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From Forms to Semigroups

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Abstract. We present a review and some new results on form methods for generating holomorphic semigroups on Hilbert spaces. In particular, we explain how the notion of closability can be avoided. As examples we include the Stokes operator, the Black–Scholes equation, degenerate differential equations and the Dirichlet-to-Neumann operator.

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Introduction

Form methods give a very efficient tool to solve evolutionary problems on Hilbert space. They were developed by T. Kato [Kat] and, in a slightly different language by J.L. Lions. In this article we give an introduction based on [AE2]. The main point in our approach is that the notion of closability is not needed anymore. In the language of Kato the form merely needs to be sectorial. Alternatively, in the setting of Lions the Hilbert space V, on which the form is defined, no longer needs to be continuously embedded in the Hilbert space H, on which the semigroup acts. Instead one merely needs a continuous linear map from V into H with dense range.

The new setting is particularly efficient for degenerate equations, since then the sectoriality condition is obvious, whilst the form is not closable, in general, or closability might be hard to verify. The Dirichlet-to-Neumann operator is normally defined on smooth domains, that is, domains with at least a Lipschitz boundary. The new form method allows us to consider the Dirichlet-to-Neumann operator on rough domains. Besides this we give several other examples.

This presentation starts by an introduction to holomorphic semigroups. Instead of the contour argument found in the literature, we give a more direct argument based on the Hille–Yosida Theorem.

1. The Hille–Yosida Theorem

A C_0-*semigroup* on a Banach space X is a mapping $T\colon (0,\infty) \to \mathcal{L}(X)$ satisfying

$$\begin{aligned} &T(t+s) = T(t)T(s) && (t,s>0) \\ &\lim_{t\downarrow 0} T(t)x = x && (x \in X)\ . \end{aligned}$$

The *generator* A of such a C_0-semigroup is defined by

$$D(A) := \{x \in X : \lim_{t\downarrow 0} \frac{T(t)x - x}{t} \text{ exists}\}$$

$$Ax := \lim_{t\downarrow 0} \frac{T(t)x - x}{t} \qquad (x \in D(A))\ .$$

Thus the domain $D(A)$ of A is a subspace of X and $A\colon D(A) \to X$ is linear. One can show that $D(A)$ is dense in X. The main interest in semigroups lies in the associated Cauchy problem

$$\text{(CP)} \quad \begin{cases} \dot u(t) = Au(t) & (t>0) \\ u(0) = x\ . \end{cases}$$

Indeed, if A is the generator of a C_0-semigroup, then given $x \in X$, the function $u(t) := T(t)x$ is the unique *mild* solution of (CP); i.e.,

$$u \in C([0,\infty);X)\ , \ \int_0^t u(s)\,ds \in D(A)$$

for all $t > 0$ and

$$\begin{aligned} u(t) &= x + A\int_0^t u(s)\,ds \\ u(0) &= x\ . \end{aligned}$$

If $x \in D(A)$, then u is a *classical solution*; i.e., $u \in C^1([0,\infty);X)$, $u(t) \in D(A)$ for all $t \geq 0$ and $\dot u(t) = Au(t)$ for all $t > 0$. Conversely, if for each $x \in X$ there exists a unique mild solution of (CP), then A generates a C_0-semigroup [ABHN, Theorem 3.1.12]. In view of this characterization of well-posedness, it is of big interest to decide whether a given operator generates a C_0-semigroup. A positive answer is given by the famous Hille–Yosida Theorem.

Theorem 1.1 (Hille–Yosida (1948)). *Let A be an operator on X. The following are equivalent.*

(i) *A generates a contractive C_0-semigroup;*
(ii) *the domain of A is dense, $\lambda - A$ is invertible for all $\lambda > 0$ and $\|\lambda(\lambda - A)^{-1}\| \leq 1$.*

Here we call a semigroup T *contractive* if $\|T(t)\| \leq 1$ for all $t > 0$. By $\lambda - A$ we mean the operator with domain $D(A)$ given by $(\lambda - A)x := \lambda x - Ax \quad (x \in D(A))$. So the condition in (ii) means that $\lambda - A \colon D(A) \to X$ is bijective and $\|\lambda(\lambda - A)^{-1}x\| \leq \|x\|$ for all $\lambda > 0$ and $x \in X$. If X is reflexive, then this existence of the *resolvent* $(\lambda - A)^{-1}$ and the contractivity $\|\lambda(\lambda - A)^{-1}\| \leq 1$ imply already that the domain is dense [ABHN, Theorem 3.3.8].

Yosida's proof is based on the Yosida-approximation: Assuming (ii), one easily sees that

$$\lim_{\lambda\to\infty} \lambda(\lambda - A)^{-1}x = x \qquad (x \in D(A)) ,$$

i.e., $\lambda(\lambda - A)^{-1}$ converges strongly to the identity as $\lambda \to \infty$. This implies that

$$A_\lambda := \lambda A(\lambda - A)^{-1} = \lambda^2(\lambda - A)^{-1} - \lambda$$

approximates A as $\lambda \to \infty$ in the sense that

$$\lim_{\lambda\to\infty} A_\lambda x = Ax \qquad (x \in D(A)) .$$

The operator A_λ is bounded, so one may define

$$e^{tA_\lambda} := \sum_{n=0}^{\infty} \frac{t^n}{n!} A_\lambda^n$$

by the power series. Note that $\|\lambda^2(\lambda - A)^{-1}\| \leq \lambda$. Since

$$e^{tA_\lambda} = e^{-\lambda t} e^{t\lambda^2(\lambda - A)^{-1}} ,$$

it follows that

$$\|e^{tA_\lambda}\| \leq e^{-\lambda t} e^{t\|\lambda^2(\lambda - A)^{-1}\|} \leq 1 .$$

The key element in Yosida's proof consists in showing that for all $x \in X$ the family $(e^{tA_\lambda}x)_{\lambda>0}$ is a Cauchy net as $\lambda \to \infty$. Then the C_0-semigroup generated by A is given by

$$T(t)x := \lim_{\lambda\to\infty} e^{tA_\lambda}x \qquad (t > 0)$$

for all $x \in X$. We will come back to this formula when we talk about holomorphic semigroups.

Remark 1.2. Hille's independent proof is based on Euler's formula for the exponential function. Note that putting $t = \frac{1}{\lambda}$ one has

$$\lambda(\lambda - A)^{-1} = (I - tA)^{-1} .$$

Hille showed that

$$T(t)x := \lim_{n\to\infty} \left(I - \frac{t}{n}A\right)^{-n} x$$

exists for all $x \in X$, see [Kat, Section IX.1.2].

2. Holomorphic semigroups

A C_0-semigroup is defined on the real half-line $(0, \infty)$ with values in $\mathcal{L}(X)$. It is useful to study when extensions to a sector

$$\Sigma_\theta := \{re^{i\alpha} : r > 0,\ |\alpha| < \theta\}$$

for some $\theta \in (0, \pi/2]$ exist. In this section X is a complex Banach space.

Definition 2.1. A C_0-semigroup T is called *holomorphic* if there exist $\theta \in (0, \pi/2]$ and a holomorphic extension

$$\widetilde{T} \colon \Sigma_\theta \to \mathcal{L}(X)$$

of T which is locally bounded; i.e.,

$$\sup_{\substack{z \in \Sigma_\theta \\ |z| \le 1}} \|\widetilde{T}(z)\| < \infty \ .$$

If $\|\widetilde{T}(z)\| \le 1$ for all $z \in \Sigma_\theta$, then we call T a *sectorially contractive holomorphic* C_0-semigroup (*of angle* θ, if we want to make precise the angle).

The holomorphic extension $\widetilde{T}$ automatically has the semigroup property

$$\widetilde{T}(z_1 + z_2) = \widetilde{T}(z_1)\widetilde{T}(z_2) \qquad (z_1, z_2 \in \Sigma_\theta) \ .$$

Because of the boundedness assumption it follows that

$$\lim_{\substack{z \to 0 \\ z \in \Sigma_\theta}} \widetilde{T}(z)x = x \qquad (x \in X) \ .$$

These properties are easy to see. Moreover, $\widetilde{T}$ can be extended continuously (for the strong operator topology) to the closure of Σ_θ, keeping these two properties. In fact, if $x = T(t)y$ for some $t > 0$ and some $y \in X$, then

$$\lim_{w \to z} T(w)x = \lim_{w \to z} T(w + t)y = T(z + t)y$$

exists. Since the set $\{T(t)y : t \in (0, \infty),\ y \in X\}$ is dense the claim follows. In the sequel we will omit the tilde and denote the extension $\widetilde{T}$ simply by T. We should add a remark on vector-valued holomorphic functions.

Remark 2.2. If Y is a Banach space and $\Omega \subset \mathbb{C}$ open, then a function $f \colon \Omega \to Y$ is called *holomorphic* if

$$f'(z) = \lim_{h \to 0} \frac{f(z + h) - f(z)}{h}$$

exists in the norm of Y for all $z \in \Omega$ and $f' \colon \Omega \to Y$ is continuous. It follows as in the scalar case that f is analytic. It is remarkable that holomorphy is the same as weak holomorphy (first observed by Grothendieck): A function $f \colon \Omega \to Y$ is holomorphic if and only if

$$y' \circ f \colon \Omega \to \mathbb{C}$$

is holomorphic for all $y' \in Y'$. In our context the space Y is $\mathcal{L}(X)$, the space of all bounded linear operators on X with the operator norm. If the function f

is bounded it suffices to test holomorphy with fewer functionals. We say that a subspace $W \subset Y'$ *separates points* if for all $x \in Y$,

$$\langle y', x \rangle = 0 \text{ for all } y' \in W \text{ implies } x = 0 \ .$$

Assume that $f\colon \Omega \to Y$ is bounded such that $y' \circ f$ is holomorphic for all $y' \in W$ where W is a separating subspace of Y'. Then f is holomorphic. This result is due to [AN], see also [ABHN, Theorem A7]. In particular, if $Y = \mathcal{L}(X)$, then a bounded function $f\colon \Omega \to \mathcal{L}(X)$ is holomorphic if and only if $\langle x', f(\cdot)x \rangle$ is holomorphic for all x in a dense subspace of X and all x' in a separating subspace of X'.

We recall a special form of Vitali's Theorem (see [AN], [ABHN, Theorem A5]).

Theorem 2.3 (Vitali). *Suppose $\Omega \subset \mathbb{C}$ is connected. For all $n \in \mathbb{N}$ let $f_n\colon \Omega \to \mathcal{L}(X)$ be holomorphic, let $M \in \mathbb{R}$ and suppose that*

a) $\|f_n(z)\| \leq M \qquad$ *for all $z \in \Omega$ and $n \in \mathbb{N}$, and;*

b) $\Omega_0 := \{z \in \Omega : \lim_{n\to\infty} f_n(z)x$ *exists for all* $x \in X\}$ *has a limit point in Ω, i.e., there exist a sequence $(z_k)_{k\in\mathbb{N}}$ in Ω_0 and $z_0 \in \Omega$ such that $z_k \neq z_0$ for all $k \in \mathbb{N}$ and $\lim\limits_{k\to\infty} z_k = z_0$.*

Then

$$f(z)x := \lim_{n\to\infty} f_n(z)x$$

exists for all $x \in X$ and $z \in \Omega$, and $f\colon \Omega \to \mathcal{L}(X)$ is holomorphic.

Now we want to give a simple characterization of holomorphic sectorially contractive semigroups. Assume that A is a densely defined operator on X such that $(\lambda - A)^{-1}$ exists and

$$\|\lambda(\lambda - A)^{-1}\| \leq 1 \qquad (\lambda \in \Sigma_\theta) \ ,$$

where $0 < \theta \leq \pi/2$. Let $z \in \Sigma_\theta$. Then for all $\lambda > 0$,

$$(zA)_\lambda = zA_{\frac{\lambda}{z}}$$

is holomorphic in z. For each $z \in \Sigma_\theta$, the operator zA satisfies Condition (ii) of Theorem 1.1. By the Hille–Yosida Theorem

$$T(z)x := \lim_{\lambda\to\infty} e^{(zA)_\lambda} x$$

exists for all $x \in X$ and $z \in \Sigma_\theta$. Since $z \mapsto e^{(zA)_\lambda} = e^{zA_{\lambda/z}}$ is holomorphic, $T\colon \Sigma_\theta \to \mathcal{L}(X)$ is holomorphic by Vitali's Theorem. If $t > 0$, then

$$T(t) = \lim_{\lambda\to\infty} e^{tA_{\lambda/t}} = T_A(t)$$

where T_A is the semigroup generated by A. Since $T_A(t+s) = T_A(t)T_A(s)$, it follows from analytic continuation that

$$T(z_1 + z_2) = T(z_1)T(z_2) \qquad (z_1, z_2 \in \Sigma_\theta) \ .$$

Thus A generates a sectorially contractive holomorphic C_0-semigroup of angle θ on X. One sees as above that

$$T_{zA}(t) = T(zt)$$

for all $t > 0$ and $z \in \Sigma_\theta$. We have shown the following.

Theorem 2.4. *Let A be a densely defined operator on X and $\theta \in (0, \pi/2]$. The following are equivalent.*

(i) *A generates a sectorially contractive holomorphic C_0-semigroup of angle θ;*
(ii) *$(\lambda - A)^{-1}$ exists for all $\lambda \in \Sigma_\theta$ and*

$$\|\lambda(\lambda - A)^{-1}\| \leq 1 \qquad (\lambda \in \Sigma_\theta) \ .$$

We refer to [AEH] for a similar approach to possibly noncontractive holomorphic semigroups.

3. The Lumer–Phillips Theorem

Let H be a Hilbert space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. An operator A on H is called *accretive* or *monotone* if

$$\operatorname{Re}(Ax|x) \geq 0 \qquad (x \in D(A)) \ .$$

Based on this notion the following very convenient characterization is an easy consequence of the Hille–Yosida Theorem.

Theorem 3.1 (Lumer–Phillips). *Let A be an operator on H. The following are equivalent.*

(i) *$-A$ generates a contraction semigroup;*
(ii) *A is accretive and $I + A$ is surjective.*

For a proof, see [ABHN, Theorem 3.4.5]. Accretivity of A can be reformulated by the condition

$$\|(\lambda + A)x\| \geq \|\lambda x\| \qquad (\lambda > 0, \ x \in D(A)) \ .$$

Thus if $\lambda + A$ is surjective, then $\lambda + A$ is invertible and $\|\lambda(\lambda + A)^{-1}\| \leq 1$. We also say that A is *m-accretive* if Condition (ii) is satisfied. If A is m-accretive and $\mathbb{K} = \mathbb{C}$, then one can easily see that $\lambda + A$ is invertible for all $\lambda \in \mathbb{C}$ satisfying $\operatorname{Re} \lambda > 0$ and

$$\|(\lambda + A)^{-1}\| \leq \frac{1}{\operatorname{Re} \lambda} \ .$$

Due to the reflexivity of Hilbert spaces, each m-accretive operator A is densely defined (see [ABHN, Proposition 3.3.8]). Now we want to reformulate the Lumer–Phillips Theorem for generators of semigroups which are contractive on a sector.

Theorem 3.2 (Generators of sectorially contractive semigroups). *Let A be an operator on a complex Hilbert space H and let $\theta \in (0, \frac{\pi}{2})$. The following are equivalent.*

(i) *$-A$ generates a holomorphic C_0-semigroup which is contractive on the sector Σ_θ;*
(ii) *$e^{\pm i\theta} A$ is accretive and $I + A$ is surjective.*

Proof. (ii)⇒(i). Since $e^{\pm i\theta}A$ is accretive the operator zA is accretive for all $z \in \Sigma_\theta$. Since $(I+A)$ is surjective, the operator A is m-accretive. Thus $(\lambda+A)$ is invertible whenever $\operatorname{Re}\lambda > 0$. Consequently $(I+zA) = z(z^{-1}+A)$ is invertible for all $z \in \Sigma_\theta$. Thus zA is m-accretive for all $z \in \Sigma_\theta$. Now (i) follows from Theorem 2.4.

(i)⇒(ii). If $-A$ generates a holomorphic semigroup which is contractive on Σ_θ, then $e^{i\alpha}A$ generates a contraction semigroup for all α with $|\alpha| \le \theta$. Hence $e^{i\alpha}A$ is m-accretive whenever $|\alpha| \le \theta$. □

If A is self-adjoint, then both conditions of Theorem 3.2 are valid for all $\theta \in (0, \frac{\pi}{2})$ and the semigroup is holomorphic on $\Sigma_{\frac{\pi}{2}}$.

4. Forms: the complete case

We recall one of our most efficient tools to solve equations, the Lax–Milgram lemma, which is just a non-symmetric generalization of the Riesz–Fréchet representation theorem from 1905.

Lemma 4.1 (Lax–Milgram (1954)). *Let V be a Hilbert space over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, and let $a\colon V \times V \to \mathbb{K}$ be sesquilinear, continuous and coercive, i.e.,*

$$\operatorname{Re} a(u) \ge \alpha \|u\|_V^2 \qquad (u \in V)$$

for some $\alpha > 0$. Let $\varphi\colon V \to \mathbb{K}$ be a continuous anti-linear form, i.e., φ is continuous and satisfies $\varphi(u+v) = \varphi(u) + \varphi(v)$ and $\varphi(\lambda u) = \overline{\lambda}\varphi(u)$ for all $u, v \in V$ and $\lambda \in \mathbb{K}$. Then there is a unique $u \in V$ such that

$$a(u,v) = \varphi(v) \qquad (v \in V)\ .$$

Of course, to say that a is continuous means that

$$|a(u,v)| \le M\|u\|_V\|v\|_V \qquad (u,v \in V)$$

for some constant M. We let $a(u) := a(u,u)$ for all $u \in V$.

In general, the range condition in the Hille–Yosida Theorem is difficult to prove. However, if we look at operators associated with a form, the Lax–Milgram Lemma implies automatically the range condition. We describe now our general setting in the complete case. Given is a Hilbert space V over $\mathbb{K}$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, and a continuous, coercive sesquilinear form

$$a\colon V \times V \to \mathbb{K}\ .$$

Moreover, we assume that H is another Hilbert space over $\mathbb{K}$ and $j\colon V \to H$ is a continuous linear mapping with dense image. Now we associate an operator A on H with the pair (a, j) in the following way. Given $x, y \in H$ we say that $x \in D(A)$ and $Ax = y$ if there exists a $u \in V$ such that $j(u) = x$ and

$$a(u,v) = (y|j(v))_H \qquad \text{for all } v \in V\ .$$

We first show that A is well defined. Assume that there exist $u_1, u_2 \in V$ and $y_1, y_2 \in H$ such that

$$j(u_1) = j(u_2) \ ,$$

$$a(u_1, v) = (y_1 | j(v))_H \qquad (v \in V), \text{ and,}$$

$$a(u_2, v) = (y_2 | j(v))_H \qquad (v \in V) \ .$$

Then $a(u_1 - u_2, v) = (y_1 - y_2 | j(v))_H$ for all $v \in V$. Since $j(u_1 - u_2) = 0$, taking $v := u_1 - u_2$ gives $a(u_1 - u_2, u_1 - u_2) = 0$. Since a is coercive, it follows that $u_1 = u_2$. It follows that $(y_1 | j(v))_H = (y_2 | j(v))_H$ for all $v \in V$. Since j has dense image, it follows that $y_1 = y_2$.

It is clear from the definition that $A\colon D(A) \to H$ is linear. Our main result is the following generation theorem. We first assume that $\mathbb{K} = \mathbb{C}$.

Theorem 4.2 (Generation theorem in the complete case). *The operator $-A$ generates a sectorially contractive holomorphic C_0-semigroup T. If a is symmetric, then A is selfadjoint.*

Proof. Let $M \geq 0$ be the constant of continuity and $\alpha > 0$ the constant of coerciveness as before. Then

$$\frac{|\operatorname{Im} a(v)|}{\operatorname{Re} a(v)} \leq \frac{M \|v\|_V^2}{\alpha \|v\|_V^2} = \frac{M}{\alpha}$$

for all $v \in V \setminus \{0\}$. Thus there exists a $\theta' \in (0, \frac{\pi}{2})$ such that

$$a(v) \in \overline{\Sigma_{\theta'}} \qquad (v \in V) \ .$$

Let $x \in D(A)$. There exists a $u \in V$ such that $x = j(u)$ and $a(u, v) = (Ax | j(v))_H$ for all $v \in V$. In particular, $(Ax | x)_H = a(u) \in \overline{\Sigma_{\theta'}}$. It follows that $e^{\pm i\theta} A$ is accretive where $\theta = \frac{\pi}{2} - \theta'$. In order to prove the range condition, consider the form $b\colon V \times V \to \mathbb{C}$ given by

$$b(u, v) = a(u, v) + (j(u) | j(v))_H \ .$$

Then b is continuous and coercive. Let $y \in H$. Then $\varphi(v) := (y | j(v))_H$ defines a continuous anti-linear form φ on V. By the Lax–Milgram Lemma 4.1 there exists a unique $u \in V$ such that

$$b(u, v) = \varphi(v) \qquad (v \in V) \ .$$

Hence $(y | j(v))_H = a(u, v) + (j(u) | j(v))_H$; i.e., $a(u, v) = (y - j(u) | j(v))_H$ for all $v \in V$. This means that $x := j(u) \in D(A)$ and $Ax = y - x$. □

The result is also valid in real Banach spaces. If T is a C_0-semigroup on a real Banach space X, then the $\mathbb{C}$-linear extension $T_{\mathbb{C}}$ of T on the complexification $X_{\mathbb{C}} := X \oplus iX$ of X is a C_0-semigroup given by $T_{\mathbb{C}}(t)(x + iy) := T(t)x + iT(t)y$. We call T *holomorphic* if $T_{\mathbb{C}}$ is holomorphic. The generation theorem above remains true on real Hilbert spaces.

In order to formulate a final result we want also allow a rescaling. Let X be a Banach space over $\mathbb{K}$ and T be a C_0-semigroup on X with generator A. Then for all $\omega \in \mathbb{K}$ and $t > 0$ define

$$T_\omega(t) := e^{\omega t} T(t) .$$

Then T_ω is a C_0-semigroup whose generator is $A + \omega$. Using this we obtain now the following general generation theorem in the complete case.

Let V, H be Hilbert spaces over $\mathbb{K}$ and $j\colon V \to H$ continuous linear with dense image. Let $a\colon V \times V \to \mathbb{K}$ be sesquilinear and continuous. We call the form a *j-elliptic* if there exist $\omega \in \mathbb{R}$ and $\alpha > 0$ such that

$$\operatorname{Re} a(u) + \omega \|j(u)\|_H^2 \geq \alpha \|u\|_V^2 \qquad (u \in V) . \tag{4.1}$$

Then we define the operator A associated with (a, j) as follows. Given $x, y \in H$ we say that $x \in D(A)$ and $Ax = y$ if there exists a $u \in V$ such that $j(u) = x$ and

$$a(u, v) = (y|j(v))_H \qquad \text{for all } v \in V .$$

Theorem 4.3. *The operator defined in this way is well defined. Moreover, $-A$ generates a holomorphic C_0-semigroup on H.*

Remark 4.4. The form a satisfies Condition (4.1) if and only if the form a_ω given by

$$a_\omega(u, v) = a(u, v) + \omega(j(u)|j(v))_H$$

is coercive. If T_ω denotes the semigroup associated with (a_ω, j) and T the semigroup associated with (a, j), then

$$T_\omega(t) = e^{-\omega t} T(t) \qquad (t > 0)$$

as is easy to see.

5. The Stokes operator

In this section we show as an example that the Stokes operator is selfadjoint and generates a holomorphic C_0-semigroup. The following approach is due to Monniaux [Mon]. Let $\Omega \subset \mathbb{R}^d$ be a bounded open set. We first discuss the Dirichlet Laplacian.

Theorem 5.1 (Dirichlet Laplacian). *Let $H = L^2(\Omega)$ and define the operator Δ^D on $L^2(\Omega)$ by*

$$D(\Delta^D) = \{u \in H_0^1(\Omega) : \Delta u \in L^2(\Omega)\}$$

$$\Delta^D u := \Delta u .$$

Then Δ^D is selfadjoint and generates a holomorphic C_0-semigroup on $L^2(\Omega)$.

Proof. Define $a\colon H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ by $a(u, v) = \int_\Omega \nabla u \nabla v$. Then a is clearly continuous. Poincaré's inequality says that a is coercive. Consider the injection j of $H_0^1(\Omega)$ into $L^2(\Omega)$. Let A be the operator associated with (a, j). We show that $A = -\Delta^D$. In fact, let $u \in D(A)$ and write $f = Au$. Then $\int_\Omega \nabla u \nabla v = \int_\Omega f v$ for all

$v \in H_0^1(\Omega)$. Taking in particular $v \in C_c^\infty(\Omega)$ we see that $-\Delta u = f$. Conversely, let $u \in H_0^1(\Omega)$ be such that $f := -\Delta u \in L^2(\Omega)$. Then $\int_\Omega f\varphi = \int_\Omega \nabla u \nabla \varphi = a(u,\varphi)$ for all $\varphi \in C_c^\infty(\Omega)$. This is just the definition of the weak partial derivatives in $H^1(\Omega)$. Since $C_c^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, it follows that $\int_\Omega fv = a(u,v)$ for all $v \in H_0^1(\Omega)$. Thus $u \in D(A)$ and $Au = f$. Now the theorem follows from Theorem 4.2. □

For our treatment of the Stokes operator it will be useful to consider the Dirichlet Laplacian also in $L^2(\Omega)^d = L^2(\Omega) \oplus \cdots \oplus L^2(\Omega)$.

Theorem 5.2. *Define the symmetric form* $a\colon H_0^1(\Omega)^d \times H_0^1(\Omega)^d \to \mathbb{R}$ *by*

$$a(u,v) = \int_\Omega \nabla u \nabla v := \sum_{j=1}^d \int_\Omega \nabla u_j \nabla v_j \ ,$$

where $u = (u_1, \ldots, u_d)$. *Moreover, let* $j\colon H_0^1(\Omega)^d \to L^2(\Omega)^d$ *be the inclusion. Then* a *is continuous and coercive. The operator* A *associated with* (a,j) *on* $L^2(\Omega)^d$ *is given by*

$$D(A) = \{u \in H_0^1(\Omega)^d : \Delta u_j \in L^2(\Omega) \text{ for all } j \in \{1,\ldots,d\}\} \ ,$$

$$Au = (-\Delta u_1, \ldots, -\Delta u_d) =: -\Delta u \ .$$

We call $\Delta^D := -A$ *the Dirichlet Laplacian on* $L^2(\Omega)^d$.

In order to define the Stokes operator we need some preparation. Let $\mathcal{D}(\Omega) := C_c^\infty(\Omega)^d$ and let $\mathcal{D}_0(\Omega) := \{\varphi \in \mathcal{D}(\Omega) : \operatorname{div}\varphi = 0\}$, where $\operatorname{div}\varphi = \partial_1\varphi_1 + \cdots + \partial_d\varphi_d$ and $\varphi = (\varphi_1, \ldots, \varphi_d)$. By $\mathcal{D}(\Omega)'$ we denote the dual space of $\mathcal{D}(\Omega)$ (with the usual topology). Each element S of $\mathcal{D}(\Omega)'$ can be written in a unique way as $S = (S_1, \ldots, S_d)$ with $S_j \in C_c^\infty(\Omega)'$ so that

$$\langle S, \varphi\rangle = \sum_{j=1}^d \langle S_j, \varphi_j\rangle$$

for all $\varphi = (\varphi_1, \ldots, \varphi_d) \in \mathcal{D}(\Omega)$.

We say that $S \in H^{-1}(\Omega)$ if there exists a constant $c \geq 0$ such that

$$|\langle S, \varphi\rangle| \leq c\,\Big(\int |\nabla\varphi|^2\Big)^{\frac12} \qquad (\varphi \in \mathcal{D}(\Omega))$$

where $|\nabla\varphi|^2 = |\nabla\varphi_1|^2 + \cdots + |\nabla\varphi_d|^2$. For the remainder of this section we assume that Ω has Lipschitz boundary. We need the following result (see [Tem, Remark 1.9, p. 14]).

Theorem 5.3. *Let* $T \in H^{-1}(\Omega)$. *The following are equivalent.*

(i) $\langle T, \varphi\rangle = 0$ *for all* $\varphi \in \mathcal{D}_0(\Omega)$*;*
(ii) *there exists a* $p \in L^2(\Omega)$ *such that* $T = \nabla p$.

Note that Condition (ii) means that

$$\langle T,\varphi\rangle=\sum_{j=1}^{d}\langle\partial_j p,\varphi_j\rangle=-\sum_{j=1}^{d}\langle p,\partial_j\varphi_j\rangle=-\langle p,\operatorname{div}\varphi\rangle\ .$$

Now the implication (ii)⇒(i) is obvious. We omit the other implication.

Consider the real Hilbert space $L^2(\Omega)^d$ with scalar product

$$(f|g)=\sum_{j=1}^{d}(f_j|g_j)_{L^2(\Omega)}=\sum_{j=1}^{d}\int_\Omega f_jg_j\ .$$

We denote by

$$H:=\mathcal{D}_0(\Omega)^{\perp\perp}=\overline{\mathcal{D}_0(\Omega)}$$

the closure of $\mathcal{D}_0(\Omega)$ in $L^2(\Omega)^d$. We call H the space of all *divergence free vectors* in $L^2(\Omega)^d$. The orthogonal projection P from $L^2(\Omega)^d$ onto H is called the *Helmholtz projection*. Now let V be the closure of $\mathcal{D}_0(\Omega)$ in $H^1(\Omega)^d$. Thus $V\subset H_0^1(\Omega)^d$ and $\operatorname{div}u=0$ for all $u\in V$. One can actually show that

$$V=\{u\in H_0^1(\Omega)^d:\operatorname{div}v=0\}\ .$$

We define the form $a\colon V\times V\to\mathbb{R}$ by

$$a(u,v)=\sum_{j=1}^{d}(\nabla u_j|\nabla v_j)_{L^2(\Omega)}\quad(u=(u_1,\dots,u_d),v=(v_1,\dots,v_d)\in V)\ .$$

Then a is continuous and coercive. The space V is dense in H since it contains $\mathcal{D}_0(\Omega)$. We consider the inclusion $j\colon V\to H$. Let A be the operator associated with (a,j). Then A is selfadjoint and $-A$ generates a holomorphic C_0-semigroup. The operator can be described as follows.

Theorem 5.4. *The operator A has the domain*

$$D(A)=\{u\in V:\exists\,\pi\in L^2(\Omega)\ \textit{such that}\ -\Delta u+\nabla\pi\in H\}$$

and is given by

$$Au=-\Delta u+\nabla\pi\ ,$$

where $\pi\in L^2(\Omega)$ is such that $-\Delta u+\nabla\pi\in H$.

If $u\in H_0^1(\Omega)^d$, then $\Delta u\in H^{-1}(\Omega)$. In fact, for all $\varphi\in\mathcal{D}(\Omega)$,

$$|\langle-\Delta u,\varphi\rangle|=|-\langle u,\Delta\varphi\rangle|=\left|\sum_{j=1}^{d}\int_\Omega\nabla u_j\nabla\varphi_j\right|\le\|u\|_{H_0^1(\Omega)^d}\|\varphi\|_{H_0^1(\Omega)^d}\ .$$

Proof of Theorem 5.4. Let $u\in D(A)$ and write $f=Au$. Then $f\in H$, $u\in V$ and $a(u,v)=(f|v)_H$ for all $v\in V$. Thus, the distribution $-\Delta u\in H^{-1}(\Omega)$ coincides with f on $\mathcal{D}_0(\Omega)$. By Theorem 5.3 there exists a $\pi\in L^2(\Omega)$ such that $-\Delta u+\nabla\pi=$

f. Conversely, let $u \in V$, $f \in H$, $\pi \in L^2(\Omega)$ and suppose that $-\Delta u + \nabla \pi = f$ in $\mathcal{D}(\Omega)'$. Then for all $\varphi \in \mathcal{D}_0(\Omega)$,

$$a(u,\varphi) = \int_\Omega \nabla u \nabla \varphi = \int_\Omega \nabla u \nabla \varphi + \langle \nabla \pi, \varphi \rangle = (f|\varphi)_{L^2(\Omega)^d} \ .$$

Since $\mathcal{D}_0(\Omega)$ is dense in V, it follows that $a(u,\varphi) = (f|\varphi)_{L^2(\Omega)^d}$ for all $\varphi \in V$. Thus, $u \in D(A)$ and $Au = f$. □

The operator A is called the *Stokes operator*. We refer to [Mon] for this approach and further results on the Navier–Stokes equation. We conclude this section by giving an example where j is not injective. Further examples will be seen in the sequel.

Proposition 5.5. *Let $\widetilde{H}$ be a Hilbert space and $H \subset \widetilde{H}$ a closed subspace. Denote by P the orthogonal projection onto H. Let $\widetilde{V}$ be a Hilbert space which is continuously and densely embedded into $\widetilde{H}$ and let $a\colon \widetilde{V} \times \widetilde{V} \to \mathbb{R}$ be a continuous, coercive form. Denote by A the operator on $\widetilde{H}$ associated with (a, j) where j is the injection of $\widetilde{V}$ into $\widetilde{H}$ and let B be the operator on H associated with $(a, P \circ j)$. Then*

$$D(B) = \{Pw : w \in D(A) \text{ and } Aw \in H\} \ ,$$

$$BPw = Aw \qquad (w \in D(A),\ Aw \in H) \ .$$

This is easy to see. In the context considered in this section we obtain the following example.

Example 5.6. Let $\widetilde{H} = L^2(\Omega)^d$, $H = \overline{\mathcal{D}_0(\Omega)}$ and $\widetilde{V} := H^1_0(\Omega)^d$. Define $a\colon \widetilde{V} \times \widetilde{V} \to \mathbb{R}$ by

$$a(u,v) = \int_\Omega \nabla u \nabla v \ .$$

Moreover, define $j\colon \widetilde{V} \to \widetilde{H}$ by $j(u) = u$. Then the operator associated with (a, j) is $A = -\Delta^D$ as we have seen in Theorem 5.2. Now let P be the Helmholtz projection and B the operator associated with $(a, P \circ j)$. Then

$$D(B) = \{u \in H : \exists\, \pi \in L^2(\Omega) \text{ such that } u + \nabla \pi \in D(\Delta^D) \text{ and } \Delta(u + \nabla\pi) \in H\}$$

and

$$Bu = -\Delta(u + \nabla \pi) \ ,$$

if $\pi \in L^2(\Omega)$ is such that $u + \nabla\pi \in D(\Delta^D)$ and $\Delta(u + \nabla\pi) \in H$. This follows directly from Proposition 5.5 and Theorem 5.3. The operator B is selfadjoint and generates a holomorphic semigroup.

6. From forms to semigroups: the incomplete case

In the preceding sections we considered forms which were defined on a Hilbert space V. Now we want to study a purely algebraic condition considering forms whose domains are arbitrary vector spaces. At first we consider the complex case. Let H be a complex Hilbert space. A *sectorial form* on H is a sesquilinear form

$$a\colon D(a) \times D(a) \to \mathbb{C} \ ,$$

where $D(a)$ is a vector space, together with a linear mapping $j\colon D(a) \to H$ with dense image such that there exist $\omega \geq 0$ and $\theta \in (0, \pi/2)$ such that

$$a(u) + \omega \|j(u)\|_H^2 \in \overline{\Sigma_\theta} \qquad (u \in D(a)) \ .$$

If $\omega = 0$, then we call the form 0-*sectorial*. To a sectorial form, we associate an operator A on H by defining for all $x, y \in H$ that $x \in D(A)$ and $Ax = y :\Leftrightarrow$ there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $D(a)$ such that

a) $\lim_{n\to\infty} j(u_n) = x$ in H; b) $\sup_{n\in\mathbb{N}} \operatorname{Re} a(u_n) < \infty$; and

c) $\lim_{n\to\infty} a(u_n, v) = (y|j(v))_H$ for all $v \in D(a)$.

It is part of the next theorem that the operator A is well defined (i.e., that y depends only on x and not on the choice of the sequence satisfying a), b) and c)). We only consider single-valued operators in this article.

Theorem 6.1. *The operator A associated with a sectorial form (a, j) is well defined and $-A$ generates a holomorphic C_0-semigroup on H.*

The proof of the theorem consists in a reduction to the complete case by considering an appropriate completion of $D(a)$. Here it is important that in Theorem 4.2 a non-injective mapping j is allowed. For a proof we refer to [AE2, Theorem 3.2].

If $C \subset H$ is a closed convex set, we say that C is *invariant* under a semigroup T if

$$T(t)C \subset C \qquad (t > 0) \ .$$

Invariant sets are important to study positivity, L^∞-contractivity, and many more properties. If the semigroup is associated with a form, then the following criterion, [AE2, Proposition 3.9], is convenient.

Theorem 6.2 (Invariance). *Let $C \subset H$ be a closed convex set and let P be the orthogonal projection onto C. Then the semigroup T associated with a sectorial form (a, j) on H leaves C invariant if and only if for each $u \in D(a)$ there exists a sequence $(w_n)_{n\in\mathbb{N}}$ in $D(a)$ such that*

a) $\lim_{n\to\infty} j(w_n) = Pj(u)$ *in H;*

b) $\limsup_{n\to\infty} \operatorname{Re} a(w_n, u - w_n) \geq 0$; *and*

c) $\sup_{n\in\mathbb{N}} \operatorname{Re} a(w_n) < \infty$.

Corollary 6.3. *Let $C \subset H$ be a closed convex set and let P be the orthogonal projection onto C. Assume that for each $u \in D(a)$, there exists a $w \in D(a)$ such that*

$$j(w) = Pj(u) \qquad \text{and} \qquad \operatorname{Re} a(w, u - w) \geq 0 \; .$$

Then $T(t)C \subset C$ for all $t > 0$.

In this section we want to use the invariance criterion to prove a generation theorem in the incomplete case which is valid in real Hilbert spaces. Let H be a real Hilbert space. A *sectorial* form on H is a bilinear mapping

$$a \colon D(a) \times D(a) \to \mathbb{R} \; ,$$

where $D(a)$ is a real vector space, together with a linear mapping $j \colon D(a) \to H$ with dense image such that there are $\alpha, \omega \geq 0$ such that

$$|a(u,v) - a(v,u)| \leq \alpha(a(u) + a(v)) + \omega(\|j(u)\|_H^2 + \|j(v)\|_H^2) \qquad (u, v \in D(a)) \; .$$

It is easy to see that the form a is sectorial on the real space H if and only if the sesquilinear extension $a_{\mathbb{C}}$ of a to the complexification of $D(a)$ together with the $\mathbb{C}$-linear extension of j is sectorial in the sense formulated in the beginning of this section.

To such a sectorial form (a, j) we associate an operator A on H by defining for all $x, y \in H$ that $x \in D(A)$ and $Ax = y$:⇔ there exists a sequence (u_n) in $D(a)$ satisfying

a) $\lim_{n\to\infty} j(u_n) = x$ in H;
b) $\sup_{n\in\mathbb{N}} a(u_n) < \infty$; and
c) $\lim_{n\to\infty} a(u_n, v) = (y|j(v))_H$ for all $v \in D(a)$.

Then the following holds.

Theorem 6.4. *The operator A is well defined and $-A$ generates a holomorphic C_0-semigroup on H.*

Proof. Consider the complexifications $H_{\mathbb{C}} = H \oplus iH$ and $D(a_{\mathbb{C}}) := D(a) + iD(a)$. Let

$$a_{\mathbb{C}}(u,v) := a(\operatorname{Re} u, \operatorname{Re} v) + a(\operatorname{Im} u, \operatorname{Im} v) + i(a(\operatorname{Re} u, \operatorname{Im} v) + a(\operatorname{Im} u, \operatorname{Re} v))$$

for all $u = \operatorname{Re} u + i \operatorname{Im} u, v = \operatorname{Re} v + i \operatorname{Im} v \in D(a_{\mathbb{C}})$. Then $a_{\mathbb{C}}$ is a sesquilinear form. Let $J \colon D(a_{\mathbb{C}}) \to H_{\mathbb{C}}$ be the $\mathbb{C}$-linear extension of j. Let

$$b(u,v) = a_{\mathbb{C}}(u,v) + \omega(J(u)|J(v))_{H_{\mathbb{C}}} \qquad (u, v \in D(a_{\mathbb{C}})) \; .$$

Then

$$\operatorname{Im} b(u) = a(\operatorname{Im} u, \operatorname{Re} u) - a(\operatorname{Re} u, \operatorname{Im} u),$$
$$\operatorname{Re} b(u) = a(\operatorname{Re} u) + a(\operatorname{Im} u) + \omega(\|j(\operatorname{Re} u)\|_H^2 + \|j(\operatorname{Im} u)\|_H^2) \; .$$

The assumptions imply that there is a $c > 0$ such that $|\operatorname{Im} b(u)| \le c \operatorname{Re} b(u)$ for all $u \in D(a_{\mathbb{C}})$. Consequently, $b(u) \in \overline{\Sigma_\theta}$, where $\theta = \arctan c$. Thus the operator B associated with b generates a C_0-semigroup $S_{\mathbb{C}}$ on $H_{\mathbb{C}}$. It follows from Corollary 6.3 that H is invariant. The part A_ω of B in H is the generator of S, where $S(t) := S_{\mathbb{C}}(t)_{|H}$. It is easy to see that $A_\omega - \omega = A$. □

Remark 6.5. It is remarkable, and important for some applications, that Condition b) in Theorem 6.1 as well as in Theorem 6.4 may be replaced by

b′) $$\lim_{n,m\to\infty} a(u_n - u_m) = 0 \ .$$

For later purposes we carry over the invariance criterion Corollary 6.3 to the real case.

Corollary 6.6. *Let H be a real Hilbert space and (a, j) a sectorial form on H with associated semigroup T. Let $C \subset H$ be a closed convex set and P the orthogonal projection onto C. Assume that for each $u \in D(a)$ there exists a $w \in D(a)$ such that*

$$j(w) = Pj(u) \qquad \text{and} \qquad a(w, u - w) \ge 0 \ .$$

Then $T(t)C \subset C$ for all $t > 0$.

We want to formulate a special case of invariance. An operator S on a space $L^p(\Omega)$ is called

$$\textit{positive} \text{ if } \Big(f \ge 0 \text{ a.e. implies } Sf \ge 0 \text{ a.e.}\Big) \text{ and}$$
$$\textit{submarkovian} \text{ if } \Big(f \le \mathbb{1} \text{ a.e. implies } Sf \le \mathbb{1} \text{ a.e.}\Big).$$

Thus, an operator S is submarkovian if and only if it is positive and $\|Sf\|_\infty \le \|f\|_\infty$ for all $f \in L^p \cap L^\infty$. A semigroup T is called *submarkovian* if $T(t)$ is submarkovian for all $t > 0$.

Proposition 6.7. *Consider the real space $H = L^2(\Omega)$ and a sectorial form a on H. Assume that for each $u \in D(a)$ one has $u \wedge \mathbb{1} \in D(a)$ and*

$$a(u \wedge \mathbb{1}, (u - \mathbb{1})^+) \ge 0 \ .$$

Then the semigroup T associated with a is submarkovian.

Recall that $u \wedge v := \min(u, v)$ and $v^+ = \max(v, 0)$.

Proof. The set $C := \{u \in L^2(\Omega) : u \le \mathbb{1} \text{ a.e.}\}$ is closed and convex. The orthogonal projection P onto C is given by $Pu = u \wedge \mathbb{1}$. Thus $u - Pu = (u - \mathbb{1})^+$ and the result follows from Corollary 6.6. □

We conclude this section with a remark concerning closable forms.

Remark 6.8 (Forget closability). In many text books, for example [Dav], [Kat], [MR], [Ouh], [Tan] one finds the notion of a sectorial form a on a complex Hilbert space H. By this one understands a sesquilinear form $a\colon D(a) \times D(a) \to \mathbb{C}$ where

$D(a)$ is a dense subspace of H such that there are $\theta \in (0, \pi/2)$ and $\omega \geq 0$ such that $a(u) + \omega\|u\|_H^2 \in \overline{\Sigma_\theta}$ for all $u \in D(a)$. Then

$$\|u\|_a := (\operatorname{Re} a(u) + (\omega + 1)\|u\|_H^2)^{1/2}$$

defines a norm on $D(a)$. The form is called *closed* if $D(a)$ is complete for this norm. This corresponds to our complete case with $V = D(a)$ and j the inclusion. If the form is not closed, then one may consider the completion V of $D(a)$. Since the injection $D(a) \to H$ is continuous for the norm $\| \; \|_a$, it has a continuous extension $j\colon V \to H$. This extension may be injective or not. The form is called *closable* if j is injective. In the literature only for closable forms generation theorems are given, see [AE2] for precise references. The results above show that the notion of closability is not needed.

In this special setting it is easy to give the proof of Theorem 6.1. There exists a unique continuous sesquilinear form $\tilde{a}\colon V \times V \to \mathbb{C}$ such that $\tilde{a}(u, v) = a(u, v)$ for all $u, v \in D(a)$. Since the form a is sectorial, it follows that $\tilde{a}$ is j-elliptic (see (4.1)). Let $\widetilde{A}$ be the operator associated with $(\tilde{a}, j)$ from Theorem 4.3. Let $x, y \in H$ and suppose that $x \in D(A)$ and $Ax = y$. By assumption there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $D(a)$ such that $\lim u_n = x$ in H, $\sup \operatorname{Re} a(u_n) < \infty$ and $\lim a(u_n, v) = (y|v)_H$ for all $v \in D(a)$. Then $(u_n)_{n\in\mathbb{N}}$ is bounded in V, so passing to a subsequence if necessary, it is weakly convergent, say to $u \in V$. Then $\tilde{a}(u, v) = \lim \tilde{a}(u_n, v) = (y|v)_H$ for all $v \in D(a)$. Hence by density, $\tilde{a}(u, v) = (y|j(v))_H$ for all $v \in V$. So $x = j(u) \in D(\widetilde{A})$ and $\widetilde{A}x = y$. Therefore A is well defined and $\widetilde{A}$ is an extension of A. It is easy to show that also A is an extension of $\widetilde{A}$. So $A = \widetilde{A}$ and $-A$ generates a holomorphic semigroup.

It is clear that one needs to consider approximating sequences in the definition of the operator A in the incomplete case. Just consider the trivial form $a = 0$ with $D(a)$ a proper dense subspace of H. Then the associated operator is the zero operator.

There is a unique correspondence between sectorially quasi contractive holomorphic semigroups and closed sectorial forms (see [Kat, Theorem VI.2.7]). One looses uniqueness if one considers forms which are merely closable or in our general setting if one allows arbitrary maps $j\colon D(a) \to H$ with dense image. However, examples show that in many cases a natural operator is obtained by this general framework.

7. Degenerate diffusion

In this section we use our tools to show that degenerate elliptic operators generate holomorphic semigroups on the real space $L^2(\Omega)$. We start with a 1-dimensional example.

Example 7.1 (Degenerate diffusion in dimension 1). Consider the real Hilbert space $H = L^2(a, b)$, where $-\infty \leq a < b \leq \infty$, and let $\alpha, \beta, \gamma \in L^\infty_{\text{loc}}(a, b)$ be real coefficients. We assume that there is a $c_1 \geq 0$ such that

$$\gamma^- := \max(-\gamma, 0) \in L^\infty(a,b) \text{ and } \beta^2(x) \leq c_1 \cdot \alpha(x) \qquad (x \in (a,b)) \ .$$

We define the bilinear form a on $L^2(a,b)$ by

$$a(u,v) = \int_a^b \Big(\alpha(x)u'(x)v'(x) + \beta(x)u'(x)v(x) + \gamma(x)u(x)v(x)\Big)\, dx$$

with domain

$$D(a) = H^1_c(a,b) = \{u \in H^1(a,b) : \operatorname{supp} u \text{ is compact in } (a,b)\} \ .$$

We choose $j\colon H^1_c(a,b) \to L^2(a,b)$ to be the inclusion map. We next show that the form a is *sectorial*, i.e., there exist constants $c, \omega \geq 0$, such that

$$|a(u,v) - a(v,u)| \leq c(a(u) + a(v)) + \omega(\|u\|^2_{L^2} + \|v\|^2_{L^2}) \qquad (u, v \in D(a)) \ . \tag{7.1}$$

For the proof of (7.1) we use Young's inequality

$$|xy| \leq \varepsilon x^2 + \frac{1}{4\varepsilon} y^2$$

twice. Let $u, v \in D(a)$. On one hand we have for all $\delta > 0$,

$$|a(u,v) - a(v,u)| = |\int_a^b \beta(u'v - uv')| \leq \int_a^b \delta\beta^2(u'^2 + v'^2) + \frac{1}{4\delta}(u^2 + v^2) \ .$$

On the other hand, for all $c, \omega, \varepsilon > 0$ one has

$$\begin{aligned} c(a(u) &+ a(v)) + \omega(\|u\|^2_H + \|v\|^2_H) \\ &= \int_a^b c\alpha(u'^2 + v'^2) + c\beta(u'u + v'v) + (c\gamma + \omega)(u^2 + v^2) \\ &\geq \int_a^b (c\alpha - \varepsilon\beta^2)(u'^2 + v'^2) - c^2\frac{1}{4\varepsilon}(u^2 + v^2) + (c\gamma + \omega)(u^2 + v^2) \\ &\geq \int_a^b (c\alpha - \varepsilon\beta^2)(u'^2 + v'^2) + (\omega - c\|\gamma^-\|_{L^\infty} - \frac{c^2}{4\varepsilon})(u^2 + v^2) \ . \end{aligned}$$

Therefore (7.1) is valid if $(c\alpha - \varepsilon\beta^2) \geq \delta\beta^2$ and $(\omega - c\|\gamma^-\|_{L^\infty} - \frac{c^2}{4\varepsilon}) \geq \frac{1}{4\delta}$. Since $\beta^2 \leq c_1\alpha$ one can find $\delta, \varepsilon, c, \omega$ such that the conditions are satisfied.

Thus a is sectorial. As a consequence, if A is the operator associated with (a, j), then it follows from Theorem 6.4 that $-A$ generates a holomorphic C_0-semigroup T on $L^2(\Omega)$. Moreover, T is submarkovian by Proposition 6.7.

The condition $\beta^2 \leq c_1 \alpha$ shows in particular that $\{x \in (a,b) : \alpha(x) = 0\} \subset \{x \in (a,b) : \beta(x) = 0\}$. This inclusion is a natural hypothesis, since in general an operator of the form $\beta u'$ does not generate a holomorphic semigroup.

A special case is the *Black–Scholes Equation*

$$u_t + \frac{\sigma^2}{2} x^2 u_{xx} + r x u_x - r u = 0 \ ,$$

with $\sigma \in \mathbb{R}$ and $r \in L^\infty(\mathbb{R})$, together with the condition that $r = 0$ if $\sigma = 0$. This one obtains by choosing $H = L^2(0,\infty)$,

$$a(u,v) = \int_0^\infty \Big(\frac{\sigma^2}{2} x^2 u' v' + (\sigma^2 - r) x u' v + r u v \Big)$$

and $D(a) = H^1_c(0,\infty)$.

It is not difficult to extend the example above to higher dimensions.

Example 7.2. Let $\Omega \subset \mathbb{R}^d$ be open and for all $i,j \in \{1,\ldots,d\}$ let $a_{ij}, b_j, c \in L^\infty_{\mathrm{loc}}(\Omega)$ be real coefficients. Assume $c^- \in L^\infty(\Omega)$, $a_{ij} = a_{ji}$ and there exists a $c_1 > 0$ such that

$$c_1 A(x) - B^2(x) \text{ is positive semidefinite}$$

for almost all $x \in \Omega$, where

$$A(x) = (a_{ij}(x)) \text{ and } B(x) = \mathrm{diag}(b_1(x), \ldots, b_d(x)) \ .$$

Define the form a on $L^2(\Omega)$ by

$$a(u,v) = \int_\Omega \Big(\sum_{i,j=1}^d a_{ij} (\partial_i u)(\partial_j v) + \sum_{j=1}^d b_j (\partial_j u) v + c u v \Big)$$

with domain

$$D(a) = H^1_c(\Omega) \ .$$

Then a is sectorial. The associated semigroup T on $L^2(\Omega)$ is submarkovian.

This and the previous example incorporate Dirichlet boundary conditions. In the next one we consider a degenerate elliptic operator with Neumann boundary conditions.

Example 7.3. Let $\Omega \subset \mathbb{R}^d$ be an open, possibly unbounded subset of $\mathbb{R}^d$. For all $i,j \in \{1,\ldots,d\}$ let $a_{ij} \in L^\infty(\Omega)$ be *real* coefficients and assume that there exists a $\theta \in (0,\pi/2)$ such that

$$\sum_{i,j=1}^d a_{ij}(x) \xi_i \overline{\xi_j} \in \overline{\Sigma_\theta} \qquad (\xi \in \mathbb{C}^d, \ x \in \Omega) \ .$$

Consider the form a on $L^2(\Omega)$ given by

$$a(u,v) = \int_\Omega \sum_{i,j=1}^d a_{ij} (\partial_i u)(\partial_j v)$$

with domain $D(a) = H^1(\Omega)$. Then a is sectorial. Let T be the associated semigroup. Our criteria show right away that T is submarkovian. Therefore T extends consistently to a semigroup T_p on $L^p(\Omega)$ for all $p \in [1, \infty]$, the semigroup T_p is strongly continuous for all $p < \infty$ and T_∞ is the adjoint of a strongly continuous semigroup on $L^1(\Omega)$. It is remarkable that even

$$T_\infty(t)\mathbb{1}_\Omega = \mathbb{1}_\Omega \qquad (t > 0) .$$

For bounded Ω this is easy to prove, but otherwise more sophisticated tools are needed (see [AE2, Corollary 4.9]).

We want to mention an abstract result which shows that our solutions are some kind of *viscosity solutions*. This is illustrated particularly well in the situation of Example 7.3.

Proposition 7.4 ([AE2, Corollary 3.9]). *Let V, H be real Hilbert spaces such that $V \underset{d}{\hookrightarrow} H$. Let $j \colon V \to H$ be the inclusion map. Let $a \colon V \times V \to \mathbb{R}$ be continuous and sectorial. Assume that $a(u) \geq 0$ for all $u \in V$. Let $b \colon V \times V \to \mathbb{R}$ be continuous and coercive. Then for each $n \in \mathbb{N}$ the form*

$$a + \frac{1}{n} b \colon V \times V \to \mathbb{R}$$

is continuous and coercive. Let A_n be the operator associated with $(a + \frac{1}{n} b, j)$ and A with (a, j). Then

$$\lim_{n\to\infty} (A_n + \lambda)^{-1} f = (A + \lambda)^{-1} f \text{ in } H$$

for all $f \in H$ and $\lambda > 0$. Moreover, denoting by T_n and T the semigroup generated by $-A_n$ and $-A$ one has

$$\lim_{n\to\infty} T_n(t) f = T(t) f \text{ in } H$$

for all $f \in H$.

The essence in the result is that the form a is merely sectorial and may be degenerate. For instance, in Example 7.3 $a_{ij}(x) = 0$ is allowed. If we perturb by the Laplacian, we obtain a coercive form

$$a_n \colon H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$$

given by

$$a_n(u, v) = a(u, v) + \frac{1}{n} \int_\Omega \nabla u \nabla v \ .$$

Then Proposition 7.4 says that in the situation of Example 7.3 for this perturbation one has $\lim_{n\to\infty} (A_n + \lambda)^{-1} f = (A + \lambda)^{-1} f$ in $L^2(\Omega)$ for all $f \in L^2(\Omega)$.

8. The Dirichlet-to-Neumann operator

The following example shows how the general setting involving non-injective j can be used. It is taken from [AE1] where also the interplay between trace properties and the semigroup generated by the Dirichlet-to-Neumann operator is studied. Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with boundary $\partial\Omega$. Our point is that we do not need any regularity assumption on Ω, except that we assume that $\partial\Omega$ has a finite $(d-1)$-dimensional Hausdorff measure. Still we are able to define the Dirichlet-to-Neumann operator on $L^2(\partial\Omega)$ and to show that it is selfadjoint and generates a submarkovian semigroup on $L^2(\Omega)$. Formally, the Dirichlet-to-Neumann operator D_0 is defined as follows. Given $\varphi \in L^2(\partial\Omega)$, one solves the Dirichlet problem

$$\begin{cases} \Delta u = 0 \text{ in } \Omega \\ u_{|\partial\Omega} = \varphi \end{cases}$$

and defines $D_0\varphi = \frac{\partial u}{\partial \nu}$. We will give a precise definition using weak derivatives. We consider the space $L^2(\partial\Omega) := L^2(\partial\Omega, \mathcal{H}^{d-1})$ with the $(d-1)$-dimensional Hausdorff measure $\mathcal{H}^{d-1}$. Integrals over $\partial\Omega$ are always taken with respect to $\mathcal{H}^{d-1}$, those over Ω always with respect to the Lebesgue measure. Throughout this section we only assume that $\mathcal{H}^{d-1}(\partial\Omega) < \infty$ and that Ω is bounded.

Definition 8.1 (Normal derivative). Let $u \in H^1(\Omega)$ be such that $\Delta u \in L^2(\Omega)$. We say that

$$\frac{\partial u}{\partial \nu} \in L^2(\partial\Omega)$$

if there exists a $g \in L^2(\partial\Omega)$ such that

$$\int_\Omega (\Delta u)v + \int_\Omega \nabla u \nabla v = \int_{\partial\Omega} gv$$

for all $v \in H^1(\Omega) \cap C(\overline{\Omega})$. This determines g uniquely and we let $\frac{\partial u}{\partial \nu} := g$.

Recall that for all $u \in L^1_{\text{loc}}(\Omega)$ the Laplacian Δu is defined in the sense of distributions. If $\Delta u = 0$, then $u \in C^\infty(\Omega)$ by elliptic regularity. Next we define traces of a function $u \in H^1(\Omega)$.

Definition 8.2 (Traces). Let $u \in H^1(\Omega)$. We let

$$\begin{aligned} \operatorname{tr}(u) = \Big\{ g \in L^2(\partial\Omega) : \exists\, (u_n)_{n\in\mathbb{N}} \text{ in } H^1(\Omega) \cap C(\overline{\Omega}) \text{ such that} \\ \lim_{n\to\infty} u_n = u \text{ in } H^1(\Omega) \ \text{ and} \\ \lim_{n\to\infty} u_{n|\partial\Omega} = g \text{ in } L^2(\partial\Omega) \Big\} . \end{aligned}$$

For arbitrary open sets and $u \in H^1(\Omega)$ the set $\operatorname{tr}(u)$ might be empty, or contain more than one element. However, if Ω is a Lipschitz domain, then for each $u \in H^1(\Omega)$ the set $\operatorname{tr}(u)$ contains precisely one element, which we denote by $u_{|\partial\Omega} \in L^2(\partial\Omega)$. Now we are in the position to define the Dirichlet-to-Neumann operator D_0. Its domain is given by

$$D(D_0) := \Big\{\varphi \in L^2(\partial\Omega) : \exists\, u \in H^1(\Omega) \text{ such that } \Delta u = 0,\ \varphi \in \operatorname{tr}(u) \text{ and } \frac{\partial u}{\partial \nu} \in L^2(\partial\Omega)\Big\}$$

and we define

$$D_0\varphi = \frac{\partial u}{\partial \nu}$$

where $u \in H^1(\Omega)$ is such that $\Delta u = 0$, $\frac{\partial u}{\partial \nu} \in L^2(\partial\Omega)$ and $\varphi \in \operatorname{tr}(u)$. It is part of our result that this operator is well defined.

Theorem 8.3. *The operator D_0 is selfadjoint and $-D_0$ generates a submarkovian semigroup on $L^2(\partial\Omega)$.*

In the proof we use Theorem 6.4. Here a non-injective mapping j is needed. We also need Maz'ya's inequality. Let $q = \frac{2d}{d-1}$. There exists a constant $c_M > 0$ such that

$$\Big(\int_\Omega |u|^q\Big)^{2/q} \le c_M \Big(\int_\Omega |\nabla u|^2 + \int_{\partial\Omega} |u|^2\Big)$$

for all $u \in H^1(\Omega) \cap C(\overline{\Omega})$. (See [Maz, Example 3.6.2/1 and Theorem 3.6.3] and [AW, (19)].)

Proof of Theorem 8.3. We consider real spaces. Our Hilbert space is $L^2(\partial\Omega)$. Let $D(a) = H^1(\Omega) \cap C(\overline{\Omega})$, $a(u,v) = \int_\Omega \nabla u \nabla v$ and define $j\colon D(a) \to L^2(\partial\Omega)$ by $j(u) = u_{|\partial\Omega} \in L^2(\partial\Omega)$. Then a is symmetric and $a(u) \ge 0$ for all $u \in D(a)$. Thus the sectoriality condition before Theorem 6.4 is trivially satisfied. Denote by A the operator on $L^2(\partial\Omega)$ associated with (a, j). Let $\varphi, \psi \in L^2(\partial\Omega)$. Then $\varphi \in D(A)$ and $A\varphi = \psi$ if and only if there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $H^1(\Omega) \cap C(\overline{\Omega})$ such that $\lim_{n\to\infty} u_{n|\partial\Omega} = \varphi$ in $L^2(\partial\Omega)$, $\lim_{n\to\infty} a(u_n, v) = \int_{\partial\Omega} \psi v_{|\partial\Omega}$ for all $v \in D(a)$ and $\lim_{n,m\to\infty} \int_\Omega |\nabla(u_n - u_m)|^2 = 0$ (here we use Remark 6.5). Now Maz'ya's inequality implies that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $H^1(\Omega)$. Thus $\lim_{n\to\infty} u_n = u$ exists in $H^1(\Omega)$, and so $\varphi \in \operatorname{tr}(u)$. Moreover $\int_{\partial\Omega} \psi v = \lim_{n\to\infty} \int_\Omega \nabla u_n \nabla v = \int_\Omega \nabla u \nabla v$ for all $v \in H^1(\Omega) \cap C(\overline{\Omega})$. Taking as v test functions, we see that $\Delta u = 0$. Thus

$$\int_\Omega \nabla u \nabla v + \int_\Omega (\Delta u) v = \int_{\partial\Omega} \psi v$$

for all $v \in H^1(\Omega)$. Consequently, $\frac{\partial u}{\partial \nu} = \psi$. We have shown that $A \subset D_0$.

Conversely, let $\varphi \in D(D_0), D_0\varphi = \psi$. Then there exists a $u \in H^1(\Omega)$ such that $\Delta u = 0$, $\varphi \in \operatorname{tr}(u)$ and $\frac{\partial u}{\partial v} = \psi$. Since $\varphi \in \operatorname{tr}(u)$ there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $H^1(\Omega) \cap C(\overline{\Omega})$ such that $u_n \to u$ in $H^1(\Omega)$ and $u_{n|_{\partial\Omega}} \to \varphi$ in $L^2(\partial\Omega)$. It follows that $j(u_n) = u_{n|_{\partial\Omega}} \to \varphi$ in $L^2(\partial\Omega)$, the sequence $(a(u_n))_{n\in\mathbb{N}}$ is bounded and

$$a(u_n, v) = \int_\Omega \nabla u_n \nabla v \to \int_\Omega \nabla u \nabla v = \int_\Omega \nabla u \nabla v + \int_\Omega (\Delta u) v = \int_{\partial\Omega} \psi v$$

for all $v \in H^1(\Omega) \cap C(\overline{\Omega})$. Thus, $\varphi \in D(A)$ and $A\varphi = \psi$ by the definition of the associated operator. Since a is symmetric, the operator A is selfadjoint. Now the claim follows from Theorem 6.4.

Our criteria easily apply and show that semigroup generated by $-D_0$ is submarkovian. □

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