburg Math. J., **16** (2005), 413–416.

[9] L. Friedlander, *Some inequalities between Dirichlet and Neumann eigenvalues,* Arch. Rational Mech. Anal., **116** (1991), 153–160.

[10] F. Gesztesy and M. Mitrea, *Nonlocal Robin Laplacians and some remarks on a paper by Filonov on eigenvalue inequalities*, J. Diff. Eq., **247** (2009), 2871–2896.

[11] J. P. Grégoire, J. C. Nédélec and J. Planchard, *A method of finding the eigenvalues and eigenfunctions of self-adjoint elliptic operators*, Comp. Methods in Appl. Mech. and Eng., **8** (1976), 201–214.

[12] E. Hebey, "Sobolev Spaces on Riemannian Manifolds," Springer, **LN 1635** (1993).

[13] E. Hebey, "Nonlinear Analysis on Manifolds Sobolev Spaces and Inequalities," Courant Lecture Notes in Mathematics 5, New York, 1999.

[14] A. Henrot, "Extremum Problems for Eigenvalues of Elliptic Operators," Birkhäuser, Basel, 2006.

[15] A. Henrot and M. Pierre, "Variation et Optimisation de Formes," Springer, Berlin, 2005.

[16] T. Kato, "Perturbation Theory," Springer, Berlin, 1966.

[17] D. Mitrea, M. Mitrea and M. Taylor, "Layer Potentials, the Hodge Laplacian, and Global Boundary Problems in Nonsmooth Riemannian Manifolds," Memoires of Amer. Math. Soc., 713, Vol. **150** (2001).

[18] M. Mitrea and M. Taylor, *Boundary layer methods for Lipschitz domains in Riemannian manifolds*, Journal of Functional Anal., **163** (1999), 181–251.

[19] R. Mazzeo, *Remarks on a paper of L. Friedlander concerning inequalities between Neumann and Dirichlet eigenvalues,* Internat. Math. Res. Notices, **4** (1991), 41–48.

[20] R. Nagel ed., "One-parameter Semigroups of Positive Operators," Springer LN, **1184** (1986).

[21] J. Necaš, "Les Méthodes Directes en Théorie des Equations Elliptiques," Masson, Paris, 1967.

[22] E. M. Ouhabaz, "Analysis of the Heat Equation on Domains," London Mathematical Society Monographs **31**, Princeton University Press, 2005.

[23] J. Rauch and M. Taylor, *Potential and scattering theory on wildly perturbed domains*, J. Funct. Anal., **18** (1975), 27–59.

[24] Y. Safarov, *On the comparison of the Dirichlet and Neumann counting functions*, In "Spectral Theory of Differential Operators: M.Sh. Birman 80th Anniversary Collection" (T. Suslina, D. Yafaev eds.), Amer. Math. Soc. Transl. Ser. 2, vol. 225, Providence, RI (2008), 191–204.

Received February 2011.

E-mail address: wolfgang.arendt@uni-ulm.de
E-mail address: mazzeo@math.stanford.edu