Form Methods for autonomous and non-autonomous Cauchy Problems

DISSERTATION
zur Erlangung des Doktorgrades Dr. rer. nat
der Fakultät für Mathematik und
Wirtschaftswissenschaften der
Universität Ulm

vorgelegt von

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2003

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Tag der Promotion:	18. Juni 2003

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Introduction

The aim of this thesis is to present some results obtained while studying form methods, which play an important role in the theory of evolution equations.
Rewriting an evolution equation as an abstract Cauchy problem, associated with a linear operator, we regard those problems, where the linear operator is given by a sesquilinear form. This is the case for most partial differential equations. If this form is densely defined, continuous and elliptic, the associated Cauchy problem is well-posed, because the operator is the generator of an analytic semigroup of contractions. Then one is interested in properties of the solution, which can be directly deduced from the sesquilinear form, such as positivity, contractivity and regularity.
The situation is more delicate for non-autonomous Cauchy problems, when the form also depends on the time parameter. Under a measurability assumption, the family of associated operators defines a multiplication operator.

The first chapter is of a preliminary nature and recalls some important definitions and results used in the following chapters of the thesis. First we introduce vector valued function spaces and define the Bochner integral. Then we give the basic facts to abstract Cauchy problems and semigroup theory. We conclude the chapter with an introduction to form methods.

In spite of the common point of departure, the research has lead into three different directions. They shall be presented in independent chapters.

In the second chapter we study multiplication operators. In scalar function spaces, they provide easy examples, whereas operator valued multiplication operators on vector valued spaces are more complicated. Their importance arises from non-autonomous Cauchy problems.
After an introduction to vector lattices, we study operators in vector and Banach lattices with an emphasis on the center and its properties. On the Banach lattice of scalar p-integrable functions, the center operators coincide with bounded multiplication operators. We give an appropriate definition of the center in vector valued function spaces to obtain an analogous characterization with respect to bounded multiplication operators.
Then a consideration of unbounded multiplication operators leads over some spectral aspects to multiplication semigroups.

Based on these results, we obtain a characterization of multiplication operators associates with sesquilinear forms, in the scalar setting as well as for operator valued multiplication operators.

The third chapter covers non-autonomous variational Cauchy problems, which are associated with a family of time dependent linear operators, each being defined by a continuous elliptic sesquilinear form. After an introduction to the underlying spaces and their properties, we give equivalent formulations of the considered problem. Then we recall an ingenious representation theorem due to J. L. Lions, which provides well-posedness. Given the existence of a unique solution, which depends continuously on the given data, we are interested in its properties.
A study of lattice operations on certain underlying spaces leads to sufficient conditions on the forms, such that the solution is positive or sub-Markovian. Namely, we require the same properties for the family of forms as in the Beurling-Deny criteria, which provide a characterization in the autonomous case.
The investigations on regularity lead to an alternative proof of maximal regularity for the autonomous case. These methods can only be applied in a very particular non-autonomous situation.
Non-autonomous Cauchy problems can be examined with semigroup methods as well, where we encounter a more restrictive notion of well-posedness. We give an introduction to evolution semigroups and families. Our characterization of well-posedness extends the known result for continuous functions. A generation theorem for surjective and dissipative operators leads to well-posedness of the non-autonomous variational Cauchy problem in a larger space, and we can restrict the obtained solution to the original space. We conclude this section with some invariance considerations of closed convex sets.
In order to use form methods directly for the non-autonomous Cauchy problem, one is lead to generalized forms, to which we give a short introduction. We characterize invariance of closed convex sets under the semigroup with respect to properties of the generalized form. As a consequence we obtain the Beurling-Deny criteria, which we then apply to the non-autonomous evolution equation.

In the last chapter we treat partial differential equations in an infinite dimensional setting. We do not use form methods there, but their application to second order differential operators inspired the work. After an introduction to Gaussian measures we examine Gaussian semigroups and their relation to the infinite dimensional heat equation.
On this basis, we study well-posedness of second order partial differential equations in infinite dimensional spaces. The idea is to diagonalize the matrix of coefficients, where the diagonal matrix satisfies the conditions for the heat equation, and thus provides a Gaussian measure. The image measure under the transformation matrix then defines a strongly continuous semigroup, which solves the second order problem.
Then we generalize these results and replace the derivatives by general group

generators. As for the differential operators, we start with the construction of a semigroup and give conditions, such that its generator coincides with the desired operator.
Finally, we consider order continuous linear forms on the space of bounded uniformly continuous functions on a Banach space. The presented characterization is used to construct a Gaussian measure for the representation of the heat semigroup and therefore given in this context.

Acknowledgement

I am very grateful to my supervisor, Prof. Wolfgang Arendt, who encouraged me strongly in the research of form methods. In several discussions he brought up new aspects which finally lead to nice results. In particular, I wish to thank him for carefully reading this thesis and giving helpful suggestions.
To all members of "Abteilung Angewandte Analysis" I wish to express my gratitude: especially to Sibylle, for all her (not only mathematical) advises, to Stefan and Mahamadi, who shared the interest in forms, and to Ralph and all three Markus, for answering numerous questions. Additionally, I have profited from discussions with our guests Prof. Charles Batty, Prof. Daniel Daners and Prof. Shanquan Bu.
Moreover, I have appreciated the cooperation of TULKA. In this context I thank Prof. Rainer Nagel for his enthusiasm and the creation of the INTERNET SEMINAR. Without the experience as a participant I might never have decided to do a PhD; special thanks to Franziska, Susanna and Cornelia.
For the kind hospitality in Italy I wish to thank Prof. Giorgio Metafune, Prof. Diego Pallara and Prof. Silvia Romanelli. The discussion in Lecce lead to the study of local forms and multiplication operators.
Most interesting experiences I collected during my stays in Morocco, which were supported by the DFG. In particular I thank Prof. Abdelaziz Rhandi for the kind hospitality in Marrakech and the discussions on non-autonomous forms and the generalized version of the Beurling-Deny criteria. However, I am most grateful to Prof. Omar El-Mennaoui, not only for the perfect integration into Moroccan life, but also for the many discussions, which lead to the chapter on infinite dimensional evolution equations. His steady encouragement and his additional visit to Ulm made it possible to conclude our project in a satisfying way.
Last but not least, I thank all my friends and my family, in particular Joey, Boris, Tinka and my parents, for their support and advises, their confidence and encouragement.

Chapter 1

Preliminaries

In this first chapter we recall some functional analytic background used in this thesis. First we introduce vector valued L^p-spaces assuming the reader to be familiar with the scalar situation. Then we define solutions to abstract Cauchy problems and explain their relation to one parameter semigroups. Finally, we consider linear operators associated to sesquilinear forms and recall the Beurling-Deny criteria for abstract Cauchy problems associated to such operators.

1.1 Vector Valued L^p-spaces

This section is meant to present some properties of the Bochner integral of vector-valued functions. Let (Ω, Σ, μ) be a measure space and E a complex or real Banach space.
A subset $N \subset \Omega$ is called **nullset**, if N is measurable and $\mu(N) = 0$. We say that a property holds **almost everywhere (a.e.)**, if there exists a nullset N, such that the property holds for all $\omega \in \Omega \setminus N$. For a measurable subset $A \subset \Omega$, the map

$$\chi_A : \omega \to \chi_A(\omega) := \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{otherwise} \end{cases}$$

is called the **characteristic function** of the set A.
A function $f : \Omega \to E$ is **simple**, if it is of the form $f(\omega) = \sum_{j=1}^{n} x_j \chi_{A_j}(\omega)$ for some $n \in \mathbb{N}$, $x_j \in E$ and measurable sets $A_j \subset \Omega$ with finite measure $\mu(A_j)$. In the representation of a simple function, the sets A_j may always be arranged to be disjoint, and then

$$f(\omega) = \begin{cases} x_j & (\omega \in A_j;\ j = 1, 2, ..., n) \\ 0 & \text{otherwise.} \end{cases}$$

Definition 1.1.1. A function $f : \Omega \to E$ is **measurable** if there is a sequence of simple functions g_n such that $f(\omega) = \lim_{n\to\infty} g_n(\omega)$ for almost all $\omega \in \Omega$.

When $E = \mathbb{C}$, this definition agrees with the usual definition of Lebesgue measurable functions but not with Borel measurable functions. Recall, that a function $f : \Omega \to \mathbb{C}$ is Borel measurable, if for every open set $O \subset \mathbb{C}$, $f^{-1}(O) \in \mathcal{B}$, where $\mathcal{B}$ denotes the σ-algebra on Ω. However, if we add to $\mathcal{B}$ all subsets of nullsets, we obtain again a σ-algebra Σ, which we call **complete** and a function $f : \Omega \to \mathbb{C}$ is Borel measurable with respect to Σ, if and only if f is Lebesgue measurable. In order to have the same notations, we assume in the scalar case, where $E = \mathbb{C}$, that the underlying measure space is complete, i.e. the σ-algebra is complete.

Lemma 1.1.2. *Let $f : \Omega \to E, g : \Omega \to E$ and $h : \Omega \to \mathbb{C}$ be measurable and $k : E \to F$ be continuous, where F is any Banach space. Then $f + g, h \cdot f$ and $k \circ f$ are measurable.*

Proof. Let $f_n, g_n : \Omega \to E$ and $h_n : \Omega \to \mathbb{C}$ be simple functions that approximate f, g and h respectively pointwise almost everywhere. Then $f_n + g_n$, $h_n \cdot f_n$ and $k \circ f_n$ approximate $f + g, h \cdot f$ and $k \circ f$ respectively pointwise almost everywhere. We only have to show that $f_n + g_n$, $h_n \cdot f_n$ and $k \circ f_n$ are simple functions. Fix $n \in \mathbb{N}$. Let $f_n(\omega) = \sum_{j=1}^m x_j \chi_{A_j}(\omega)$, $g_n(\omega) = \sum_{k=1}^l y_k \chi_{B_k}(\omega)$ and $h_n(\omega) = \sum_{k=1}^\nu z_k \chi_{C_k}(\omega)$ be representations, such that A_j, respectively B_k or C_k are disjoint. Then

$$(f_n + g_n)(\omega) = \begin{cases} x_j + y_k & (\omega \in A_j \cap B_k;\ j = 1, 2, ..., m,\ k = 1, 2, , ..., l) \\ x_j & (\omega \in A_j \setminus \bigcup_k B_k;\ j = 1, 2, ..., m) \\ y_k & (\omega \in B_k \setminus \bigcup_j A_j;\ k = 1, 2, ..., l) \\ 0 & \text{otherwise}, \end{cases}$$

$$(h_n \cdot f_n)(\omega) = \begin{cases} z_k \cdot x_j & (\omega \in A_j \cap B_k;\ j = 1, 2, ..., m,\ k = 1, 2, , ..., l) \\ 0 & \text{otherwise}, \end{cases}$$

and

$$(k \circ f_n)(\omega) = \begin{cases} k(x_j) & (\omega \in A_j;\ j = 1, 2, ..., m) \\ k(0) & \text{otherwise}. \end{cases}$$

Hence they are simple. □

Definition 1.1.3. For a simple function $g : \Omega \to E$, $g = \sum_{j=1}^n x_j \chi_{A_j}$ we define

$$\int_\Omega g(\omega)\, d\mu := \sum_{j=1}^n x_j \mu(A_j).$$

This definition is independent of the representation of $g = \sum_{j=1}^n x_j \chi_{A_j}$ and the integral is linear.

Definition 1.1.4. A function $f : \Omega \to E$ is called **Bochner integrable** if there exist simple functions g_n such that $g_n \to f$ pointwise a.e. and and one has for the scalar integral $\lim_{n\to\infty}\int_\Omega \|f(\omega)-g_n(\omega)\|\,d\mu = 0$. If $f:\Omega\to E$ is Bochner integrable, then the **Bochner integral** of f on Ω is

$$\int_\Omega f(\omega)\,d\mu := \lim_{n\to\infty}\int_\Omega g_n(\omega)\,d\mu.$$

The following theorem gives an easy characterization for Bochner integrable functions.

Theorem 1.1.5. (Bochner) *A function $f:\Omega\to E$ is Bochner integrable if and only if f is measurable and $\|f\|$ is integrable. If f is Bochner integrable, then*

$$\left\|\int_\Omega f(\omega)\,d\mu\right\| \le \int_\Omega \|f(\omega)\|\,d\mu.$$

For the proof we refer to [ABHN], Theorem 1.1.4.

Definition 1.1.6. For $1\le p<\infty$, we denote by $L^p(\Omega,E)$ the space of all measurable functions $f:\Omega\to E$ such that

$$\|f\|_p := \left(\int_\Omega \|f(\omega)\|^p\,d\mu\right)^{1/p} < \infty.$$

Further, $L^\infty(\Omega,E)$ denotes the space of all measurable functions $f:\Omega\to E$ such that

$$\|f\|_\infty := \operatorname*{ess\,sup}_{\omega\in\Omega}\|f(\omega)\| < \infty,$$

where $\operatorname{ess\,sup}_{\omega\in\Omega}\|f(\omega)\| = \inf_{\mathcal{N}:\mu(\mathcal{N})=0}\sup_{\omega\in\Omega\backslash\mathcal{N}}\|f(\omega)\|$.

In the usual way, we identify functions, which differ only on a set of measure zero. Then $(L^p(\Omega,E),\|.\|_p)$ is a Banach space for $1\le p\le\infty$.
Note that $L^p(\Omega,\mathbb{C})$ are the usual L^p-space, which we shall denote simply by $L^p(\Omega)$.

Remark 1.1.7. As in the scalar-valued case, for each $f\in L^\infty(\Omega,E)$, there exists a null set $\mathcal{N}\subset\Omega$ such that

$$\|f\|_\infty := \operatorname*{ess\,sup}_{\omega\in\Omega}\|f(\omega)\| = \sup_{\omega\in\Omega\backslash\mathcal{N}}\|f(\omega)\|.$$

Hence for $f,g\in L^\infty(\Omega,E)$ we have $\|f\cdot g\|_\infty \le \|f\|_\infty\cdot\|g\|_\infty$. Indeed,

$$\begin{aligned}
\operatorname*{ess\,sup}_{\omega\in\Omega}\|f(\omega)\|\cdot\operatorname*{ess\,sup}_{\omega\in\Omega}\|g(\omega)\| &= \sup_{\omega\in\Omega\backslash\mathcal{N}_f}\|f(\omega)\|\cdot\sup_{\omega\in\Omega\backslash\mathcal{N}_g}\|g(\omega)\| \\
&\ge \sup_{\omega\in\Omega\backslash(\mathcal{N}_f\cup\mathcal{N}_g)}\|f(\omega)\|\cdot\sup_{\omega\in\Omega\backslash(\mathcal{N}_f\cup\mathcal{N}_g)}\|g(\omega)\| \\
&= \sup_{\omega\in\Omega\backslash(\mathcal{N}_f\cup\mathcal{N}_g)}(\|f(\omega)\|\cdot\|g(\omega)\|) \\
&\ge \inf_{\mathcal{N}:\mu(\mathcal{N})=0}\sup_{\omega\in\Omega\backslash\mathcal{N}}(\|f(\omega)\|\cdot\|g(\omega)\|) \qquad (1.1) \\
&= \operatorname*{ess\,sup}_{\omega\in\Omega}(\|f(\omega)\|\cdot\|g(\omega)\|).
\end{aligned}$$

1.2 Cauchy Problems and Semigroups

In physical and economical problems one is often confronted with partial differential equations, e.g. the heat equation in $\mathbb{R}^3$ is given as

$$(HE_3) \qquad \begin{cases} \dfrac{\partial}{\partial t} u(t,x) &= \displaystyle\sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} u(t,x), \quad t>0, \\ u(0,x) &= u_0(x). \end{cases}$$

Regarding $u(t,.)$ as an element of an appropriate function space E and then $\sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} =: A$ as a linear operator, the differential equation can be written as an **abstract Cauchy problem** on E

$$(ACP) \qquad \begin{cases} u'(t) = Au(t), \quad t>0, \\ u(0) = u_0 \quad \in E, \end{cases}$$

where we assume, that E is a Banach space, A a closed linear operator on E and $u : \mathbb{R}_+ \to E,\ u(t) = u(t,.)$.

Definition 1.2.1. (i) A function $u : \mathbb{R}_+ \to E$ is called a**(classical) solution** of (ACP) if u is continuously differentiable with respect to t and with values in X, $u(t) \in D(A)$ for all $t \geq 0$, and (ACP) holds.

(ii) A continuous function $u : \mathbb{R}_+ \to E$ is called a **mild solution** of (ACP) if $\int_0^t u(s)\,ds \in D(A)$ for all $t \geq 0$ and

$$u(t) = A\int_0^t u(s)\,ds + x.$$

Existence and uniqueness of solutions for abstract Cauchy problems is strongly related to the theory of one parameter semigroups. We refer to the monographs [EN], [Paz] and [Ta] among many others for more details.

Definition 1.2.2. A family $(T(t))_{t\geq 0}$ of bounded linear operators on a Banach space E is called **strongly continuous (one-parameter) semigroup** (or **C_0-semigroup**) if it satisfies the functional equation

$$(FE) \qquad \begin{cases} T(t+s) = T(t)T(s), \quad \text{for all } t,s \geq 0 \\ T(0) = Id \end{cases}$$

holds and the orbit maps

$$T(\cdot)x : \mathbb{R}_+ \to E, t \mapsto T(t)x$$

are continuous for every $x \in E$.

For the relation to abstract Cauchy problems we will need the generator of a semigroup defined as follows.

Definition 1.2.3. The **generator** $A : D(A) \subset E \to E$ of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space E is the operator

$$Ax := \lim_{t\downarrow 0} \frac{1}{t}(T(t)x - x)$$

defined for every x in its domain

$$D(A) = \{x \in E : \lim_{t\downarrow 0} \frac{1}{t}(T(t)x - x) \text{ exists}\}.$$

We just want to give some basic properties of C_0-semigroups and their generators. We cite [EN], Lemma II.1.3 and Theorem II.1.4, to which we also refer for the proof.

Proposition 1.2.4. *For the generator $(A, D(A))$ of a strongly continuous semigroup $(T(t))_{t\geq 0}$, the following properties hold.*

(i) $A : D(A) \subset E \to E$ is a closed densely defined linear operator.

(ii) If $x \in D(A)$, then $T(t)x \in D(A)$ and

$$\frac{d}{dt}T(t)x = T(t)Ax = AT(t)x \quad \textit{for all } t \geq 0.$$

(iii) For every $t \geq 0$ and $x \in E$, one has

$$\int_0^t T(s)x\, ds \in D(A).$$

(iv) For every $t \geq 0$, one has

$$\begin{aligned} T(t)x - x &= A\int_0^t T(s)x\, ds \quad \textit{if } x \in E, \\ &= \int_0^t T(s)Ax\, ds \quad \textit{if } x \in D(A). \end{aligned}$$

Therefore we get the following characterization.

Corollary 1.2.5. $x \in D(A)$ and $Ax = y$ if and only if

$$T(t)x - x = \int_0^t T(x)y\, ds.$$

Now we get as an immediate consequence the relation to the abstract Cauchy problem. For more details on the proof, see [EN], Proposition II.6.2 and II.6.4.

Proposition 1.2.6. *Let $(A, D(A))$ be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$. Then,*

(i) for every $x \in D(A)$, the function

$$u : t \mapsto u(t) := T(t)x$$

is the unique classical solution of (ACP).

(ii) for every $x \in E$, the orbit map

$$u : t \mapsto u(t) := T(t)x$$

is the unique mild solution of the abstract Cauchy problem (ACP).

In view of this proposition, one is interested in a characterization of operators that generate a strongly continuous semigroup. The following generation theorem is due to Hille and Yosida.
Note that $\rho(A) := \{\lambda \in \mathbb{C} : (\lambda - A) \text{ is invertible and } (\lambda - A)^{-1} \in \mathcal{L}(E)\}$ is called the **resolvent set** and $R(\lambda, A) := (\lambda - A)^{-1}$ for $\lambda \in \rho(A)$ the **resolvent operator** of A.

Theorem 1.2.7. *For a linear operator $(A, D(A))$ on a Banach space E, the following properties are equivalent.*

(i) $(A, D(A))$ generates a strongly continuous semigroup of contractions, i.e. $\|T(t)\| \leq 1$ for all $t \geq 0$.

(ii) $(A, D(A))$ is closed, densely defined, and for every $\lambda > 0$ one has $\lambda \in \rho(A)$ and

$$\|\lambda R(\lambda, A)\| \leq 1.$$

(iii) $(A, D(A))$ is closed, densely defined, and for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$ one has $\lambda \in \rho(A)$ and

$$\|R(\lambda, A)\| \leq \frac{1}{\operatorname{Re}\lambda}.$$

We shall also recall the generation theorem for dissipative operators. We first give the definition.

Definition 1.2.8. A linear operator $(A, D(A))$ on a Banach space X is called **dissipative** if

$$\|(\lambda - A)x\| \geq \lambda \|x\|$$

for all $\lambda > 0$ and $x \in D(A)$.

The following generation theorem for dissipative operators is due to Lumer and Phillips.

Theorem 1.2.9. *For a densely defined, dissipative operator* $(A, D(A))$ *on a Banach space* X *the following statements are equivalent.*

(i) The closure $\overline{A}$ *of* A *generates a contraction semigroup.*

(ii) $\mathrm{Im}(\lambda - A)$ *is dense in* X *for some (hence all)* $\lambda > 0$.

We have the following more convenient characterization of dissipativity, see [EN], Proposition II.3.23. For a Banach space X we denote by X' its dual space and for every $x \in X$ its **duality set** by $J(x) := \{x' \in X' :< x, x' >= \|x\|^2 = \|x'\|^2\}$, which is nonempty by the Hahn-Banach theorem.

Proposition 1.2.10. *A linear operator* $(A, D(A))$ *on a Banach space* X *is dissipative if and only if for every* $x \in D(A)$ *there exists* $j(x) \in J(x)$ *such that*

$$\mathrm{Re} < Ax, j(x) > \leq 0. \tag{1.2}$$

If A *is the generator of a strongly continuous contraction semigroup, then* (1.2) *holds for all* $x \in D(A)$ *and arbitrary* $x' \in J(x)$.

Remark 1.2.11. On a Hilbert space H, which we identify with its dual $H' \cong H$, one has for every $x \in H$ that $J(x) = \{x\}$. Thus, a linear operator $(A, D(A))$ on a Hilbert space H is dissipative if and only if

$$\mathrm{Re}(Ax, x) \leq 0$$

for all $x \in D(A)$.

For the sake of completeness we also give the definition of an analytic semigroup. In the complex plane we denote by $\Sigma_\delta := \{\lambda \in \mathbb{C} : |\arg \lambda| < \delta\} \setminus \{0\}$ the open sector of angle δ.

Definition 1.2.12. A family of operators $(T(z))_{z \in \Sigma_\delta \cup \{0\}} \subset \mathcal{L}(E)$ is called an **analytic semigroup** (of angle $\delta \in (0, \pi/2]$) if

(i) $T(0) = Id$ and $T(z_1 + z_2) = T(z_1)T(z_2)$ for all $z_1, z_2 \in \Sigma_\delta$.

(ii) The map $z \mapsto T(z)$ is analytic in Σ_δ.

(iii) $\lim_{\Sigma_{\delta'} \ni z \to 0} T(z)x = x$ for all $x \in E$ and $0 < \delta' < \delta$.

If, in addition

(iv) $\|T(z)\|$ is bounded in $\Sigma_{\delta'}$ for every $0 < \delta' < \delta$,

we call $(T(z))_{z \in \Sigma_\delta \cup \{0\}}$ a **bounded analytic semigroup**.

In the following section, we regard a certain class of operators, which turn out to be generators of bounded analytic semigroups.

1.3 Sesquilinear Forms and Associated Operators

For a long time, sesquilinear forms have been studied in the context of evolution equations. We refer to [DL2], Chapter VII, and [LM], Chapter 3, for further details. Note that the non linear theory is presented in [Br]. Here we give a short introductions to form methods with an emphasis on those results, that we shall use later in this thesis. We mainly follow the approach in [Ta].

Let H and V be two Hilbert spaces. We denote by $\|.\|_H$, $\|.\|_V$ and $(.,.)_H$, $(.,.)_V$ the norm and scalar product in H and V respectively.

Assume that V is a dense, continuously embedded subspace of H, i.e. there exists a constant C such that $\|v\|_H \leq C\|v\|_V$ for all $v \in V$. We shortly write $V \overset{d}{\hookrightarrow} H$.

Let V' and H' be the anti dual spaces of V and H respectively, i.e. the spaces of all continuous anti linear forms, where anti linear means that $\Phi(\lambda v) = \overline{\lambda}\Phi(v)$.

Then after identifying H with H' by the Riesz-Fréchet Theorem, we obtain

$$V \overset{d}{\hookrightarrow} H \cong H' \overset{d}{\hookrightarrow} V'. \tag{1.3}$$

Indeed, let $\Phi_{|V}$ denote the restriction of $\Phi \in H'$ to V, then

$$|\Phi_{|V}(v)| = |\Phi(v)| \leq \|\varphi\|_{H'}\|v\|_H \leq \|\varphi\|_{H'}C\|v\|_V. \tag{1.4}$$

Hence $\Phi_{|V} \in V'$. Since V is dense in H, the correspondence $\Phi \mapsto \Phi_{|V}$ is injective, so that by identifying Φ with $\Phi_{|V}$ we may consider $H' \subset V'$. From (1.4) we conclude $\|\Phi_{|V}\|_{V'} \leq C\|\Phi\|_{H'}$, hence the embedding is continuous. We have that V is dense in V', if $\Psi_{|V} = 0$ implies $\Psi = 0$ for all $\Psi \in V''$. Note that $V'' \cong V$, since V is reflexive as a Hilbert space. Then, if we take $v \in V$, we obtain $\Psi(v) = <\Psi, v>_{V'',V'} = \overline{<v, \Psi>_{V',V}} = \overline{(v, \Psi)_H}$, since $v \in V \subset H$. Hence $\Psi(v) = 0$ for all $v \in V$ implies $\Psi = 0$ by taking $v = \Psi$. Therefore V is dense in V' and in particular H is dense in V'.

Now consider a **sesquilinear form** a on H with **domain** V. That is a mapping $a : V \times V \to \mathbb{C}$ which is linear in the first component and anti linear in the second:

$$\begin{aligned} a(u_1 + u_2, v) &= a(u_1, v) + a(u_2, v) \qquad & a(\lambda u, v) &= \lambda a(u, v) \\ a(u, v_1 + v_2) &= a(u, v_1) + a(u, v_2) \qquad & a(u, \lambda v) &= \overline{\lambda} a(u, v) \end{aligned} \tag{1.5}$$

Assume that a is **continuous**, i.e. there exist a constant $M > 0$, such that for all $u, v \in V$

$$|a(u, v)| \leq M\|u\|_V\|v\|_V \tag{1.6}$$

and **elliptic**, i.e. there exist constants $\lambda \in \mathbb{R}$ and $\alpha > 0$, such that for all $u \in V$

$$\operatorname{Re} a(u, u) + \lambda\|u\|_H^2 \geq \alpha\|u\|_V^2. \tag{1.7}$$

This inequality is called *Gårding's inequality*. Under these assumptions the norm $\|.\|_V$ in V is equivalent to $(\operatorname{Re} a(u,u) + \lambda\|u\|_H^2)^{1/2}$.
In this situation we can associate with a an Operator A given by

$$\begin{aligned} D(A) &= \{u \in V : \exists v \in H \text{ satisfying } a(u,\varphi) = (v,\varphi)_H \text{ for all } \varphi \in V\} \\ Au &= v \end{aligned}$$

The sesquilinear form is called **coercive** if there exists a constant $\alpha > 0$, such that for all $u \in V$

$$\operatorname{Re} a(u,u) \geq \alpha\|u\|_V^2, \tag{1.8}$$

i.e. λ can be chosen equal to zero in (1.7).
To a sesquilinear form a we define the adjoint form by

$$a^*(.,.) : V \times V \to \mathbb{C} : (u,v) \mapsto a^*(u,v) := \overline{a(v,u)}.$$

Then a^* inherits the properties of a. It is easy to see, that if A^* is the operator associated with a^*, then A^* is the adjoint operator to A, the operator associated with a.
If a is a continuous elliptic form with form domain V, then the form a_ω defined by

$$a_\omega : V \times V \to \mathbb{C} : (u,v) \mapsto a(u,v) + \omega(u,v)_H \tag{1.9}$$

is also continuous with continuity constant $M_\omega = M + |\omega|C^2$, since by the Cauchy-Schwarz inequality $|(u,v)| \leq \|u\|_H\|v\|_H \leq C^2\|u\|_V\|v\|_V$, and for $\omega \geq \lambda$ (the constant in (1.7) for a), a_ω is coercive. Further, if A is the operator associated with a, then $A + \omega : u \mapsto Au + \omega u$ is the operator associated with a_ω.
Hence it is not a severe restriction to assume from now on, that A is the operator associated with a continuous coercive form a.

Remark 1.3.1. The operator A associated with a continuous coercive form a is surjective as a consequence of the Lax-Milgram Theorem, see [Ta], Lemma 2.2.1.

Proposition 1.3.2. *If a is a continuous coercive form, the associated operator A is densely defined, closed and for every $\lambda > 0$, one has $\lambda \in \rho(A)$ and the estimate $\|\lambda R(\lambda, A)\| \leq 1$ holds.*

Proof. Assume that there exists a $v \in H$ such that $(u,v)_H = 0$ for all $u \in D(A)$. Since A^* is surjective, there exists a $w \in D(A^*) \subset V$ such that $v = A^*w$. Hence

$$\begin{aligned} 0 &= (u,v)_H = (u,A^*w)_H = \overline{(A^*w,u)_H} = \overline{a^*(w,u)} \\ &= a(u,w) = (Au,w). \end{aligned}$$

Since A is surjective, this implies $w = 0$, hence $v = 0$, which shows that $D(A)$ is dense in H.
The closedness of A can be immediately deduced from the fact that $\rho(A) \neq \emptyset$.

For $\lambda > 0$, the operator $(\lambda + A)$ is associated with the form a_λ, which is again a continuous coercive form. Hence by Remark 1.3.1, $(\lambda + A)$ is surjective.
Since a is coercive, we obtain for every $u \in D(A)$

$$\operatorname{Re}(-Au, u)_H = -\operatorname{Re} a(u, u) \leq 0.$$

Hence for $\lambda > 0$

$$\begin{aligned} \|(\lambda + A)u\|_H^2 &= ((\lambda + A)u, (\lambda + A)u)_H \\ &= (\lambda u, \lambda u)_H + 2\operatorname{Re}(\lambda u, Au)_H + (Au, Au)_H \\ &\geq \lambda^2 \|u\| + 2(-\lambda)\operatorname{Re}(-Au, u) \\ &\geq \lambda^2 \|u\| \end{aligned}$$

Hence $(\lambda + A)$ is injective, therefore invertible and

$$\lambda \|(\lambda + A)^{-1} u\| \leq \|u\|.$$

This shows $\lambda \in \rho(A)$ and $\|\lambda R(\lambda, A)\| \leq 1$. □

Remark 1.3.3. By [Ta],Lemma 2.2.2, $D(A)$ is even dense in V.

From the above proposition and by Theorem 1.2.7 it follows, that $-A$ is the generator of a contraction semigroup $(T(t)_{t\geq 0}$. We even have that it is an analytic semigroup of contractions, see [Ta], Theorem 3.6.1.
Note that if a is a continuous elliptic form, then for $\omega \geq \lambda$, a_ω is coercive, hence $-(A+\omega)$ is the generator of an analytic semigroup of contractions $T(t)_{t\geq 0}$. Then $-A = -(A + \omega) + \omega$ is the generator of the semigroup $T_\omega(t) := e^{\omega t}\, T(t)$.

In the literature as [Da2], [FOT], [Ka2] or [MR], one often finds a different approach to sesquilinear forms. Let $a : D(a) \times D(a) \to \mathbb{C}$ be a sesquilinear form on H, where the form domain $D(a)$ is a linear dense subspace of H. Define a norm on $D(a)$ by

$$\|u\|_a^2 := \operatorname{Re} a(u, u) + \|u\|_H^2.$$

Assume that a is **positive**, i.e. for all $u \in D(a)$

$$\operatorname{Re} a(u, u) \geq 0,$$

continuous, i.e. there exists constant $M > 0$ such that for all $u, v \in D(a)$

$$|a(u, v)| \leq M \|u\|_a \|v\|_a,$$

and **closed**, i.e. the space $(D(a), \|.\|_a)$ is complete.
The following proposition shows, that in a way, the two approaches to sesquilinear forms are equivalent.

Proposition 1.3.4. *Let H be a Hilbert space. A sesquilinear form a on H with dense domain $D(a)$ is positive, continuous and closed if and only if the form a_1 (defined by (1.9) for $\omega = 1$) with domain $V = D(a)$ is continuous and coercive with constant $\alpha = C$, where $\|u\|_H \leq C\|v\|_V$ for all $v \in V$.*

Proof. Let a be positive, continuous and closed. Then $(D(a), \|.\|_a)$ is complete, hence we can suppose $\|u\|_V = \|u\|_a = (\operatorname{Re} a(u,u) + \|u\|_H^2)^{1/2} \geq \|u\|_H$ and the embedding from V to H is continuous, in particular $C = 1$. Then

$$\begin{aligned} |a_1(u,v)| &\leq |a(u,v)| + |(u,v)_H| \leq M\|u\|_V\|v\|_V + \|u\|_H\|v\|_H \\ &\leq (M+1)(\|u\|_V\|v\|_V, \end{aligned}$$

hence a_1 is continuous. Since $\operatorname{Re} a(u,u) + \|u\|_H^2 = \|u\|_V^2$, a_1 is coercive with constant $\alpha = 1 = C$.
Conversely, suppose that a_1 is continuous and coercive. Then $(\operatorname{Re} a_1(u,u))^{1/2} = (\operatorname{Re} a(u,u) + \|u\|_H^2)^{1/2} = \|u\|_a$ is an equivalent norm on V, hence $(D(a), \|.\|_a)$ is complete. Further

$$\begin{aligned} |a(u,v)| &= |a_1(u,v) - (u,v)_H| \\ &\leq |a_1(u,v)| + |(u,v)_H| \leq M\|u\|_V\|v\|_V + \|u\|_H\|v\|_H \\ &\leq (M+1)\|u\|_V\|v\|_V, \end{aligned}$$

which shows that a is continuous. Finally

$$\operatorname{Re} a(u,u) = \operatorname{Re} a_1(u,u) - (u,u)_H \geq \alpha\|u\|_V^2 - \|u\|_H^2 \geq (\alpha - C)\|u\|_V = 0,$$

hence a is positive. □

The theory of sesquilinear forms can be applied to many differential operators. Since we obtain the generator of a semigroup, the Cauchy problem is well-posed. Further advantages of form methods in the context of partial differential equations is the fact, that in most cases the form domain is known explicitly, whereas the domain of the operator is difficult to determine.
Even properties of the semigroup, hence of the solution, can be directly characterized by the form. In this context, the following theorem, which is due to E.-M. Ouhabaz, see [Ou], Theorem 2.1 and Proposition 2.3, plays an important role for this thesis.
Let H be a Hilbert space. Assume that a is a continuous elliptic form with domain $V \overset{d}{\hookrightarrow} H$, and denote by A the associated operator and by T the semigroup generated by $-A$.

Theorem 1.3.5. *Let K be a closed convex subset of the Hilbert space H and denote by P the projection of H on K. The following assertions are equivalent*

(i) *$T(t)K \subset K$ for all $t \geq 0$.*

(ii) $\lambda R(\lambda, A)K \subset K$ *for all* $\lambda < 0$.

(iii) $u \in V$ *implies* $Pu \in V$ *and* $\operatorname{Re} a(u, u - Pu) \geq 0$.

Remark 1.3.6. The assertions of the above theorem are equivalent to

(iv) $u \in V$ implies $Pu \in V$ and $\operatorname{Re} a(Pu, u - Pu) \geq 0$.

The proof is implicitly contained in [Ou], see also [Bar] or [Th].

As a direct consequence, we obtain the so-called *Beurling-Deny* criteria. Let Ω be a measure space and $H = L^2(\Omega, \mathbb{C})$. We denote $H_{\mathbb{R}} := L^2(\Omega, \mathbb{R})$, for $u \in H_{\mathbb{R}}$, let $u^+ := \sup\{u, 0\} \in H_{\mathbb{R}}$, and $H_+ := \{u \in H_{\mathbb{R}} : u \geq 0\}$.
We call a semigroup on H **positive**, if $T(t)H_+ \subset H_+$ for all $t \geq 0$ and **sub-Markovian**, if $T(t)f \leq 1$, whenever $f \leq 1$ for all $t \geq 0$.
We suppose, that the sesquilinear form a is **real**, i.e.

$$u \in V \text{ implies } \operatorname{Re} u \in V, \text{ and } a(u, v) \in \mathbb{R} \text{ for all } u, v \in V \cap H_{\mathbb{R}}.$$

Proposition 1.3.7. *(Beurling-Deny I) The following assertions are equivalent.*

(i) The semigroup T *is positive.*

(ii) For all $u \in V \cap H_{\mathbb{R}}$, *we have* $u^+ \in V$ *and* $a(u^+, u^-) \leq 0$.

Proof. The set $K := \{u \geq 0\}$ is closed an convex with orthogonal projection given by $u \mapsto (\operatorname{Re} u)^+$. □

Proposition 1.3.8. *(Beurling-Deny II) Assume, that the semigroup* T *is positive. Then the following assertions are equivalent.*

(i) The semigroup T *is sub-Markovian.*

(ii) For all $0 \leq u \in V \cap H_{\mathbb{R}}$, *we have* $u \wedge 1 \in V$ *and* $a(u, u \wedge 1) \leq a(u, u)$.

Proof. The set $K := \{u \leq 1\}$ is closed an convex with orthogonal projection given by $u \mapsto (\operatorname{Re} u) \wedge 1$. □

These last results have been generalized in [Th], to the case, where the form is not densely defined. Then one has to consider the semigroup obtained through a composition of the semigroup on the closure of the domain and the orthogonal projection onto this closure. This semigroup is not strongly continuous in the origin. However, this generalization can immediately be deduced by the results for a non-linear setting established in [Bar]. Finally, as was shown in [MVV], it is even enough to verify the conditions on a dense subset of the form domain.

Chapter 2

Multiplication Operators

In operator theory one often wishes to have examples of a simple structure. Multiplication operators often play this role in function spaces, in particular in L^p-spaces. Moreover, non-autonomous Cauchy problems are associated to a family of operators, which in most cases define an operator valued multiplication operator.
First we will treat the more general case of operators belonging to the center of a Banach lattice. For that we recall some basic definitions and properties of vector and Banach lattices, which we took from the monographs [Me] and [Scha] and the survey article [BR]. Then we study bounded and unbounded multiplication operators and their relation to center operators, which we mainly took from [Me] for the scalar case, but developed some new results in the vector valued situation. Finally we will see that under certain assumptions on a sesquilinear form and its domain, the associated operator is automatically a multiplication operator.

2.1 Operators in Vector Lattices

This section is meant to give some background information on vector lattices and their center. As an application we will treat in particular the Banach lattice L^p. The results shall be generalized to vector-valued L^p-spaces, although those are in general no vector lattices.

2.1.1 Vector Lattices and Normal Cones

In the following definitions X is supposed to be an ordered vector space, i.e. X is a real vector space with an order $\leq$ satisfying

- $x \leq y$ implies $x + z \leq y + z$ for all x, y, z in X.
- $x \geq 0$ implies $\lambda x \geq 0$ for all x in X and $\lambda \geq 0$.

The **positive cone** $X_+ := \{x \in X : x \geq 0\}$ determines the order completely.

Definition 2.1.1. An ordered vector space X is called a **vector lattice** (or *Riesz space*) if any two elements x, y in X have a supremum, which is denoted by $x \vee y = \sup(x, y)$, and an infimum, denoted by $x \wedge y = \inf(x, y)$.
A vector lattice X is called **order complete** (or *Dedekind complete*), if for each non-void majorized set $B \subset X$, $\sup B$ exists in X.
A vector lattice X is called **countably order complete** (or *σ-Dedekind complete*), if for each non-void countable majorized set $B \subset X$, $\sup B$ exists in X.

A **vector sub-lattice** Y of a vector lattice X is a vector subspace of X such that $x, y \in Y$ implies $\sup(x, y) \in Y$ where the supremum is formed in X.
For $x \in X$ we use the usual notations $x^+ := \sup(x, 0)$ and $x^- := \sup(-x, 0)$ for the positive, respectively negative part of x, and $|x| := x^+ + x^-$ for the absolute value.

Definition 2.1.2. Two elements x, y of a vector lattice are called **orthogonal** or **lattice disjoint** and noted $x \perp y$, if $\inf(|x|, |y|) = 0$.

Certain subspaces of a vector lattice will play an important role in the sequel. We briefly recall the definitions.

Definition 2.1.3. A linear subspace I of a vector lattice X is called an **ideal** if $x \in I, |y| \leq |x|$ implies $y \in I$. A **band** in a vector lattice X is an ideal, which contains arbitrary suprema. For any subset B in X the **disjoint complement** $B^\perp := \{y \in X : \inf(|y|, |x|) = 0 \text{ for all } x \in B\}$ is a band in X. A band B is called a **projection band** if it does have a complemented ideal, i.e. $E = B \oplus B^\perp$, and the projection of X onto B with kernel $B^\perp$ is called the **band projection** belonging to B.

The smallest band containing a subset B is called the **band generated by** B. A band generated by a singleton $\{x\}$ is called **principal band**, and the lattice is said to have the **principal projection property**, if each principal band of X is a projection band.
In particular countably order complete lattices have the principal projection property (see [Scha], p. 64), and are therefore **Archimedian**, i.e. $x, y \in X$ and $nx \leq y$ for all $n \in \mathbb{N}$ implies $x \leq 0$. In Archimedian lattices the Band generated by a subset B is given by $B^{dd} := \{B^\perp\}^\perp$.

Remark 2.1.4. It is clear that an ideal is a vector sublattice. Conversely a sublattice I of X is an ideal, if $0 \leq y \leq x,\ x \in I$ implies $y \in I$.

In the following, let E be an ordered Banach space with norm $\|.\|$ and positive cone E_+, not necessarily a lattice. Since the definition of an ordered Banach space does not require any direct relation between the order and the topological structure involved, it is necessary to impose further restrictions. The most useful restriction of this sort is to require the cone to be normal with respect to the topology.

Definition 2.1.5. In an ordered Banach space E, the positive cone E_+ is said to be **normal** if there exist an $\alpha \geq 1$ such that the inequality $y \leq x \leq z$ always implies $\|x\| \leq \alpha(\|y\| \vee \|z\|)$.

Remark 2.1.6. This condition is equivalent to the requirement that order bounded sets, i.e. sets of the form $\{x \in E : y \leq x \leq z\} =: [y, z]$ for arbitrary $y, z \in E$, are norm bounded.

For more details on Banach spaces with normal cone see [BR].
There exist of course stronger conditions on the compatibility of the order structure and the norm. However, now we suppose the underlying space to be a lattice.

Definition 2.1.7. A norm on a vector lattice X is called a **lattice norm**, if it satisfies

$$|x| \leq |y| \quad \text{implies} \quad \|x\| \leq \|y\|.$$

Then X is called a **normed vector lattice**. A **Banach lattice** is a Banach space E endowed with an order $\leq$ such that $(E, \leq)$ is a vector lattice and the norm on E is a lattice norm.

By [Scha], II, Proposition 5.2. we have that in a normed vector lattice the lattice operations

$$x \mapsto x^+ \quad x \mapsto x^- \quad x \mapsto |x| \quad (x, y) \mapsto x \wedge y \quad (x, y) \mapsto x \vee y$$

are continuous.
In the sequel we will use the **Riesz decomposition theorem**, see [Scha], Theorem 2.10.

Theorem 2.1.8. *For any subset A of an order complete vector lattice E, E is the direct sum of the band generated by A and of the band $A^\perp$. In particular each band of E is a projection band.*

2.1.2 The Banach Lattice $L^p(Y, \Sigma, \mu)$

In order to illustrate the properties of Banach lattices, we give a well known example. We refer to [Me], Section 1.1 for more details.
Let (Y, Σ, μ) be an arbitrary measure space, and define on the real valued $L^p(Y)$, $1 \leq p \leq \infty$, the order pointwise. Then $L^p(Y)$ is an Archimedian Banach lattice. Two elements $f, g \in L^p(Y)$ are disjoint, $f \perp g$, if and only if $f = 0$ almost everywhere (a.e.) on $\{y : g(y) \neq 0\}$. Further for $1 \leq p < \infty$, $L^p(Y)$ is Dedekind complete, and if (Y, Σ, μ) is σ-finite, then also $L^\infty(Y)$ is Dedekind complete. We have the following characterization of ideals in $L^p(Y)$.

Lemma 2.1.9. *For a sublattice $I \subset L^p(Y)$ the following assertions are equivalent.*

(i) I is an ideal of $L^p(Y)$.

(ii) $L^\infty I \subset I$, where $L^\infty I := \{\varphi f : \varphi \in L^\infty(Y),\ f \in L^p(Y)\}$.

Proof. If I is an ideal, let $\varphi \in L^\infty(Y)$ and $f \in I$. Then for any $\alpha \in \mathbb{R}$, $\alpha f \in I$, hence $|\varphi f| = |\varphi||f| \leq \|\varphi\|_\infty |f| = |\|\varphi\|_\infty f|$ implies $\varphi f \in I$.
Conversely, let $f \in I$ and $g \in L^p(Y)$ such that $0 \leq g \leq f$. Let $\varphi(y) = \frac{g(y)}{f(y)}$ on $\{y : f(y) \neq 0\}$ and 0 otherwise. Then $\varphi \in L^\infty(Y)$, hence $g = \varphi f \in I$. □

For $1 \leq p < \infty$ we also have a characterization of the bands in $L^p(Y)$.

Lemma 2.1.10. *For a subspace $B \subset L^p(Y, \Sigma, \mu)$ the following assertions are equivalent.*

(i) B is a band in $L^p(Y, \Sigma, \mu)$.

(ii) $B = L^p(A, \Sigma_A, \mu_A)$, for some measurable subset $A \subset Y$ and the induced σ-algebra Σ_A and measure μ_A.

Proof. Obviously, $L^p(A, \Sigma_A, \mu_A)$ is a band in $L^p(Y, \Sigma, \mu)$. For the converse, let B be a band in $L^p(Y)$. Since $L^p(Y)$ is Dedekind complete, by Theorem 2.1.8, every band is a projection band. Thus $L^p(Y) = B \oplus B^\perp$. If $B^\perp = \{0\}$, then $B = L^p(Y, \Sigma, \mu)$ and $A = Y$. If $0 \neq g \in B^\perp$, then for all $f \in B = B^{\perp\perp}$, $f = 0$ a.e. on $\{y : g(y) \neq 0\}$. Since $L^p(Y)$ is separable, there exists $\{g_n : n \in \mathbb{N}\}$ dense in $B^\perp$. Then $A^C := \bigcup_{n\in\mathbb{N}} \{y : g_n(y) \neq 0\}$ is measurable and for all $f \in B$, $f = 0$ a.e. on A^C, i.e. $B \subset L^p(A, \Sigma_A, \mu_A)$. On the other hand, for $g \in B^\perp$, there exists $g_{n_k} \to g$ in $L^p(Y)$ as $k \to \infty$, hence, passing to a subsequence, $g_{n_k}(y) \to g(y)$ for almost all $y \in Y$. Thus, if $f \in L^p(A, \Sigma_A, \mu_A)$, then $f(y)g_{n_k}(y) = 0$ a.e. and therefore $f(y)g(y) = 0$ a.e., which yields $f \perp g$. Consequently, $f \in B^{\perp\perp} = B$. □

We have used the fact, that every band is a projection band. The projection onto the band $L^p(A, \Sigma_A, \mu_A)$ is obviously given by $\chi_A : f \mapsto \chi_A \cdot f$.
Moreover, $L^p(Y)$ has the principal projection property, which can be seen as follows. Let $f \in L^p(Y)$, then the principal band generated by f is given as the set $B(f) = \{f\}^{dd} = \{g \in L^p(Y) : g = 0 \text{ (a.e.) on } \{y : f(y) = 0\}\}$, and the corresponding band projection by $Pg = g \cdot \chi_{\{f\neq 0\}}$.

2.1.3 The Center of Vector Lattices

In this section we will treat special classes of operators on vector lattices. Mainly we are interested in band preserving operators and the center. We use the notation of [Me], Section 3.1.
Throughout this section let X and Y be Archimedean vector lattices.

Definition 2.1.11. Assume that $T : X \to Y$ is a linear operator.

(i) T is called **positive**, if $TX_+ \subset Y_+$. We will denote this by $T \geq 0$. Let $L_+(X,Y)$ be the collection of all positive linear operators from X into Y.

(ii) T is called **regular**, if T is the difference of two positive operators. We denote by $L^r(X,Y)$ the collection of all regular operators ordered by $T \geq S$ if and only if $T - S \geq 0$.

(iii) A linear operator $T : X \to X$ satisfying $TB \subset B$ for every band $B \subset X$ is called **band preserving**.

(iv) A band preserving operator $T \in L^r(X)$ is called an **orthomorphism**. We denote by $\mathrm{Orth}(X)$ the collection of all orthomorphisms on X.

(v) The **center** $\mathcal{Z}(X)$ consists of all linear operators $T : X \to X$ such that $-a\,Id \leq T \leq a\,Id$ for some $0 < a \in \mathbb{R}$.

It is clear, that every positive operator is regular and that one has a decomposition $L^r(X,Y) = L_+(X,Y) - L_+(X,Y)$. Further, observe that $\mathcal{Z} \subset L^r(X)$, because $T = (T + a\,Id) - (a\,Id - T)$ is the difference of two positive operators. Finally, $\mathcal{Z}(X) \subset \mathrm{Orth}(X)$ and $\mathcal{Z}(X)$ is the ideal in $L^r(X)$ generated by the identity.
We want to recall some properties of these operators.

Proposition 2.1.12. *For every linear operator $T : X \to X$ the following assertions are equivalent.*

(i) T is band preserving.

(ii) $Tx \in B_x$ (the principal band generated by x) for every $x \in X$

(iii) $Tx \perp y$ for all $x, y \in X$ satisfying $x \perp y$.

Proof. $(i) \Rightarrow (ii)$ $x \in B_x$, hence from $TB_x \subset B_x$ follows $Tx \in B_x$.
$(ii) \Rightarrow (iii)$ Since X is Archimedian, $B_x = \{x\}^{dd}$ and $x \perp y \Leftrightarrow y \in \{x\}^\perp$. Therefore $Tx \in B_x = \{x\}^{dd} \Rightarrow Tx \perp y$.
$(iii) \Rightarrow (i)$ Let $B \subset X$ be a band, then $B^{dd} = B$. For $x \in B$ arbitrary, we get $x \perp y\ \forall y \in B^\perp$. By (iii) $Tx \perp y\ \forall y \in B^\perp$, hence $Tx \in B^{dd} = B$. □

Proposition 2.1.13. *Suppose that X has the principal projection property. A linear operator $T : X \to X$ is band preserving if and only if T commutes with every band projection.*

Proof. Assume that T is band preserving. Let P be a band projection and $B = P(X)$ the corresponding projection band. Then $X = B \oplus B^\perp$, and $B^\perp$ is a projection band with corresponding band projection $Id - P$. By assumption $TB^\perp \subset B^\perp$ and $TB \subset B$, hence $PTx = PT(Px + (Id - P)x) = PTPx = TPx$, for all $x \in X$.

Conversely, assume that T commutes with every band projection. Since X has the principal projection property, for every $x \in X$, the principal band B_x generated by x is a projection band. If P_x denotes the band projection, then $P_xT = TP_x$. Therefore, one gets $Tx = TP_xx = P_xTx \subset B_x$, and T satisfies (ii) of the preceding proposition and is therefore band preserving. □

Remark 2.1.14. T commutes with every band projection is equivalent to saying that $TB \subset B$ for every projection band.

As $\mathcal{Z}(X) \subset \mathrm{Orth}(X)$, each $T \in \mathcal{Z}(X)$ is band preserving. But the following even stronger property holds.

Lemma 2.1.15. *Let $T \in \mathcal{Z}(X)$, then $TI \subset I$ for each ideal in X.*

Proof. Let I be an arbitrary ideal in X and $-a\, Id \leq T \leq a\, Id$ for some $0 < a \in \mathbb{R}$. Then for every $x \in I$, $ax^+, ax^- \in I$ and $|T(x^+)| \leq |ax^+|$, $|T(x^-)| \leq |ax^-|$ imply $Tx = T(x^+) - T(x^-) \in I$. □

2.1.4 The Center of Banach Lattices

We have seen the basic properties of band preserving and center operators on vector lattices. In most applications, the underlying space is a normed vector lattice or even a Banach lattice. It is well known that center operators on a normed vector lattice are always bounded and that they coincide with band preserving operators on Banach lattices. We recall these results in this section. Throughout this section let E denote a normed vector lattice.

Proposition 2.1.16. *If E is a normed vector lattice, then every $T \in \mathcal{Z}(E)$ is bounded, i.e. $\mathcal{Z}(E) \subset \mathcal{L}(E)$.*

Proof. Let $T \in \mathcal{Z}(E)$, then $-a\, Id \leq T \leq a\, Id$ for some $0 < a \in \mathbb{R}$. Hence for all $f \in E_+$ we have $|Tf| \leq af = |af|$. As E is a normed vector lattice, this implies $\|Tf\| \leq \|af\| = a\|f\|$ for all $f \in E_+$. Now $f^+, f^- \leq |f|$ holds for all $f \in E$, thus $\|Tf\| = \|T(f^+ - f^-)\| \leq \|T(f^+)\| + \|T(f^-)\| \leq a\|f^+\| + a\|f^-\| \leq 2a\|f\|$. □

Assume now, that E is complete, i.e. E is a Banach lattice. The following result is due to B. de Pagter, see [Pag], for the proof we refer to [Me], Proposition 3.1.12.

Proposition 2.1.17. *If E is a Banach lattice, then every band preserving operator $T : E \to E$ is in the center $\mathcal{Z}(E)$, hence bounded. In particular, one has $\mathcal{Z}(E) = \mathrm{Orth}(E)$.*

Therefore on a Banach lattice, we get the following characterization of center operators.

Theorem 2.1.18. *Let E be a Banach lattice and $T : E \to E$ a linear operator. Then the following assertions are equivalent.*

(i) $T \in \mathcal{Z}(E)$,
i.e. there exists $0 < a \in \mathbb{R}$ *such that* $-a\ Id \leq T \leq a\ Id$.

(ii) $TI \subset I$ *for every ideal* $I \subset E$.

(iii) T *is band preserving.*

(iv) $Tx \perp y$ *for all* $x, y \in E$ *satisfying* $x \perp y$.

In particular, if T *satisfies one of the above assertions, then* T *is bounded, i.e.* $T \in \mathcal{L}(E)$. *If in addition* E *has the principal projection property, the above assertions are equivalent to*

(v) $TB \subset B$ *for every projection band* $B \subset E$, *i.e.* T *commutes with every band projection.*

The proof is contained in the above results, and the fact that a band is in particular an ideal.

2.1.5 The Center of Scalar- and Vector-Valued L^p-spaces

As we have seen before, typical examples for Banach lattices are the L^p-spaces. We will recall in the subsequent section, that the center of $L^p(\Omega)$ coincides with the set of bounded multiplication operators. Since there is a sense to multiplication operators on vector-valued L^p-spaces, we wish to give a sense to center operators on these spaces as well, although vector-valued L^p-spaces are in general not even vector lattices.
Throughout this section, let (Ω, Σ, μ) be a σ-finite measure space, i.e. $\Omega = \bigcup_n \Omega_n$ with $\mu(\Omega_n) < \infty$.
First we want to examine some of the assertions of Theorem 2.1.18 for $E = L^p(\Omega)$. See section 2.1.2 for the lattice properties of $L^p(\Omega)$.

Lemma 2.1.19. *For a linear operator* $T : L^p(\Omega) \to L^p(\Omega)$ *the following assertions are equivalent.*

(i) $Tf \perp g$ *for all* $f, g \in L^p(\Omega)$ *satisfying* $f \perp g$

(ii) $Tf = 0$ *a.e. on* $\{\omega : g(\omega) \neq 0\}$ *for all* $f, g \in L^p(\Omega)$ *with* $f = 0$ *a.e. on* $\{\omega : g(\omega) \neq 0\}$

(iii) $Tf = 0$ *a.e. on* $\{\omega : f(\omega) = 0\}$

Proof. *(i)* $\Leftrightarrow$ *(ii)* Evident, by the characterization of orthogonality in $L^p(\Omega)$.
(ii) $\Rightarrow$ *(iii)* Let $f \in L^p(\Omega)$. First assume that $\mu(\{\omega : f(\omega) = 0\}) < \infty$. Then $g := \chi_{\{f=0\}} \in L^p(\Omega)$ and $f = 0$ a.e. on $\{g \neq 0\}$. Hence $Tf = 0$ a.e. on $\{g \neq 0\} = \{f = 0\}$. Now, if $\mu(\{\omega : f(\omega) = 0\}) = \infty$, then set $g_n = \chi_{\{f=0\} \cap \Omega_n} \in$

L^p. Then $f = 0$ a.e. on $\{g_n \neq 0\}$ for all $n \in \mathbb{N}$, hence $Tf = 0$ a.e. on $\bigcup_n \{g_n \neq 0\} = \{f = 0\}$.
$(iii) \Rightarrow (ii)$ Let $f, g \in L^p(\Omega)$ with $f = 0$ a.e. on $\{\omega : g(\omega) \neq 0\}$, i.e. $f \perp g$. Then $\mu(\{g \neq 0\} \setminus \{f = 0\}) = \mu(\{g \neq 0\} \cap \{f \neq 0\}) = 0$. Therefore we obtain $\{g \neq 0\} = (\{g \neq 0\} \cap \{f \neq 0\}) \cup (\{g \neq 0\} \cap \{f = 0\})$, where the first set has measure 0 and the second is a subset of $\{f = 0\}$. Hence $Tf = 0$ a.e. on $\{f = 0\}$ implies $Tf = 0$ a.e. on $\{g \neq 0\}$, i.e. $Tf \perp g$. □

Now we can rewrite Theorem 2.1.18 for scalar L^p-spaces in the following way. Recall that we write χ_A also for the multiplication with χ_A, i.e. the band projection onto $L^p(A)$, and $L^\infty I = \{\varphi f : \varphi \in L^\infty, f \in I\}$.

Theorem 2.1.20. *Let $T : L^p(\Omega) \to L^p(\Omega)$ be a linear operator. Then the following assertions are equivalent.*

- *(i)* $T \in \mathcal{Z}(L^p(\Omega))$,
  i.e. $T \in L^r(L^p(\Omega)$ *and there exists* $0 < a \in \mathbb{R}$ *such that* $-a\, Id \leq T \leq a\, Id$.
- *(ii)* $TI \subset I$ *for every* $I \subset L^p(\Omega)$, *satisfying* $L^\infty(\Omega)I \subset I$.
- *(iii)* $TL^p(A) \subset L^p(A)$ *for every measurable subset* $A \subset \Omega$.
- *(iv)* $Tf = 0$ *a.e. on the set* $\{\omega \in \Omega : f(\omega) = 0\}$, *for all* $f \in L^p(\Omega)$.
- *(v)* $T\chi_A = \chi_A T$ *for every measurable subset* $A \subset \Omega$.

In particular, if T satisfies one of the above assertions, then T is bounded, i.e. $T \in \mathcal{L}(L^p(\Omega))$.

Now we shall generalize the above results on vector valued L^p-spaces as introduced in Section 1.1. For a Banach space E, and for $1 \leq p < \infty$, let

$$L^p(\Omega, E) = \left\{ f : \Omega \to E : \|f\|_p := \left(\int_\Omega \|f(\omega)\|_E^p \, d\mu(\omega) \right)^{1/p} < \infty \right\},$$

and for $p = \infty$, let

$$L^\infty(\Omega, E) = \left\{ f : \Omega \to E : \|f\|_\infty := \operatorname*{ess\,sup}_{\omega \in \Omega} \|f(\omega)\|_E < \infty \right\}.$$

As usual we identify functions that are only different on a set of measure zero. Then for $1 \leq p \leq \infty$, $(L^p(\Omega, E), \|.\|_p)$ is a Banach space.
Obviously, if E is not ordered, $L^p(\Omega, E)$ cannot be a vector lattice. Hence there is no sense to the order of operators on $L^p(\Omega, E)$, nor to an ideal nor to a band in $L^p(\Omega, E)$.
However, the reformulated assertion $(ii) - (v)$ of Theorem 2.1.20 still have a meaning in $L^p(\Omega, E)$, but are not all equivalent. Moreover, we do not get boundedness

for the operator directly from de Pagter's Theorem 2.1.17. But we shall give a generalization to this result, see Theorem 2.1.23 below. Note that here again χ_A denotes the multiplication with χ_A and $L^\infty(\Omega)I = \{\varphi f : \varphi \in L^\infty(\Omega), f \in I\}$, where φ is an element of the scalar L^∞.

Theorem 2.1.21. *For a linear operator $T : L^p(\Omega, E) \to L^p(\Omega, E)$ consider the following assertions.*

(i) $TI \subset I$ for every $I \subset L^p(\Omega, E)$, satisfying $L^\infty(\Omega)I \subset I$.

(ii) $TL^p(A, E) \subset L^p(A, E)$ for every measurable subset $A \subset \Omega$.

(iii) $Tf = 0$ a.e. on the set $\{\omega \in \Omega : f(\omega) = 0\}$.

(iv) $T\chi_A = \chi_A T$ for every measurable subset $A \subset \Omega$.

Then (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv).

Proof. $(i) \Rightarrow (ii)$ $L^\infty(\Omega)L^p(A, E) \subset L^p(A, E)$ for every measurable subset $A \subset \Omega$.
$(ii) \Rightarrow (iii)$ Let $A^C = \{\omega : f(\omega) = 0\}$, then $f \in L^p(A, E)$. Hence $Tf \in L^p(A, E)$ and $Tf = 0$ a.e. on $A^C = \{\omega : f(\omega) = 0\}$.
$(iii) \Rightarrow (iv)$ For $f \in L^p(\Omega, E)$ and every measurable set $A \subset \Omega$, $\chi_{A^C} T \chi_A f = 0$ by $iii)$. Hence $T\chi_A f = \chi_A T \chi_A f + \chi_{A^C} T \chi_A f = \chi_A T \chi_A f + \chi_A T \chi_{A^C} f = \chi_A T f$.
$(iv) \Rightarrow (ii)$ Let $f \in L^p(A, E)$, then one has $f = \chi_A f$ and therefore gets that $Tf = T\chi_A f = \chi_A T f \in L^p(A, E)$. □

The following example show, that indeed assertion (i) of the above theorem is not necessary for the other equivalent assertions (ii), (iii) and (iv).

Example 2.1.22. Let $E = \mathbb{R}^2$ and $T : L^p(\Omega, \mathbb{R}^2) \to L^p(\Omega, \mathbb{R}^2)$, $f \mapsto Mf$, where M is a fixed 2×2-matrix. Then T satisfies $(ii) - (iv)$. However, (i) is not satisfied for $M = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $I = L^p(\Omega, \mathbb{R} \times \{0\})$, although $L^\infty(\Omega)I \subset I$.

In order to obtain some analogy to the scalar $L^p(\Omega)$ we will define the center of $L^p(\Omega, E)$ by means of the last three equivalent assertions.
We have seen in Section 2.1.4 that a band preserving operator on a Banach lattice is automatically bounded, which immediately gives that a linear operator $T : L^p(\Omega) \to L^p(\Omega)$ satisfying $Tf = 0$ almost everywhere on $\{f = 0\}$, is bounded. We will generalize this result to vector-valued L^p-spaces, but we have to distinguish cases with respect to the underlying measure space.
Assume that (Ω, Σ, μ) is a non atomic measure space. Then for every measurable $A \subset \Omega$, such that $\mu(A) \neq 0$, there exists a measurable $B \subset A$, such that $\mu(B) \neq 0$ and $\mu(A \setminus B) \neq 0$.

Theorem 2.1.23. *Let (Ω, Σ, μ) be a non atomic measure space and E be a separable Banach space. Assume that $T : L^p(\Omega, E) \to L^p(\Omega, E)$ is a linear operator satisfying $Tf = 0$ almost everywhere on the set $\{\omega \in \Omega : f(\omega) = 0\}$ for every $f \in L^p(\Omega, E)$. Then T is bounded, i.e. $T \in \mathcal{L}(L^p(\Omega, E))$.*

Proof. Note that since $Tf = 0$ almost everywhere on the set $\{\omega \in \Omega : f(\omega) = 0\}$ we have $TL^p(A, E) \subset L^p(A, E)$ for every measurable $A \subset \Omega$. In particular $\|T\| \leq 2\max\{\|T_{|L^p(A,E)}\|, \|T_{|L^p(A^C,E)}\|\}$.
We proof the theorem by contradiction. Assume that T is unbounded. Then for every $M \in \mathbb{N}$ there exist two disjoint measurable sets $\Omega_1, \Omega_2 \subset \Omega$ such that $\mu(\Omega_i) > 0$ for $i = 1, 2$, and $\|T_{|L^p(\Omega_1,E)}\| > M$ and $T_{|L^p(\Omega_2,E)}$ is unbounded. Indeed, if this is not true, then there exists an $M \in \mathbb{N}$ such that for every two disjoint sets $\Omega_1, \Omega_2 \subset \Omega$ with $\mu(\Omega_i) > 0$ for $i = 1, 2$, if $T_{|L^p(\Omega_2,E)}$ is unbounded, then $\|T_{|L^p(\Omega_1,E)}\| \leq M$. Let $\Omega_1 \subset \Omega$ be measurable and $\Omega_2 := \Omega_1^C$, then $T_{|L^p(\Omega_i,E)}$ is unbounded for one $i = 1, 2$, otherwise T were bounded. Suppose $T_{|L^p(\Omega_2,E)}$ is unbounded, then $\|T_{|L^p(\Omega_1,E)}\| \leq M$. Further let $\Omega_3 \subset \Omega_2$ and $\Omega_4 := \Omega_2 \setminus \Omega_3$, then again $T_{|L^p(\Omega_i,E)}$ is unbounded for one $i = 3, 4$. Suppose $T_{|L^p(\Omega_4,E)}$ is unbounded, then $\|T_{|L^p(\Omega_3,E)}\| \leq M$. Continuing in the same way, we get $\|T_{|L^p(\Omega_{2n-1},E)}\| \leq M$ for all $n \in \mathbb{N}$ and $T_{|L^p(\Omega_{2n},E)}$ is unbounded for all $n \in \mathbb{N}$. Further one obtains $\Omega_{2n} = \Omega \setminus \bigcup_{k=1}^{n} \Omega_{2n-1}$, hence $\Omega = \bigcup_{n\in\mathbb{N}} \Omega_{2n-1}$, where the Ω_{2n-1} are disjoint. Thus $\|T\| = \max_{n\in\mathbb{N}} \|T_{|L^p(\Omega_{2n-1},E)}\| \leq M$, which contradicts that T is unbounded.
Now for $n = 1$, let $M = 2^n = 2$. Then there exist $\Omega_1, \Omega_1' \subset \Omega$ disjoint of positive measure such that $\|T_{|L^p(\Omega_1,E)}\| > M = 2$ and $T_{|L^p(\Omega_1',E)}$ is unbounded. Then $T_{|L^p(\Omega_1',E)} : L^p(\Omega_1', E) \to L^p(\Omega_1', E)$ is linear, unbounded and satisfies for $f \in L^p(\Omega_1', E)$, $T_{|L^p(\Omega_1',E)}f = 0$ almost everywhere on the set $\{f = 0\}$. Hence with the same argument as above, for $n = 2$ and $M = 2^n = 4$, there exist $\Omega_2, \Omega_2' \subset \Omega_1'$ disjoint and of positive measure such that $\|T_{|L^p(\Omega_2,E)}\| > M = 4$ and $T_{|L^p(\Omega_2',E)}$ is unbounded. Continuing this procedure, we obtain $\Omega_n \subset \Omega$ disjoint, such that $\|T_{|L^p(\Omega_n,E)}\| > 2^n$ and $T_{|L^p(\Omega_n',E)}$ is unbounded, where Ω_n' is disjoint to Ω_k for $k = 1, 2, ..., n$.
Then for all $n \in \mathbb{N}$, as $\|T_{|L^p(\Omega_n,E)}\| > 2^n$, there exists $f_n \in L^p(\Omega_n, E)$ such that $\|f_n\|_{L^p(\Omega_n,E)} = 1$ and $\|T_{|L^p(\Omega_n,E)}f_n\| > 2^n$. We can regard f_n as an element of $L^p(\Omega, E)$, if we set $f_n = 0$ on Ω_n^C.
Let $f := \sum_n \frac{f_n}{2^n}\chi_{\Omega_n}$. Then

$$\|f\|^p_{L^p(\Omega,E)} = \sum_n \frac{1}{2^n}\|f_n\|^p_{L^p(\Omega_n,E)} = 1,$$

hence $f \in L^p(\Omega, E)$. But

$$\|Tf\|^p_{L^p(\Omega,E)} = \sum_n \frac{1}{2^n}\|T_{|L^p(\Omega_n,E)}f_n\|^p_{L^p(\Omega_n,E)} > \sum_n \frac{2^{np}}{2^n} = \infty,$$

which contradict the fact that $T : L^p(\Omega, E) \to L^p(\Omega, E)$, hence T has to be bounded. □

The assumption that the measure space is non atomic is essential in the previous theorem. Indeed, if $\Omega_1 \subset \Omega$ is an atom, then every $f \in L^p(\Omega, E)$ has to be constant on Ω_1 as a measurable function. Hence, either $\mu(\Omega_1) < \infty$ or $L^p(\Omega_1, E) = \{0\}$.
Assume $\mu(\Omega_1) < \infty$ and let $A : E \to E$ be a linear unbounded operator. For $f \in L^p(\Omega, E)$ define

$$(Tf)(\omega) = A(f\chi_{\Omega_1}(\omega)) + f\chi_{\Omega_1^C}(\omega).$$

Then $T : L^p(\Omega, E) \to L^p(\Omega, E)$ is a linear operator satisfying $Tf = 0$ almost everywhere on $\{f = 0\}$. But obviously T is unbounded.
Since every measure space can be decomposed into a non atomic and a purely atomic subset, we immediately get the following result.

Corollary 2.1.24. Let (Ω, Σ, μ) be a measure space and E be a separable Banach space. Assume that $T : L^p(\Omega, E) \to L^p(\Omega, E)$ is a linear operator which satisfies $Tf = 0$ almost everywhere on the set $\{\omega \in \Omega : f(\omega) = 0\}$ for every $f \in L^p(\Omega, E)$. Then there exists $\Omega_1 \subset \Omega$ measurable, such that $T_{|L^p(\Omega_1,E)} : L^p(\Omega_1, E) \to L^p(\Omega_1, E)$ is bounded and $\Omega \setminus \Omega_1$ is purely atomic.

This result explains the following notation.

Definition 2.1.25. We say that a linear operator $T : L^p(\Omega, E) \to L^p(\Omega, E)$ belongs to the **center** of $L^p(\Omega, E)$, and write $T \in \mathcal{Z}(L^p(\Omega, E))$, if $Tf = 0$ a.e. on $\{\omega : f(\omega) = 0\}$ and $T_{|L^p(\tilde{\Omega},E)}$ is bounded for every purely atomic subset $\tilde{\Omega}$ of Ω.

Thus, every center operator on $L^p(\Omega, E)$ is bounded.

2.2 Bounded Multiplication Operators

This section is devoted to the discussion of bounded multiplication operators in the Banach space $L^p(\Omega)$ and bounded operator valued multiplication operators on $L^p(\Omega, E)$. In the first case, we recall its relation to center operators. Moreover, we consider the special case, where the underlying measure space is locally compact Hausdorff space provided with the Borel σ-algebra. Finally, we generalize the characterization to the vector valued setting.
Our considerations were strongly motivated by the results obtained in [Gr] for the space of continuous functions. However, we encounter difficulties, whenever an evaluation of a function at one point is involved.
Again we suppose, that (Ω, Σ, μ) is a σ-finite measure space, i.e. $\Omega = \bigcup_n \Omega_n$ with $\mu(\Omega_n) < \infty$, and E is a separable Banach space.

2.2.1 Scalar Multiplication Operators

First we want to make precise what we understand by a multiplication operator on $L^p(\Omega)$.

Definition 2.2.1. For a function $m : \Omega \to \mathbb{C}$ we define on $L^p(\Omega)$ the **scalar multiplication operator** M_m associated to m by

$$\begin{aligned} D(M_m) &= \{f \in L^p(\Omega) : mf \in L^p(\Omega)\}, \\ M_m f &= mf. \end{aligned}$$

Clearly M_m is a linear operator, but in general unbounded. The following proposition gives an answer to the natural question, under which assumptions on m, the operator M_m is bounded.

Proposition 2.2.2. *With the notation above the following assertions are equivalent.*

(i) $M_m \in \mathcal{L}(L^p(\Omega))$.

(ii) $m \in L^\infty(\Omega)$.

In this case $\|M_m\| = \|m\|_\infty$.

Proof. *(i)* $\Rightarrow$ *(ii)* First assume that $\mu(\Omega) < \infty$, then $\chi_\Omega \in L^p(\Omega)$ and thus one has $m = m\chi_\Omega = M_m\chi_\Omega \in L^p(\Omega)$, hence m is measurable. If $\mu(\Omega) = \infty$, let $\Omega = \bigcup_n \Omega_n$, with Ω_n disjoint and $\mu(\Omega_n) < \infty$. Then $m(\omega) = (M_m\chi_{\Omega_n})(\omega)$ if $\omega \in \Omega_n$ and m is measurable.
Let $A = \{\omega \in \Omega : |m(\omega)| > \|M_m\|\}$, and suppose $\mu(A) < \infty$, otherwise consider a subset of finite measure.. Then $f := \chi_A \in L^p(\Omega)$ and $\|f\| = (\mu(A))^{1/p}$. If $\mu(A) > 0$, then $f \neq 0$ and by the boundedness of M_m

$$\begin{aligned} \|M_m\|\,\|f\| &\geq \|M_m f\| = (\int_\Omega |m(\omega)\chi_A(\omega)|^p)^{1/p} \\ &= (\int_A |m(\omega)|^p)^{1/p} > \|M_m\|(\mu(A))^{1/p} = \|M_m\|\,\|f\|, \end{aligned}$$

which is a contradiction, hence $\mu(A) = 0$. Therefore $|m(\omega)| \leq \|M_m\|$ a.e., thus $m \in L^\infty(\Omega)$ and $\|m\|_\infty \leq \|M_m\|$.
(ii) $\Rightarrow$ *(i)* $m \in L^\infty(\Omega)$ implies $\|M_m f\|_p = \left(\int_\Omega |mf|^p\right)^{1/p} \leq \|m\|_\infty \|f\|_p$, for all $f \in L^p(\Omega)$, hence $M_m \in \mathcal{L}(L^p(\Omega))$ and $\|M_m\| \leq \|m\|_\infty$. □

Now we can show that on $L^p(\Omega)$ the bounded multiplication operators coincide with the center.

Theorem 2.2.3. *For a linear operator* $T : L^p(\Omega) \to L^p(\Omega)$ *the following assertions are equivalent.*

(i) *T is a bounded multiplication operator,*
i.e. there exists $m \in L^\infty(\Omega)$ such that $T = M_m$.

(ii) *$T(\varphi f) = \varphi(Tf)$ for all $f \in L^p(\Omega)$ and $\varphi \in L^\infty(\Omega)$.*

(iii) *$Tf = 0$ a.e. on the set $\{\omega \in \Omega : f(\omega) = 0\}$ for all $f \in L^p(\Omega)$,*
i.e. $T \in \mathcal{Z}(L^p(\Omega))$.

Proof. *(i)* $\Rightarrow$ *(ii)* $T(\varphi f) = m\varphi f = \varphi m f = \varphi(Tf)$.
(ii) $\Rightarrow$ *(iii)* Let $A = \{\omega \in \Omega : f(\omega) \neq 0\}$, then $Tf = T(\chi_A f) = \chi_A(Tf) = 0$ a.e. on $A^C = \{\omega \in \Omega : f(\omega) = 0\}$.
(iii) $\Rightarrow$ *(i)* Let T be a linear operator on $L^p(\Omega)$, such that $(Tf)(\omega) = 0$ a.e. on $\{\omega \in \Omega : f(\omega) = 0\}$ for all $f \in L^p(\Omega)$, then T is band preserving and therefore $T \in \mathcal{L}(L^p(\Omega))$ by Proposition 2.1.17. Further $T(\chi_A f) = \chi_A Tf$ for each measurable subset $A \subset \Omega$ and all $f \in L^p(\Omega)$. Indeed, $A^C \subset \{\omega \in \Omega : \chi_A f = 0\}$, hence $T(\chi_A f) = 0$ a.e. on A^C, which implies $\chi_{A^C} T(\chi_A f) = 0$. Therefore

$$\begin{aligned} T(\chi_A f) &= \chi_A T(\chi_A f) + \chi_{A^C} T(\chi_A f) \\ &= \chi_A T(\chi_A f) + \chi_A T(\chi_{A^C} f) = \chi_A Tf. \end{aligned}$$

Let $0 \neq f \in L^p(\Omega)$, let $c > 0$ and let $A = \{\omega \in \Omega : |Tf(\omega)| > c|f(\omega)|\}$. Assume that $\mu(A) > 0$ and let $g = \chi_A f$. Then $0 \neq g \in L^p(\Omega)$ and for almost all $\omega \in A$, $c|g(\omega)| = c|f(\omega)| < |Tf(\omega)|$, whereas $c|g(\omega)| = 0$ almost everywhere on A^C. Therefore $c|g| < \chi_A|Tf| = |\chi_A Tf| = |T(\chi_A f)| = |Tg|$ a.e., hence $c\|g\|_{L^p} < \|Tg\|_{L^p} \leq \|T\|\ \|g\|_{L^p}$. Since $g \neq 0$, it follows that $c < \|T\|$. This shows that

$$|Tf(\omega)| \leq \|T\|\ |f(\omega)| \quad \text{a.e. for all } f \in L^p. \tag{2.1}$$

Assume next that $\mu(\Omega) < \infty$. Then $\chi_\Omega \in L^p(\Omega)$. Let $m = T\chi_\Omega$. Then $m \in L^p(\Omega)$ is measurable. It follows from (2.1) that $m \in L^\infty(\Omega, \mu)$ and $\|m\|_\infty \leq \|T\|$. Let A be measurable. Then $T\chi_A = T\chi_A\chi_\Omega = \chi_A T\chi_\Omega = m\chi_A$. Thus $Tf = mf$ for all simple functions. Consequently, $Tf = mf$ for all $f \in L^p(\Omega)$ by density.
Finally, if $\mu(\Omega) = \infty$, choose $\Omega_k \subset \Omega$ measurable such that $\Omega_k \cap \Omega_l = \emptyset$ for $k \neq l$, $\mu(\Omega_k) < \infty$ and $\bigcup_{k\in\mathbb{N}} \Omega_k = \Omega$. Let $m_k = T\chi_{\Omega_k}$ and $m(\omega) = m_k(\omega)$ if $\omega \in \Omega_k$. Then m is measurable and as above, $|m(\omega)| \leq \|T\|$ a.e. and $Tf = mf$ for all $f \in L^p(\Omega)$. □

In the following we consider $(\Omega, \mathcal{B}, \mu)$, where Ω is a locally compact Hausdorff space, $\mathcal{B}$ the Borel σ-algebra and μ a Borel measure on Ω.
In the latter case we assume μ to satisfy the following properties:

(i) μ is positive,

(ii) $\mu(K) < \infty$ for every compact set $K \subset \Omega$,

(iii) for every $B \subset \mathcal{B}$, we have

$$\mu(B) = \inf\{\mu(O) : B \subset O \text{ open}\},$$

(iv) the relation

$$\mu(B) = \sup\{\mu(K) : B \supset K \text{ compact}\}$$

holds for every open set B, and for every $B \subset \mathcal{B}$ with $\mu(B) < \infty$.

Further we require Ω to be σ-compact, i.e. there exist $K_n \subset \Omega$ compact, such that $\Omega = \bigcup_n K_n$.

Remark 2.2.4. By [Ru], Theorem 2.18, if each open subset O of Ω is σ-compact, in particular if $\Omega = \mathbb{R}^n$, then it suffices to assume (i) and (ii) above, in order that the measure μ is regular, i.e. (iii) and (iv) hold for every $B \subset \mathcal{B}$.

Definition 2.2.5. For $f : \Omega \to \mathbb{R}$ measurable, we call the **support** of f the set $\operatorname{supp} f := \Omega \setminus O_f$ with $O_f := \{\omega \in \Omega : \exists U \in \mathcal{U}(\omega) : f_{|U} = 0 \text{ } a.e.\}$, which is open in Ω.

Proposition 2.2.6. *For $f : \Omega \to \mathbb{R}$ measurable, $f = 0$ almost everywhere on O_f.*

Proof. By property (iv) we find compact subsets $K_n \subset K_{n+1} \subset O_f$, such that $\bigcup_n K_n = O_f$ almost everywhere. Since every open covering admits a finite subcovering, $f = 0$ almost everywhere on K_n for all n. Since a countable union of null sets is a null set, we obtain that $f = 0$ almost everywhere on O_f. □

Our aim is again to characterize multiplication operators in this situation, where the underlying measure space has a topology. Note that here we assume a priori, that the linear operator is bounded.

Theorem 2.2.7. *Let $1 \leq p \leq \infty$. For an operator $T \in \mathcal{L}(L^p(\Omega))$, the following assertions are equivalent.*

(i) There exists an $m \in L^\infty(\Omega)$ such that $Tf = mf$.

(ii) $\operatorname{supp} Tf \subset \operatorname{supp} f$ *for all $f \in C_c(\Omega)$.*

Proof. Let T be a bounded multiplication operator, i.e. there exists $m \in L^\infty(\Omega)$ such that $Tf = mf$ for all $f \in L^p(\Omega)$. Let $f \in C_c(\Omega)$. Then by Theorem 2.2.3, $Tf = 0$ on the set $\{f = 0\}$, which contains almost all of O_f. Therefore $Tf = 0$ almost everywhere on O_f, hence $\operatorname{supp} Tf \subset \operatorname{supp} f$ for all $f \in C_c(\Omega)$.
Conversely, suppose that $\operatorname{supp} Tf \subset \operatorname{supp} f$ for all $f \in C_c(\Omega)$. Thus for every open subset $O \subset \Omega$, $f = 0$ on O implies $Tf = 0$ a.e. on O, for all $f \in C_c(\Omega)$.
First we want to show, that for all $f \in L^p(\Omega)$, $\operatorname{supp} Tf \subset \operatorname{supp} f$, i.e. $Tf = 0$ a.e. on O_f. Let $f \in L^p(\Omega)$, and choose $\omega \in O_f$ and let $K \subset O_f$ be a compact neighborhood of ω. For $\Psi \in C_c(\Omega)$, such that $0 \leq \Psi \leq 1$, $\Psi \equiv 1$ on K and

$\operatorname{supp}\Psi \subset O_f$ and $f_n \in C_c(\Omega)$ such that $f_n \to f$ in $L^p(\Omega)$, we obtain that $C_c(\Omega) \ni f_n - \Psi f_n \to f - \Psi f = f$ (a.e.).
As $f_n - \Psi f_n = 0$ a.e. on $\overset{\circ}{K}$, the open interior of K, by assumption $T(f_n - \Psi f_n) = 0$ a.e. on $\overset{\circ}{K}$. Hence as $n \to \infty$, and if necessary by passing to a subsequence, we obtain that $Tf = 0$ a.e. on $\overset{\circ}{K}$. Since $\omega \in O_f$ was arbitrary, we get $Tf = 0$ a.e. on O_f.
We deduce, that for every open subset $O \subset \Omega$, $f(\omega) = 0$ almost everywhere on O implies $Tf(\omega) = 0$ almost everywhere on O, for every $f \in L^p(\Omega)$.
Let $A = \{\omega \in \Omega : f(\omega) = 0\}$, then $A^C = \{\omega \in \Omega : f(\omega) \neq 0\}$. First assume, that $\mu(A^C) < \infty$. Then by property (iii) there exist $O_n \subset \Omega$ open such that $\mu(O_n) < \infty$, $A^C \subset O_{n+1} \subset O_n$ and $\mu(O_n \setminus A^C) = \mu(O_n \cap A) \to 0$ for $n \to \infty$, and by property (iv) there exist $C_n \subset O_n$ compact, such that $\mu(O_n \setminus C_n) < \frac{1}{n}$.
Let $\varphi_n = \chi_{C_n} \in L^p(\Omega)$. Then $\varphi_n f \to f$ in measure and hence for a suitable subsequence in $L^p(\Omega)$ by the dominated convergence theorem. Now we have $\varphi_n(\omega)f(\omega) = 0$ on C_n^C, which is open and hence $T(\varphi_n f) = 0$ (a.e.) on $O_n^C \subset C_n^C$. Letting $n \to \infty$, as $T(\varphi_n f) \to T(f)$ almost everywhere for a suitable subsequence, we obtain $T(f) = 0$ (a.e.) on $\bigcup_n O_n^C$. But $O_n^C \subset O_{n+1}^C \subset A$ and therefore $\mu(A \setminus O_n^C) = \mu(A \cap O_n) \to 0$ yields $T(f) = 0$ (a.e.) on A, if $\mu(A^C) < \infty$.
Now let $\mu(A^C) = \infty$. Since $\Omega = \bigcup_n K_n$, with K_n compact, we obtain that $A^C = \bigcup_n (A^C \cap K_n)$, with $\mu(A^C \cap K_n) < \infty$. Consequently $A = \bigcap_n (A^C \cap K_n)^C$. As above we can show that $T(f) = 0$ a.e. on $(A^C \cap K_n)^C$ for all $n \in \mathbb{N}$. Hence $T(f) = 0$ a.e. on $\bigcap_n (A^C \cap K_n)^C = A$. □

2.2.2 Operator Valued Multiplication Operators

Similar to multiplication operators on $L^p(\Omega)$ we study operator valued multiplication operators on $L^p(\Omega, E)$. We denote as usual by $\mathcal{L}(E)$ the space of bounded linear operators on E.
The role of $L^\infty(\Omega)$ will now be played by an operator valued L^∞.

Definition 2.2.8. Let $\mathcal{L}_s(E)$ denote the space of all bounded linear operators on E, provided with the strong operator topology and

$$L^\infty(\Omega, \mathcal{L}_s(E)) := \{[M : \Omega \to \mathcal{L}(E)] : \omega \mapsto M(\omega)x \in L^\infty(\Omega, E) \text{ for all } x \in E\}.$$

Lemma 2.2.9. *Let $M \in L^\infty(\Omega, \mathcal{L}_s(E))$ and $f : \Omega \to E$ be measurable, then $Mf : \Omega \to E$ defined by $\omega \mapsto M(\omega)f(\omega)$ is measurable.*

Proof. First assume that $f = \chi_A \cdot x$ for some measurable subset $A \subset \Omega$ and $x \in E$. Then $M(\omega)f(\omega) = M(\omega)\chi_A(\omega)x = \chi_A(\omega)M(\omega)x$ for all $\omega \in \Omega$. Hence Mf is measurable as the product of the two measurable functions $M(\cdot)x \in L^\infty(\Omega, E)$ and χ_A. Further $M(\omega)$ is linear for every $\omega \in \Omega$ and consequently Mf is measurable for a step function $f = \sum_{j=1}^n \chi_{A_j} x_j$ since $M(\omega)f(\omega) = \sum_{j=1}^n \chi_{A_j} M(\omega)x_j$.

If f is an arbitrary measurable function, then there exist a sequence of step functions f_n such that $f(\omega) = \lim_{n\to\infty} f_n(\omega)$ for almost every $\omega \in \Omega$. As the operators $M(\omega)$ are continuous for every $\omega \in \Omega$, we obtain the convergence $M(\omega)f(\omega) = M(\omega)(\lim_{n\to\infty} f_n(\omega)) = \lim_{n\to\infty} M(\omega)f_n(\omega)$. Hence Mf is measurable as the almost everywhere pointwise limit of measurable functions. $\square$

Lemma 2.2.10. *Let $M \in L^\infty(\Omega, \mathcal{L}_s(E))$, then*

(i) there exists a constant C and a nullset $\mathcal{N}$, such that for all $\omega \in \Omega \setminus \mathcal{N}$ and for all $x \in E$, $\|M(\omega)x\|_E \leq C\|x\|_E$.

(ii) $\omega \mapsto \|M(\omega)\|_{E\to E}$ is measurable.

Proof. (i)Define an operator $\Phi : E \to L^\infty(\Omega, E)$ by $x \mapsto M(\cdot)x$. Obviously Φ is linear. Further Φ is closed, because if $x_n \to x$ in E and $M(\cdot)x_n \to f$ in $L^\infty(\Omega, E)$, then $M(\omega)x_n \to f(\omega)$ almost everywhere. But $M(\omega)$ is continuous for every $\omega \in \Omega$, hence $M(\omega)x_n \to M(\omega)x$ everywhere. This shows $f = M(\cdot)x$, thus Φ is closed and therefore bounded by the closed graph theorem. Hence there exists a constant C such that $\|M(\cdot)x\|_\infty \leq C\|x\|_E$, therefore $\|M(\omega)x\| \leq C\|x\|_E$ almost everywhere, where the exceptional null set depends on x. However, since E is separable, there exists $\{x_n : n \in \mathbb{N}\}$ dense in E. For all $n \in \mathbb{N}$ let N_n be the exceptional null set for x_n, then $\mathcal{N} := \bigcup_{n\in\mathbb{N}} N_n$ is a null set and

$$\|M(\omega)x_n\|_E \leq C\|x_n\|_E \quad \text{for all } n \in \mathbb{N} \quad \text{and all } \omega \notin \mathcal{N}.$$

As $\{x_n : n \in \mathbb{N}\}$ is dense in E, this implies $\|M(\omega)x\|_E \leq C\|x\|_E$ for all $x \in E$ and all $\omega \notin \mathcal{N}$, i.e. almost all $\omega \in \Omega$.
(ii) Observe, that we identify the two elements M and $\tilde{M} := M\chi_{\Omega\setminus\mathcal{N}}$ in the space $L^\infty(\Omega, \mathcal{L}_s(E))$, where $\mathcal{N}$ is the nullset of part (i). For all $x \in E$, the map $\omega \mapsto \tilde{M}(\omega)x \in L^\infty(\Omega, E)$ is measurable, hence $\omega \mapsto \|\tilde{M}(\omega)x\|_E$ is measurable, as $\|.\|_E$ is continuous. Since E is a separable Banach space, there exist $x_n \in B_E := \{x \in E : \|x\| \leq 1\}$ such that $\{x_n : n \in \mathbb{N}\}$ is dense in B_E. Then $\|\tilde{M}(\omega)\| = \sup_n \|\tilde{M}(\omega)x_n\|$, hence $\omega \mapsto \|\tilde{M}(\omega)\|$ is measurable, as $\omega \mapsto \|\tilde{M}(\omega)x_n\|_E$ is measurable for every $n \in \mathbb{N}$ and the supremum exists because by the choice of $\tilde{M}$ and part (i), $\|\tilde{M}(\omega)x\|_E \leq C\|x\|_E$ for all $\omega \in \Omega$ $\square$

Proposition 2.2.11. $L^\infty(\Omega, \mathcal{L}_s(E))$ *provided with the essential supremum norm* $\|M\|_\infty := \operatorname{ess\,sup}_{\omega\in\Omega} \|M(\omega)\|_{E\to E}$ *is a Banach algebra under pointwise multiplication.*

Proof. $L^\infty(\Omega, \mathcal{L}_s(E))$ is obviously a vector space over $\mathbb{C}$.
For every element $M \in L^\infty(\Omega, \mathcal{L}_s(E))$ the essential supremum norm is finite, hence well defined. Indeed by Lemma 2.2.10(i) there exists a nullset $\mathcal{N}$, such that $\|M(\omega)x\|_E \leq C\|x\|_E$ for all $x \in E$ and all $\omega \notin \mathcal{N}$, thus $\|M(\omega)\|_{E\to E} \leq C$ for almost all $\omega \in \Omega$. Therefore $\|M\|_\infty := \operatorname{ess\,sup}_{\omega\in\Omega} \|M(\omega)\|_{E\to E} < \infty$.

Obviously $\|.\|_\infty$ defines a norm on $L^\infty(\Omega, \mathcal{L}_s(E))$.
$L^\infty(\Omega, \mathcal{L}_s(E))$ is complete, hence a Banach space. Indeed, let M_n be a Cauchy sequence in $L^\infty(\Omega, \mathcal{L}_s(E))$, i.e. for all $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $n, m \geq n_0$ implies $\|M_n - M_m\|\infty < \varepsilon$. Further by Remark 1.1.7 for all $(n, m) \in \mathbb{N} \times \mathbb{N}$ there exists a null set $N_{n,m}$ such that

$$\|M_n - M_m\|_\infty = \sup_{\omega \in \Omega \setminus N_{n,m}} \|M_n(\omega) - M_m(\omega)\|.$$

Hence for the null set $\mathcal{N} := \bigcup_{n,m \in \mathbb{N}} N_{n,m}$ we obtain

$$\sup_{\omega \in \Omega \setminus \mathcal{N}} \|M_n(\omega) - M_m(\omega)\| < \varepsilon \quad \text{for all } n, m > n_0(\varepsilon). \tag{2.2}$$

Thus for all $\omega \in \Omega \setminus \mathcal{N}$, $M_n(\omega)$ is a Cauchy sequence in $\mathcal{L}(E)$, hence convergent. For $\omega \in \Omega \setminus \mathcal{N}$ let $M(\omega) := \lim_{n\to\infty} M_n(\omega)$ and for $\omega \in \mathcal{N}$ let $M(\omega) = 0$. Then $M(\omega) \in \mathcal{L}(E)$ for all $\omega \in \Omega$. Further, for all $x \in E$,

$$M(\omega)x := \chi_{\Omega \setminus \mathcal{N}}(\lim_{n\to\infty} M_n)(\omega)x = \lim_{n\to\infty} \chi_{\Omega \setminus \mathcal{N}} M_n(\omega)x$$

is measurable as the pointwise limit of measurable functions. Moreover, from (2.2) we conclude, that

$$\sup_{\omega \in \Omega \setminus \mathcal{N}} \|M_n(\omega) - M(\omega)\| < \varepsilon \quad \text{for all } n > n_0(\varepsilon). \tag{2.3}$$

Then for all $x \in E$,

$$\sup_{\omega \in \Omega \setminus \mathcal{N}} \|M_n(\omega)x - M(\omega)x\| < \varepsilon\|x\| \quad \text{for all } n > n_0(\varepsilon).$$

Therefore $\|M_n(\cdot)x - M(\cdot)x\|_\infty \to 0$, which implies $M(\cdot)x \in L^\infty(\Omega, E)$. Hence $M \in L^\infty(\Omega, \mathcal{L}_s(E))$. On the other hand (2.3) implies $\|M_n(\cdot) - M(\cdot)\|_\infty \to 0$, which proves that $L^\infty(\Omega, \mathcal{L}_s(E))$ is complete.
For $M_1,\ M_2 \in L^\infty(\Omega, \mathcal{L}_s(E))$ we define the product by pointwise multiplication, hence $(M_1 \cdot M_2)(\omega) := M_1(\omega) \circ M_2(\omega) \in \mathcal{L}(E)$ for all $\omega \in \Omega$ and

$$(M_1 \cdot M_2)(\cdot)x = (M_1(\cdot) \circ M_2(\cdot))x = M_1(\cdot)(M_2(\cdot)x)$$

is measurable by Lemma 2.2.9, since $M_2(\cdot)x \in L^\infty(\Omega, E)$ is measurable. Further we have by (1.1),

$$\begin{aligned}
\|(M_1 \cdot M_2)(\cdot)x\|_{L^\infty(\Omega,E)} &= \|(M_1(\cdot) \circ M_2(\cdot))x\|_{L^\infty(\Omega,E)} \\
&= \operatorname*{ess\,sup}_{\omega \in \Omega} \|M_1(\omega)(M_2(\omega)x)\| \\
&\leq \operatorname*{ess\,sup}_{\omega \in \Omega} (\|M_1(\omega)\| \cdot \|M_2(\omega)x\|) \\
&\leq \operatorname*{ess\,sup}_{\omega \in \Omega} \|M_1(\omega)\| \cdot \operatorname*{ess\,sup}_{\omega \in \Omega} \|M_2(\omega)x\| < \infty,
\end{aligned}$$

which shows that $M_1 \cdot M_2 \in L^\infty(\Omega, \mathcal{L}_s(E))$. Thus $(M_1, M_2) \mapsto M_1 \cdot M_2$ is an associative bilinear mapping from $L^\infty(\Omega, \mathcal{L}_s(E)) \times L^\infty(\Omega, \mathcal{L}_s(E))$ to $L^\infty(\Omega, \mathcal{L}_s(E))$. Finally, again with (1.1) we obtain

$$\begin{aligned}
\|M_1 \cdot M_2\|_\infty &= \operatorname*{ess\,sup}_{\omega\in\Omega} \|(M_1 \cdot M_2)(\omega)\| \\
&= \operatorname*{ess\,sup}_{\omega\in\Omega} \|M_1(\omega) \circ M_2(\omega)\| \\
&\leq \operatorname*{ess\,sup}_{\omega\in\Omega} (\|M_1(\omega)\| \cdot \|M_2(\omega)\|) \\
&\leq \operatorname*{ess\,sup}_{\omega\in\Omega} \|M_1(\omega)\| \cdot \operatorname*{ess\,sup}_{\omega\in\Omega} \|M_2(\omega)\| \\
&= \|M_1\|_\infty \cdot \|M_2\|_\infty .
\end{aligned}$$

Therefore $L^\infty(\Omega, \mathcal{L}_s(E))$ is a Banach algebra. □

Lemma 2.2.12. *Let $M \in L^\infty(\Omega, \mathcal{L}_s(E))$ and $f \in L^p(\Omega, E)$ be measurable, then $Mf : \Omega \to E$ defined by $\omega \mapsto M(\omega)f(\omega)$ is in $L^p(\Omega, E)$.*

Proof. By Lemma 2.2.9 $\omega \mapsto M(\omega)f(\omega)$ is measurable. Further there exists a null set $\mathcal{N}$ such that $\|M\|_\infty = \sup_{\omega\in\Omega\setminus\mathcal{N}} \|M(\omega)\|$. Hence for $1 \leq p < \infty$ and all $f \in L^p(\Omega, E)$ we have

$$\begin{aligned}
\|Mf\|_{L^p(\Omega,E)} &= \left(\int_\Omega \|M(\omega)f(\omega)\|_E^p\right)^{1/p} \\
&\leq \left(\int_{\Omega\setminus\mathcal{N}} \|M(\omega)\|_{E\to E}^p \, \|f(\omega)\|_E^p\right)^{1/p} \\
&\leq \sup_{\omega\in\Omega\setminus\mathcal{N}} \|M(\omega)\| \left(\int_{\Omega\setminus\mathcal{N}} \|f(\omega)\|_E^p\right)^{1/p} \\
&= \|M\|_\infty \, \|f\|_{L^p(\Omega,E)} .
\end{aligned}$$

For $p = \infty$ and $f \in L^\infty(\Omega, E)$ we have

$$\begin{aligned}
\|Mf\|_\infty &= \operatorname*{ess\,sup}_{\omega\in\Omega} \|M(\omega)f(\omega)\| \\
&\leq \operatorname*{ess\,sup}_{\omega\in\Omega} \|M(\omega)\| \operatorname*{ess\,sup}_{\omega\in\Omega} \|f(\omega)\| \\
&= \|M\|_\infty \, \|f\|_\infty .
\end{aligned}$$

Hence for all $1 \leq p \leq \infty$

$$\|Mf\|_{L^p(\Omega,E)} \leq \|M\|_\infty \, \|f\|_{L^p(\Omega,E)}, \tag{2.4}$$

therefore $\omega \mapsto M(\omega)f(\omega)$ is in $L^p(\Omega, E)$. □

Corollary 2.2.13. For $M \in L^\infty(\Omega, \mathcal{L}_s(E))$, $\mathcal{M}_M : f \mapsto M(\cdot)f(\cdot)$ defines a bounded linear operator on $L^p(\Omega, E)$, i.e. $\mathcal{M}_M \in \mathcal{L}(L^p(\Omega, E))$.

Proof. $\mathcal{M}_M$ is obviously linear and the boundedness follows from (2.4). □

Proposition 2.2.14. *The map $\Phi : L^\infty(\Omega, \mathcal{L}_s(E)) \to \mathcal{L}(L^p(\Omega, E))$ defined by $M \mapsto \mathcal{M}_M$ is an isometric algebra homomorphism.*

Proof. Observe that for $M_1,\ M_2 \in L^\infty(\Omega, \mathcal{L}_s(E))$ and $\lambda \in \mathbb{C}$

$$\begin{aligned}\mathcal{M}_{\lambda M_1 + M_2} f &= (\lambda M_1(\cdot) + M_2(\cdot))f(\cdot) = \lambda M_1(\cdot)f(\cdot) + M_2(\cdot)f(\cdot) \\ &= \lambda \mathcal{M}_{M_1} f + \mathcal{M}_{M_2} f = (\lambda \mathcal{M}_{M_1} + \mathcal{M}_{M_2})f\end{aligned}$$

and

$$\begin{aligned}\mathcal{M}_{M_1 \cdot M_2} f &= (M_1(\cdot) \circ M_2(\cdot))f(\cdot) = M_1(\cdot)(M_2(\cdot)f(\cdot)) \\ &= \mathcal{M}_{M_1}(\mathcal{M}_{M_2} f) = (\mathcal{M}_{M_1} \circ \mathcal{M}_{M_2})f.\end{aligned}$$

Hence Φ is linear and multiplicative, i.e. an algebra homomorphism.
We still have to show that $\|\mathcal{M}_M\| = \|M\|_\infty$. By the inequality (2.4) we get $\|\mathcal{M}_M f\| = \|Mf\| \le \|M\|_\infty \|f\|$, hence $\|\mathcal{M}_M\| \le \|M\|_\infty$. For the converse inequality define for each $x \in E$ the set $A_x := \{\omega \in \Omega : \|M(\omega)x\| > \|\mathcal{M}_M\| \|x\|\}$, which is measurable, since $\omega \mapsto \|M(\omega)x\| \in L^\infty(\Omega)$ is measurable. We can assume $\mu(A_x) < \infty$, otherwise consider a subset of finite measure. Then the function $f := \chi_{A_x} \cdot x \in L^p(\Omega, E)$ for all $x \in E$ and $\|f\| = \|x\|(\mu(A_x))^{1/p}$ for $1 \le p < \infty$ and $\|f\| = \|x\|$ for $p = \infty$. Suppose that $\mu(A_x) > 0$. Then by the continuity of $\mathcal{M}_M$ we get for $1 \le p < \infty$

$$\begin{aligned}\|\mathcal{M}_M\|\,\|x\|(\mu(A_x))^{1/p} &= \|\mathcal{M}_M\|\,\|f\| \ge \|\mathcal{M}_M f\|_p \\ &= \left(\int_\Omega \|M(\omega)\chi_{A_x}(\omega)x\|_E^p\right)^{1/p} \\ &= \left(\int_{A_x} \|M(\omega)x\|_E^p\right)^{1/p} \\ &> \|\mathcal{M}_M\|\,\|x\|(\mu(A_x))^{1/p},\end{aligned}$$

and for $p = \infty$

$$\begin{aligned}\|\mathcal{M}_M\|\,\|x\| &= \|\mathcal{M}_M\|\,\|f\| \ge \|\mathcal{M}_M f\|_\infty \\ &= \operatorname*{ess\,sup}_{\omega \in \Omega} \|M(\omega)\chi_{A_x}(\omega)x\|_E \\ &= \operatorname*{ess\,sup}_{\omega \in A_x} \|M(\omega)x\|_E \\ &> \|\mathcal{M}_M\|\,\|x\|,\end{aligned}$$

which leads in both cases to a contradiction. Hence $\mu(A_x) = 0$ and therefore $\|M(\omega)x\| \le \|\mathcal{M}_M\|\,\|x\|$ almost everywhere, where the exceptional null set depends on x. Now we proceed in the same way as in the proof of Lemma 2.2.10.

Since E is separable, there exists $\{x_n : n \in \mathbb{N}\}$ dense in E. For all $n \in \mathbb{N}$ let N_n be the exceptional null set for x_n, then $\mathcal{N} := \bigcup_{n\in\mathbb{N}} N_n$ is a null set and

$$\|M(\omega)x_n\|_E \le \|\mathcal{M}_M\| \, \|x_n\|_E \quad \text{for all } n \in \mathbb{N} \quad \text{and all } \omega \notin \mathcal{N}.$$

As $\{x_n : n \in \mathbb{N}\}$ is dense in E, this implies $\|M(\omega)x\|_E \le \|\mathcal{M}_M\| \, \|x\|_E$ for all $x \in E$ and all $\omega \notin \mathcal{N}$, thus $\|M(\omega)\|_{E\to E} \le \|\mathcal{M}_M\|$ for almost all $\omega \in \Omega$. Therefore $\|M\|_\infty := \operatorname{ess\,sup}_{\omega\in\Omega} \|M(\omega)\|_{E\to E} \le \|\mathcal{M}_M\|$. □

Definition 2.2.15. We call an operator $\mathcal{M} \in \mathcal{L}(L^p(\Omega, E)$ a **bounded operator valued multiplication operator** if $\mathcal{M} = \mathcal{M}_M$ for some $M \in L^\infty(\Omega, \mathcal{L}_s(E))$, i.e. $\mathcal{M} \in \Phi(L^\infty(\Omega, \mathcal{L}_s(E))$.

Remark 2.2.16. Note that Φ is an isometric isomorphism from $L^\infty(\Omega, \mathcal{L}_s(E))$ to the bounded operator valued multiplication operators.
In particular, since $L^\infty(\Omega, \mathcal{L}_s(E))$ is complete, the set of bounded operator valued multiplication operators is closed in $\mathcal{L}(L^p(\Omega, E))$.

We have the following characterization. Here again, we write for short $\{f = 0\}$ for the measurable set $\{\omega \in \Omega : f(\omega) = 0\}$.

Theorem 2.2.17. *For a linear operator $T : L^p(\Omega, E) \to L^p(\Omega, E)$ the following assertions are equivalent.*

(i) T is a bounded operator valued multiplication operator,
i.e. there exists $M \in L^\infty(\Omega, \mathcal{L}_s(E))$ such that $T = \mathcal{M}_M$.

(ii) $T(\varphi f) = \varphi(Tf)$ for all $f \in L^p(\Omega, E)$ and $\varphi \in L^\infty(\Omega)$.

(iii) $Tf = 0$ a.e. on the set $\{f = 0\}$ for all $f \in L^p(\Omega, E)$, and $T_{|L^p(\tilde\Omega,E)}$ is bounded for every purely atomic subset $\tilde\Omega \subset \Omega$, i.e. $T \in \mathcal{Z}(L^p(\Omega, E))$.

Proof. $(i) \Rightarrow (ii)$ Let $f \in L^p(\Omega, E)$ and $\varphi \in L^\infty(\Omega)$, then

$$\begin{aligned}(T(\varphi f)(\omega)) &= M(\omega)(\varphi f)(\omega) = M(\omega)\varphi(\omega)f(\omega) \\ &= \varphi(\omega)M(\omega)f(\omega) = \varphi(\omega)(Tf)(\omega)\end{aligned}$$

for almost all $\omega \in \Omega$, hence $T(\varphi f) = \varphi(Tf)$.
$(ii) \Rightarrow (iii)$ Let $A = \{\omega \in \Omega : f(\omega) \ne 0\}$, then $T(\chi_A f) = \chi_A(Tf) = 0$ a.e. on $A^C = \{f = 0\}$.
$(iii) \Rightarrow (i)$ We proceed as in the proof of Theorem 2.2.3. Let T satisfy $Tf = 0$ a.e. on the set $\{f = 0\}$ and $T_{|L^p(\tilde\Omega,E)}$ is bounded for every purely atomic subset $\tilde\Omega$ of Ω. Then $T \in \mathcal{L}(L^p(\Omega, E))$ by Corollary 2.1.24. Further for all $A \subset \Omega$ measurable, $\chi_{A^C} T(\chi_A f) = 0$, because $A^C \subset \{\chi_A f = 0\}$ and therefore $T(\chi_A f) = 0$ on A^C.

Thus

$$\begin{aligned} T(\chi_A f) &= \chi_A T(\chi_A f) + \chi_{A^C} T(\chi_A f) \\ &= \chi_A T(\chi_A f) + \chi_A T(\chi_{A^C} f) = \chi_A T f \end{aligned}$$

Let $0 \neq f \in L^p(\Omega, E)$ and $c > 0$, define $A := \{\omega \in \Omega : \|(Tf)(\omega)\| > c\|f(\omega)\|\}$. Suppose, that $\mu(A) > 0$, then $0 \neq g := \chi_A f \in L^p(\Omega, E)$ and

$$c\|g(\omega)\| = \begin{cases} c\|f(\omega)\| < \|(Tf)(\omega)\| = \|(Tg)(\omega)\| & \omega \in A \\ 0 < \|(Tg)(\omega)\| & \omega \in A^C \end{cases}$$

and consequently $c\|g\|_{L^p} < \|Tg\|_{L^p} \leq \|T\| \, \|g\|_{L^p}$. Since $g \neq 0$ we conclude $c < \|T\|$, which shows

$$\|(Tf)(\omega)\|_E \leq \|T\| \, \|f(\omega)\|_E \quad \text{a.e. for all } f \in L^p(\Omega, E). \tag{2.5}$$

Now first assume that $\mu(\Omega) < \infty$, then for all $x \in E$, $\chi_\Omega \cdot x \in L^p(\Omega, E)$. Therefore, we can define $M_x := T(\chi_\Omega \cdot x)$, which is an element of $L^\infty(\Omega, E)$, because $T(\chi_\Omega \cdot x) \in L^p(\Omega, E)$ is measurable and $\|M_x\|_\infty \leq \|T\| \, \|x\|$ by (2.5). Then in view of

$$\begin{aligned} M_{\lambda x_1 + x_2} &= T(\chi_\Omega \cdot (\lambda x_1 + x_2)) = T(\lambda \chi_\Omega \cdot x_1 + \chi_\Omega \cdot x_2)) \\ &= \lambda T(\chi_\Omega \cdot x_1) + T(\chi_\Omega \cdot x_2) = \lambda M_{x_1} + M_{x_2} \end{aligned}$$

for all $\lambda \in \mathbb{C}$ and $x_1, x_2 \in E$, the map $E \to L^\infty(\Omega, E) : x \mapsto M_x$ is linear. In particular $\|M_{x_1} - M_{x_2}\|_\infty = \|M_{x_1 - x_2}\|_\infty \leq \|T\| \, \|x_1 - x_2\|$. Again we use the fact that E is separable, hence there exists $\{x_n : n \in \mathbb{N}\}$ dense in E. Then there exists a null set $\mathcal{N} \subset \Omega$ such that for all $\omega \in \Omega \setminus \mathcal{N}$ and all $n, m \in \mathbb{N}$

$$\|M_{x_n}(\omega)\| \leq \|T\| \, \|x_n\| \tag{2.6}$$

and

$$\|M_{x_n}(\omega) - M_{x_m}(\omega)\| \leq \|T\| \, \|x_n - x_m\| \tag{2.7}$$

and

$$M_{x_n}(\omega) + M_{x_m}(\omega) = M_{x_n + x_m}(\omega) \tag{2.8}$$

and

$$M_{-x_n}(\omega) = -M_{x_n}(\omega) \tag{2.9}$$

and

$$M_{ix_n}(\omega) = iM_{x_n}(\omega) \tag{2.10}$$

Now for $\omega \in \Omega \setminus \mathcal{N}$ define $\overline{M}_x(\omega) := \lim_{n\to\infty} M_{x_n}(\omega)$ where $x_n \to x$. Then $\overline{M}_x(\omega)$ is well defined, because for $x_n \to x$ and $\tilde{x}_n \to x$, by the linearity of $x \mapsto M_x$ and (2.7) we have

$$\|M_{x_n}(\omega) - M_{\tilde{x}_n}(\omega)\| \leq \|T\| \, \|x_n - \tilde{x}_n\| \to 0.$$

We claim, that $x \mapsto \overline{M}_x(\omega) \in \mathcal{L}(E)$. From (2.6) we deduce

$$\|\overline{M}_x(\omega)\| = \lim_{n\to\infty} \|M_{x_n}(\omega)\| \le \lim_{n\to\infty} \|T\|\,\|x_n\| = \|T\|\,\|x\| \tag{2.11}$$

for all $\omega \in \Omega \setminus \mathcal{N}$, hence $x \mapsto \overline{M}_x(\omega)$ is bounded. In the same way we obtain from (2.7)

$$\|\overline{M}_x(\omega) - \overline{M}_{\tilde{x}}(\omega)\| \le \|T\|\,\|x - \tilde{x}\|$$

for all $\omega \in \Omega \setminus \mathcal{N}$.
We still have to show that $x \mapsto \overline{M}_x(\omega)$ is linear for all $\omega \in \Omega \setminus \mathcal{N}$. For the additivity let $x_n \to x$ and $\tilde{x}_n \to \tilde{x}$, then by (2.8) and (2.9)

$$\begin{aligned}\overline{M}_{x\pm\tilde{x}}(\omega) &= \lim_{n\to\infty} M_{x_n\pm\tilde{x}_n}(\omega) = \lim_{n\to\infty} (M_{x_n}(\omega) \pm M_{\tilde{x}_n}(\omega)) \\ &= \overline{M}_x(\omega) \pm \overline{M}_{\tilde{x}}(\omega).\end{aligned}$$

From the additivity we immediately deduce $\overline{M}_{zx}(\omega) = z\overline{M}_x(\omega)$ for all $z \in \mathbb{Z}$. Then for all rational numbers $q = \frac{z}{n}$ with $z \in \mathbb{Z}$ and $n \in \mathbb{N}$ we obtain that $n\overline{M}_{qx}(\omega) = \overline{M}_{nqx}(\omega) = \overline{M}_{zx}(\omega) = z\overline{M}_x(\omega)$, thus $\overline{M}_{qx}(\omega) = \frac{z}{n}\overline{M}_x(\omega) = q\overline{M}_x(\omega)$. Further every real number r is the limit of some sequence of rational numbers q_n. Since $\|\overline{M}_{q_nx}(\omega) - \overline{M}_{rx}(\omega)\| \le \|T\|\,\|q_nx - rx\| \le \|T\|\,|q_n - r|\,\|x\| \to 0$, we get $\overline{M}_{rx}(\omega) = \lim_{n\to\infty} \overline{M}_{q_nx}(\omega) = \lim_{n\to\infty} q_n\overline{M}_x(\omega) = r\overline{M}_x(\omega)$. Analogously we get for $x_n \to x$ by (2.10) $M_{ix}(\omega) = \lim_{n\to\infty} M_{ix_n}(\omega) = \lim_{n\to\infty} iM_{x_n}(\omega) = iM_{ix}(\omega)$. Hence for all $\lambda \in \mathbb{C}$,

$$\overline{M}_{\lambda x}(\omega) = \overline{M}_{(\operatorname{Re}\lambda)x + i(\operatorname{Im}\lambda)x}(\omega) = \operatorname{Re}\lambda\overline{M}_x(\omega) + i\operatorname{Im}\lambda\overline{M}_x(\omega) = \lambda\overline{M}_x(\omega)$$

Therefore $x \mapsto \overline{M}_x(\omega) \in \mathcal{L}(E)$ for all $\omega \in \Omega\setminus\mathcal{N}$. Set $\overline{M}_x(\omega) = 0$ for $\omega \in \mathcal{N}$, then $x \mapsto \overline{M}_x(\omega) \in \mathcal{L}(E)$ for all $\omega \in \Omega$.
We claim that for all $x \in E$, $\omega \mapsto \overline{M}_x(\omega) \in L^\infty(\Omega, E)$. Let $x \in E$ be fixed, and let $x_n \to x$ for $n \to \infty$. Then $\overline{M}_x(\omega) = \lim_{n\to\infty} \chi_{\Omega\setminus\mathcal{N}} M_{x_n}(\omega)$, hence measurable as the pointwise limit of measurable functions. In order to obtain that the mapping $\omega \mapsto \overline{M}_x(\omega) \in L^\infty(\Omega, E)$, it suffices to show that $\overline{M}_x(\omega) = M_x(\omega)$ almost everywhere.
For the moment, assume that $\mu(\Omega) < \infty$. Let $x_n \to x$ in E, then $\chi_\Omega \cdot x_n \to \chi_\Omega \cdot x$ in $L^p(\Omega, E)$. Indeed, for $1 \le p < \infty$

$$\begin{aligned}\|\chi_\Omega \cdot x_n - \chi_\Omega \cdot x\|_{L^p(\Omega,E)} &= \left(\int_\Omega \|\chi_\Omega(\omega) \cdot x_n - \chi_\Omega(\omega) \cdot x\| d\mu\right)^{1/p} \\ &= \left(\int_\Omega \|x_n - x\| d\mu\right)^{1/p} \\ &= \|x_n - x\|\,(\mu(\Omega))^{1/p} \to 0\end{aligned}$$

and for $p = \infty$

$$\|\chi_\Omega \cdot x_n - \chi_\Omega \cdot x\|_\infty = \operatorname*{ess\,sup}_{\omega\in\Omega} \|\chi_\Omega(\omega) \cdot x_n - \chi_\Omega(\omega) \cdot x\| = \|x_n - x\| \to 0.$$

Since $T \in \mathcal{L}(L^p(\Omega, E))$ is continuous, we also have $T(\chi_\Omega \cdot x_n) \to T(\chi_\Omega \cdot x)$ in $L^p(\Omega, E)$, and therefore for a subsequence $T(\chi_\Omega \cdot x_n)(\omega) \to T(\chi_\Omega \cdot x)(\omega) = M_x(\omega)$ almost everywhere. But $T(\chi_\Omega \cdot x_n)(\omega) = M_{x_n}(\omega) \to \overline{M}_x(\omega)$ for all $\omega \in \Omega \setminus \mathcal{N}$. Hence $\overline{M}_x(\omega) = M_x(\omega)$ almost everywhere.
Now for all $\omega \in \Omega$ define $M(\omega) := [x \mapsto \overline{M}_x(\omega)] \in \mathcal{L}(E)$. Then $M : \Omega \to \mathcal{L}(E)$ defined by $\omega \mapsto M(\omega)$ is an element of $L^\infty(\Omega, \mathcal{L}_s(E))$, because we have that $\omega \mapsto M(\omega)x = \overline{M}_x(\omega) \in L^\infty(\Omega, E)$ for all $x \in E$.
We still have to show, that $Tf(\omega) = M(\omega)f(\omega)$ almost everywhere for every $f \in L^p(\Omega, E)$. Indeed

$$T(\chi_\Omega \cdot x)(\omega) = M_x(\omega) = \overline{M}_x(\omega) = M(\omega)x,$$

where each equality holds for almost all $\omega \in \Omega$. Since $T(\chi_A f) = \chi_A Tf$ for all $A \subset \Omega$ measurable and all $f \in L^p(\Omega, E)$, we deduce

$$\begin{aligned} T(\chi_A \cdot x)(\omega) &= T(\chi_A \chi_\Omega \cdot x)(\omega) = (\chi_A T(\chi_\Omega \cdot x))(\omega) \\ &= \chi_A(\omega) T(\chi_\Omega \cdot x)(\omega) = \chi_A(\omega) M_x(\omega) = \chi_A(\omega)\overline{M}_x(\omega) \\ &= \chi_A(\omega) M(\omega) x = M(\omega)(\chi_A(\omega) \cdot x). \end{aligned}$$

By linearity, for all simple functions $Tf(\omega) = M(\omega)f(\omega)$ almost everywhere, and since the simple functions are dense in $L^p(\Omega, E)$ and by continuity of the operators, we obtain $Tf(\omega) = M(\omega)f(\omega)$ almost everywhere for all $f \in L^p(\Omega, E)$.
Finally, if $\mu(\Omega) = \infty$, choose measurable $\Omega_k \subset \Omega$, such that $\Omega_k \cap \Omega_j = \emptyset$ for $k \neq j$, $\mu(\Omega_k) < \infty$ for all $k \in \mathbb{N}$ and $\Omega = \bigcup_k \Omega_k$. Then $T \in \mathcal{L}(L^p(\Omega_k, E))$ for all $k \in \mathbb{N}$ and satisfies (iii). Hence for $\omega \in \Omega_k$ we can define $M_k(\omega)$ as before, such that $\omega \mapsto M_k(\omega) \in L^\infty(\Omega_k, \mathcal{L}_s(E))$ and $Tf(\omega) = M_k(\omega)f(\omega)$ almost everywhere on Ω_k and for all $f \in L^p(\Omega_k, E)$. Now let $M(\omega) = M_k(\omega)$ if $\omega \in \Omega_k$, then $\omega \mapsto M(\omega) \in L^\infty(\Omega, \mathcal{L}_s(E))$ and $Tf(\omega) = M(\omega)f(\omega)$ almost everywhere on Ω and for all $f \in L^p(\Omega, E)$. □

We immediately obtain an analogue to Proposition 2.2.2.

Proposition 2.2.18. *Denote by $\mathcal{O}(E)$ the set of linear operators on E, not necessarily bounded. Let $M : \Omega \to \mathcal{O}(E)$ be a function and define a linear operator*

$$\begin{aligned} T : \ L^p(\Omega, E) &\to L^p(\Omega, E) \\ f &\mapsto [\omega \mapsto M(\omega) f(\omega)]. \end{aligned} \tag{2.12}$$

Then T is bounded if and only if $M \in L^\infty(\Omega, \mathcal{L}_s(E))$

Proof. If T is given by (2.12), then for every $f \in L^p(\Omega, E)$, $Tf = 0$ a.e. on $\{f = 0\}$. If T is bounded, then in particular $T_{|L^p(\tilde{\Omega}, E)}$ is bounded for every purely atomic subset $\tilde{\Omega}$ of Ω. Hence T satisfies (iii) of Theorem 2.2.17 and get that $M \in L^\infty(\Omega, \mathcal{L}_s(E))$. The converse is contained in Corollary 2.2.13. □

2.3 Unbounded Multiplication Operators

After the discussion of bounded multiplication operators, we are certainly interested in unbounded multiplication operators on scalar and vector valued L^p-spaces. We will study some basic properties, which lead us over spectral considerations to multiplication semigroups. Again, we have been inspired by [Gr], where such operators are treated on the space of continuous functions. But once more, we could not transfer the results, whenever an evaluation of a function at one point is involved.

2.3.1 Unbounded Multiplication Operators on L^p-spaces

Throughout this section, let Ω be a σ-finite measure space and E a Banach space. We will not treat the scalar case where $E = \mathbb{R}$ or $\mathbb{C}$ separately.
We study those unbounded linear operators on $L^p(\Omega, E)$, which are defined through pointwise multiplication with linear operators on E.

Definition 2.3.1. Let $(\mathcal{A}, D(\mathcal{A}))$ be an unbounded linear operator on $L^p(\Omega, E)$. It is called an **unbounded operator valued multiplication operator**, if there exists a family $(A(\omega), D(A(\omega))_{\omega\in\Omega}$ of linear operators on E, such that

$$\begin{aligned} D(\mathcal{A}) &= \{f \in L^p(\Omega, E) : f(\omega) \in D(A(\omega) \text{ for almost all } \omega \in \Omega \\ &\qquad\qquad \text{and } \omega \mapsto A(\omega)f(\omega) \in L^p(\Omega, E)\} \\ (\mathcal{A}f)(\omega) &= A(\omega)f(\omega) \text{ for all } f \in D(\mathcal{A}) \text{ and almost all } \omega \in \Omega. \end{aligned}$$

In this case the operators $(A(\omega), D(A(\omega))_{\omega\in\Omega}$ are called the **fiber operators** of $\mathcal{A}$.

Lemma 2.3.2. *An unbounded operator valued multiplication operator $(\mathcal{A}, D(\mathcal{A}))$ on $L^p(\Omega, E)$ satisfies*

(i) for all $f \in D(\mathcal{A})$ and all $\varphi \in L^\infty(\Omega)$, $\varphi f \in D(\mathcal{A})$ and $\mathcal{A}(\varphi f) = \varphi(\mathcal{A}f)$,

(ii) for all $f \in D(\mathcal{A})$, $\mathcal{A}f = 0$ a.e. on the set $\{\omega \in \Omega : f(\omega) = 0\}$.

Proof. (i) Let $(A(\omega), D(A(\omega))_{\omega\in\Omega}$ be the fiber operators of $\mathcal{A}$. Let $f \in D(\mathcal{A})$ and $\varphi \in L^\infty(\Omega)$. Then $f(\omega) \in D(A(\omega))$ for almost all $\omega \in \Omega$ and the mapping $\omega \mapsto A(\omega)f(\omega) \in L^p(\Omega, E)$. Further $\varphi(\omega)$ is a bounded scalar for almost all $\omega \in \Omega$. Hence $\varphi(\omega)f(\omega) \in D(A(\omega))$ for almost all $\omega \in \Omega$, and

$$\omega \mapsto A(\omega)\varphi(\omega)f(\omega) = \varphi(\omega)A(\omega)f(\omega) \in L^p(\Omega, E),$$

i.e. $\varphi f \in D(\mathcal{A})$. In particular $\mathcal{A}(\varphi f) = \varphi(\mathcal{A}f)$.
(ii) Let $\varphi = \chi_{\{f\neq 0\}} \in L^\infty(\Omega)$. Then $f = \varphi f$ and by (i), we obtain the equality $\mathcal{A}f = \mathcal{A}(\varphi f) = \varphi(\mathcal{A}f) = \chi_{\{f\neq 0\}}(\mathcal{A}f) = 0$ a.e. on $\{\omega \in \Omega : f(\omega) = 0\}$. □

Remark 2.3.3. In contrast to bounded operators, the properties in the above lemma do not characterize multiplication operators. For example on $L^2(\Omega)$ the Laplace operator Δ satisfies (ii) of the above lemma, but not (i) nor is it a multiplication operator.

For a discussion of spectral properties we are interested in closed operator valued multiplication operators. The following lemma shows, how this property can be deduced from the fiber operators.

Lemma 2.3.4. *If $A(\omega)$ is closed for almost every $\omega \in \Omega$, then $\mathcal{A}$ is closed.*

Proof. Let $D(\mathcal{A}) \ni f_n \to f$ and $\mathcal{A}f_n \to g$ in $L^p(\Omega, E)$. Then for a subsequence $f_n(\omega) \to f(\omega)$ a.e. and $A(\omega)f_n(\omega) = (\mathcal{A}f_n)(\omega) \to g(\omega)$ almost everywhere. As $A(\omega)$ is closed for almost every $\omega \in \Omega$, we obtain $f(\omega) \in D(A(\omega))$ and $A(\omega)f(\omega) = g(\omega)$ almost everywhere. Hence, $f \in D(\mathcal{A})$ and $\mathcal{A}f = g$. □

2.3.2 The Resolvent of a Multiplication Operator

In the sequel we will always assume $(\mathcal{A}, D(\mathcal{A}))$ to be a closed operator valued multiplication operator with closed fiber operators $(A(\omega), D(A(\omega))_{\omega\in\Omega}$. We shall establish a relationship between an operator and its resolvent with respect to the property of being a multiplication operator.

Lemma 2.3.5. *Let $\mathcal{A}$ be a multiplication operator, and assume $\lambda \in \rho(\mathcal{A})$. Then the operator $R(\lambda, \mathcal{A})$ is a bounded multiplication operator.*

Proof. For all $f \in L^p(\Omega, E)$, we have $R(\lambda, \mathcal{A})f \in D(\mathcal{A})$. Hence for all $\varphi \in L^\infty(\Omega)$ by Lemma 2.3.2, as $\mathcal{A}$ is a multiplication operator, $\varphi R(\lambda, \mathcal{A})f \in D(\mathcal{A})$. Further $(\lambda - \mathcal{A})$ is a multiplication operator with fiber operators $(\lambda - A(\omega))$ and also with Lemma 2.3.2, we obtain

$$(\lambda - \mathcal{A})(\varphi R(\lambda, \mathcal{A})f) = \varphi(\lambda - \mathcal{A})(R(\lambda, \mathcal{A})f) = \varphi f.$$

Therefore $R(\lambda, \mathcal{A})\varphi f = R(\lambda, \mathcal{A})(\lambda - \mathcal{A})(\varphi R(\lambda, \mathcal{A})f) = \varphi R(\lambda, \mathcal{A})f$, which implies by Theorem 2.2.17 that $R(\lambda, \mathcal{A})$ is a bounded operator valued multiplication operator, i.e. there exists an $M \in L^\infty(\Omega, \mathcal{L}_s(E))$ such that $R(\lambda, \mathcal{A}) = \mathcal{M}_M$. □

Since we have a characterization for bounded multiplication operators, the converse result would be a strong tool. With the following theorem we can conclude from the fact, that the resolvent is a bounded multiplication operator, that also $\mathcal{A}$ is a multiplication operator.

Theorem 2.3.6. *Let $\mathcal{A}$ be a densely defined closed operator on $L^p(\Omega, E)$. Assume that there exists an unbounded sequence $(\lambda_k)_{k\in\mathbb{N}} \subset \rho(\mathcal{A})$ such that for all $f \in L^p(\Omega, E)$, $\lim_{k\to\infty} \lambda_k R(\lambda_k, \mathcal{A})f = f$. If $R(\lambda_k, \mathcal{A})$ is a bounded multiplication operator for every $k \in \mathbb{N}$, then there exists a family $(A(\omega))_{\omega\in\Omega}$ of densely defined* *closed operators on E, such that $(\mathcal{A}, D(\mathcal{A}))$ is a multiplication operator with fiber operators $(A(\omega), D(A(\omega)))$. Further there exists a null-set $\mathcal{N}$ such that for every $\omega \in \Omega \setminus \mathcal{N}$ and for every $k \in \mathbb{N}$ one has $\lambda_k \in \rho(A(\omega))$.*

Proof. Let $M_{\lambda_k} : \Omega \to \mathcal{L}(E), \omega \mapsto M_{\lambda_k}(\omega) \in L^\infty(\Omega, \mathcal{L}_s(E))$ be determined by the bounded multiplication operator $R(\lambda_k, \mathcal{A})$, i.e. for all $f \in L^p(\Omega, R)$, $(R(\lambda_k, \mathcal{A})f)(\omega) = M_{\lambda_k}(\omega)f(\omega)$ for almost all $\omega \in \Omega$.
1. Step: For almost all $\omega \in \Omega$, $(M_{\lambda_k}(\omega))_{k\in\mathbb{N}}$ is a pseudo resolvent.
The resolvent equation

$$(\lambda_l - \lambda_k)(R(\lambda_l, \mathcal{A})R(\lambda_k, \mathcal{A})f)(\omega) = (R(\lambda_k, \mathcal{A})f)(\omega) - (R(\lambda_l, \mathcal{A})f)(\omega)$$

holds for all $f \in L^p(\Omega, E)$ and almost all $\omega \in \Omega$, where the exceptional null-set depends on f.
Since Ω is σ-finite, there exist Ω_n disjoint, such that $\Omega = \bigcup_n \Omega_n$ and $\mu(\Omega_n) < \infty$. Further there exists $\{x_m : m \in \mathbb{N}\}$ dense in E, because E is separable. Then define $f_{n,m} := \chi_{\Omega_n} \cdot x_m \in L^p(\Omega, E)$.
As the countable union of null-sets is again a null-set, there exists a null-set $\mathcal{N}_1$, such that for all $m \in \mathbb{N}$ and for all $\omega \in \Omega \setminus \mathcal{N}_1$, there exist an $n \in \mathbb{N}$ such that $\omega \in \Omega_n \setminus \mathcal{N}_1$ and

$$(\lambda_l - \lambda_k)(R(\lambda_l, \mathcal{A})R(\lambda_k, \mathcal{A})f_{n,m})(\omega) = (R(\lambda_k, \mathcal{A})f_{n,m})(\omega) - (R(\lambda_l, \mathcal{A})f_{n,m})(\omega)$$

and

$$(R(\lambda_k, \mathcal{A})f_{n,m})(\omega) = M_{\lambda_k}(\omega)f_{n,m}(\omega) = M_{\lambda_k}(\omega)x_m$$

and

$$(R(\lambda_l, \mathcal{A})R(\lambda_k, \mathcal{A})f_{n,m})(\omega) = M_{\lambda_l}(\omega)M_{\lambda_k}(\omega)x_m.$$

Therefore, we obtain for all $m \in \mathbb{N}$ and for all $\omega \in \Omega \setminus \mathcal{N}_1$ if we choose n such that $\omega \in \Omega_n \setminus \mathcal{N}_1$,

$$\begin{aligned}
&(\lambda_l - \lambda_k)M_{\lambda_l}(\omega)M_{\lambda_k}(\omega)x_m \\
&\quad = (\lambda_l - \lambda_k)(R(\lambda_l, \mathcal{A})R(\lambda_k, \mathcal{A})f_{n,m})(\omega) \\
&\quad = (R(\lambda_k, \mathcal{A})f_{n,m})(\omega) - (R(\lambda_l, \mathcal{A})f_{n,m})(\omega) \\
&\quad = M_{\lambda_k}(\omega)x_m - M_{\lambda_l}(\omega)x_m.
\end{aligned}$$

Since $\{x_m : m \in \mathbb{N}\}$ is dense in E we obtain by continuous extension, that $(\lambda_l - \lambda_k)M_{\lambda_l(\omega)}M_{\lambda_k}(\omega)x = M_{\lambda_k}(\omega)x - M_{\lambda_l}(\omega)x$ for all $x \in E$ and all $\omega \in \Omega \setminus \mathcal{N}_1$, i.e. for almost all $\omega \in \Omega$, $(M_{\lambda_k}(\omega))_{k\in\mathbb{N}}$ is a pseudo resolvent.
2. Step: For almost every $\omega \in \Omega$, $\lim_{k\to\infty} \lambda_k M_{\lambda_k}(\omega)x = x$ for all $x \in E$.
Since $\lambda_k R(\lambda_k, \mathcal{A})f \to f$ for all $f \in L^p(\Omega, E)$, $\|\lambda_k R(\lambda_k, \mathcal{A})f\|$ is bounded and therefore by the principle of uniform boundedness, $\|\lambda_k R(\lambda_k, \mathcal{A})\| \le C$ for some constant C.

Recall that $\|\lambda_k R(\lambda_k, \mathcal{A})\| = \|M_{\lambda_k}(\cdot)\|_{L^\infty(\Omega,\mathcal{L}_s(E))}$ by Proposition 2.2.14. By Remark 1.1.7 there exists a nullset $\mathcal{N}_2$ such that

$$\sup_{\omega\in\Omega\setminus\mathcal{N}_2} \|\lambda_k M_{\lambda_k}(\omega)\| = \|\lambda_k M_{\lambda_k}(\cdot)\|_{L^\infty(\Omega,\mathcal{L}_s(E))}.$$

Hence for all $\omega \in \Omega\setminus\mathcal{N}_2$ and for all $k \in \mathbb{N}$,

$$\|\lambda_k M_{\lambda_k}(\omega)\|_{E\to E} \leq |\lambda_k|\|M_{\lambda_k}(\cdot)\|_{L^\infty(\Omega,\mathcal{L}_s(E))} = |\lambda_k|\|R(\lambda_k, \mathcal{A})\| \leq C.$$

Further $\lambda_k R(\lambda_k, \mathcal{A})f(\omega) \to f(\omega)$ for a subsequence, which we shall again denote by λ_k, and almost all $\omega \in \Omega$, where the subsequence and the exceptional null-set depend on f. Again for the countable $f_{n,m}$ we obtain a subsequence and a null-set $\mathcal{N}_3$, such that for all $m \in \mathbb{N}$ and all $\omega \in \Omega\setminus(\mathcal{N}_1\cup\mathcal{N}_3)$, there exists $n \in \mathbb{N}$ such that $\omega \in \Omega_n \setminus(\mathcal{N}_1\cup\mathcal{N}_3)$ and

$$\begin{aligned}\lim_{k\to\infty} \lambda_k M_{\lambda_k}(\omega)x_m &= \lim_{k\to\infty} \lambda_k R(\lambda_k, \mathcal{A})f_{n,m}(\omega)\\ &= f_{n,m}(\omega) = x_m\end{aligned}$$

Let $\mathcal{N} = \mathcal{N}_1\cup\mathcal{N}_2\cup\mathcal{N}_3$, then for all $\omega \in \Omega\setminus\mathcal{N}$ and all $x \in E$

$$\|\lambda_k M_{\lambda_k}(\omega)x - x\| \leq \|\lambda_k M_{\lambda_k}(\omega)\|\|x - x_m\| + \|\lambda_k M_{\lambda_k}(\omega)x_m - x_m\| + \|x_m - x\|,$$

which implies since $\{x_m : m \in \mathbb{N}\}$ is dense in E, that $\lim_{k\to\infty}\lambda_k M_{\lambda_k}(\omega)x = x$ for all $x \in E$.

3. Step: Construction of fiber operators.

Combining Step 1 and 2, for almost all $\omega \in \Omega$, namely all $\omega \in \Omega\setminus\mathcal{N}$, $(M_{\lambda_k}(\omega))_{k\in\mathbb{N}}$ is a pseudo resolvent satisfying $\lim_{k\to\infty}\lambda_k M_{\lambda_k}(\omega)x = x$ for all $x \in E$. By [EN], Corollary III.4.7, there exist densely defined closed operators $(A(\omega), D(A(\omega))$ such that $\lambda_k \in \rho(A(\omega))$ and $(M_{\lambda_k}(\omega)) = R(\lambda_k, A(\omega))$ for all $k \in \mathbb{N}$. We set $A(\omega) = 0$ for $\omega \in \mathcal{N}$.

Define an operator B by

$$\begin{aligned}D(B) &= \{f \in L^p(\Omega, E) : f(\omega) \in D(A(\omega)) \text{ for almost all } \omega \in \Omega\\ &\qquad \text{and } \omega \mapsto A(\omega)f(\omega) \in L^p(\Omega, E)\}\\ Bf(\omega) &= A(\omega)f(\omega).\end{aligned}$$

4. Step: $B = \mathcal{A}$.

Let $\lambda \in \{\lambda_k : k \in \mathbb{N}\}$ be arbitrary, but fixed. Then

$$\begin{aligned}D(\mathcal{A}) &= R(\lambda, \mathcal{A})E\\ &= \{f \in L^p(\Omega, E) : \exists\varphi \in L^p(\Omega, E), f = R(\lambda, \mathcal{A})\varphi\}\\ &\subset D(B).\end{aligned}$$

Indeed, if $f \in L^p(\Omega, E)$ such that there exists $\varphi \in L^p(\Omega, E)$ and $f = R(\lambda, \mathcal{A})\varphi$, then for almost all $\omega \in \Omega$,

$$\begin{aligned} f(s) &= R(\lambda, \mathcal{A})\varphi(s) = M_\lambda(\omega)\varphi(\omega) \\ &= R(\lambda, A(\omega))\varphi(\omega) \in D(A(\omega) \end{aligned}$$

and

$$\begin{aligned} A(\cdot)f(\cdot) &= A(\cdot)R(\lambda, A(\cdot))\varphi(\cdot) \\ &= \lambda R(\lambda, A(\cdot))\varphi(\cdot) - \varphi(\cdot) \\ &= \lambda f(\cdot) - \varphi(\cdot) \in L^p(\Omega, E). \end{aligned}$$

And for $f \in D(\mathcal{A})$ and almost all $\omega \in \Omega$,

$$\begin{aligned} \mathcal{A}f(\omega) &= \mathcal{A}R(\lambda, \mathcal{A})\varphi(\omega) = \lambda f(\omega) - \varphi(\omega) = (\lambda R(\lambda, A(\omega)) - Id)\varphi(\omega) \\ &= A(\omega)R(\lambda, A(\omega))\varphi(\omega) = A(\omega)f(\omega) = Bf(\omega). \end{aligned}$$

Hence $\mathcal{A} \subset B$.
Conversely,

$$\begin{aligned} D(B) &= \{f \in L^p(\Omega, E) : f(\omega) \in D(A(\omega)) \text{ for almost all } \omega \in \Omega \\ &\qquad \text{and } \omega \mapsto A(\omega)f(\omega) \in L^p(\Omega, E)\} \\ &\subset \{f \in L^p(\Omega, E) : \exists \varphi \in L^p(\Omega, E), f = R(\lambda, \mathcal{A})\varphi\} \\ &= D(\mathcal{A}). \end{aligned}$$

Indeed, if $f(\omega) \in D(A(\omega))$ for almost all $\omega \in \Omega$ and $A(\cdot)f(\cdot) \in L^p(\Omega, E)$, then $\varphi := \lambda f - A(\cdot)f(\cdot) \in L^p(\Omega, E)$. Moreover, for almost all $\omega \in \Omega$, the equality $f(\omega) = R(\lambda, A(\omega))\varphi(\omega) = M_\lambda(\omega)\varphi(\omega) = R(\lambda, \mathcal{A})\varphi(\omega)$ holds. Thus $B = \mathcal{A}$. $\square$

If we already assume, that $\mathcal{A}$ is a multiplication operator, we get the following relationship between the resolvent of unbounded operator valued multiplication operators and the resolvent of their fiber operators.

Proposition 2.3.7. *Let $(\mathcal{A}, D(\mathcal{A}))$ be a closed multiplication operator with closed fiber operators $(A(\omega), D(A(\omega)))_{\omega\in\Omega}$.*

(a) If $\lambda \in \rho(A(\omega))$ for almost all $\omega \in \Omega$ and $R(\lambda, A(\cdot)) : \Omega \ni \omega \mapsto R(\lambda, A(\omega))$ is in $L^\infty(\Omega, \mathcal{L}_s(E))$, then $\lambda \in \rho(\mathcal{A})$ and $R(\lambda, \mathcal{A}) = \mathcal{M}_{R(\lambda, A(\cdot))}$.

(b) If there exists an unbounded sequence $(\lambda_k)_{k\in\mathbb{N}} \subset \rho(\mathcal{A})$, such that for all $f \in L^p(\Omega, E)$, one has $\lambda_k R(\lambda_k, \mathcal{A})f \to f$ as $k \to \infty$, then for almost all $\omega \in \Omega$ and all $k \in \mathbb{N}$, $\lambda_k \in \rho(A(\omega))$ and $\mathcal{M}_{R(\lambda_k, A(\cdot))} = R(\lambda_k, \mathcal{A})$.

Proof. (a) Since $(\lambda - \mathcal{A})\mathcal{M}_{R(\lambda, A(\cdot))} = Id = \mathcal{M}_{R(\lambda, A(\cdot))}(\lambda - \mathcal{A})$, one has $(\lambda - \mathcal{A})$ is invertible with bounded inverse $\mathcal{M}_{R(\lambda, A(\cdot))}$.
(b) From $\lambda_k R(\lambda_k, \mathcal{A})f \to f$ we conclude that A is densely defined. Further by Lemma 2.3.5, $R(\lambda_k, \mathcal{A})$ is a bounded multiplication operator for each $k \in \mathbb{N}$. Hence the conditions of Theorem 2.3.6 are satisfied and the claim follows. $\square$

2.3.3 Multiplication Semigroups

After having established a relationship between the resolvents, it is natural to seek similar results for the semigroups generated by multiplication operators and their fiber operators respectively. An important role will be played by the following operators.

Definition 2.3.8. We call a C_0-semigroup $(\mathcal{T}(t))_{t\geq 0}$ on $L^p(\Omega, E)$ a **multiplication semigroup** if for every $t \geq 0$ the operator $\mathcal{T}(t)$ is a bounded multiplication operator, i.e. for every $t \geq 0$ there exists $T_{(\cdot)}(t) : \Omega \to \mathcal{L}(E), \omega \mapsto T_\omega(t) \in L^\infty(\Omega, \mathcal{L}_s(E))$ such that $(\mathcal{T}(t)f)(\omega) = T_\omega(t)f(\omega)$ for almost all $\omega \in \Omega$.

In order to use the previous results for resolvents, the following characterization for multiplication semigroups is quite helpful.

Lemma 2.3.9. *Let $(\mathcal{T}(t))_{t\geq 0}$ be a C_0-semigroup with generator $(\mathcal{A}, D(\mathcal{A}))$ on $L^p(\Omega, E)$, that satisfies $\|\mathcal{T}(t)\| \leq M\,\mathrm{e}^{\mathrm{w}t}$. The following assertions are equivalent.*

(i) $(\mathcal{T}(t))_{t\geq 0}$ *is a multiplication semigroup.*

(ii) $R(\lambda, A)$ *is a bounded multiplication operator, whenever* $\operatorname{Re}\lambda > \mathrm{w}$.

Proof. *(i)* $\Rightarrow$ *(ii)* As $\mathcal{T}(t)$ is a bounded operator valued multiplication operator, by Theorem 2.2.17, $\mathcal{T}(t)f = 0$ a.e. on the set $\{\omega \in \Omega : f(\omega) = 0\}$ for all $t \geq 0$. Hence for all $\operatorname{Re}\lambda > \mathrm{w}$, $R(\lambda, \mathcal{A})f = \int_0^\infty \mathrm{e}^{-\lambda t}\mathcal{T}(t)f\,dt = 0$ a.e. on the set $\{\omega \in \Omega : f(\omega) = 0\}$, i.e. $R(\lambda, \mathcal{A})$ is a bounded operator valued multiplication operator.
(ii) $\Rightarrow$ *(i)* As $R(\lambda, A)$ is a bounded operator valued multiplication operator for all $\operatorname{Re}\lambda > \mathrm{w}$, $\mathcal{T}(t)f = \lim_{n\to\infty}(\frac{n}{t}R(\frac{n}{t}, \mathcal{A}))^n f = 0$ a.e. on the set $\{\omega \in \Omega : f(\omega) = 0\}$, hence $\mathcal{T}(t)$ is a bounded operator valued multiplication operator for all $t \geq 0$, i.e. $(\mathcal{T}(t))_{t\geq 0}$ is a multiplication semigroup. ☐

Starting from a family of generators, we shall examine the property of defining a bounded operator valued multiplication operator in the sense of Definition 2.2.15 for the semigroups and resolvents. However, we do not assume beforehand that the family of operators defines a multiplication operator.

Lemma 2.3.10. *Let $(A(\omega), D(A(\omega)))_{\omega\in\Omega}$ be a family of operators on E, that generate C_0-semigroups $(T_\omega(t))_{t\geq 0}$ satisfying $\|T_\omega(t)\| \leq M\,\mathrm{e}^{\mathrm{w}t}$ for some constants $M \geq 1$ and $\mathrm{w} \in \mathbb{R}$.. Then the following assertions are equivalent.*

(i) For every $t \geq 0$ the map $\omega \mapsto T_\omega(t)$ belongs to $L^\infty(\Omega, \mathcal{L}_s(E))$.

(ii) For every $\lambda > \mathrm{w}$ the map $\omega \mapsto R(\lambda, A(\omega))$ belongs to $L^\infty(\Omega, \mathcal{L}_s(E))$.

Proof. (*i*) ⇒ (*ii*) We have for $\omega \in \Omega$ the representation

$$R(\lambda, A(\omega)) = \int_0^\infty \mathrm{e}^{-\lambda t}\, T_\omega(t)\, dt.$$

Hence from $\omega \mapsto T_\omega(t) \in L^\infty(\Omega, \mathcal{L}_s(E))$, we deduce that for all $x \in E$, the mapping $\omega \mapsto R(\lambda, A(\omega))x = \int_0^\infty \mathrm{e}^{-\lambda t}\, T_\omega(t)x\, dt$ is measurable. Further

$$\begin{aligned} \|R(\lambda, A(\omega))x\| &= \left\| \int_0^\infty \mathrm{e}^{-\lambda t}\, T_\omega(t)x\, dt \right\| \\ &\le \int_0^\infty \mathrm{e}^{-\lambda t}\, \|T_\omega(t)\|\, \|x\|\, dt \\ &\le \int_0^\infty M\, \mathrm{e}^{(\mathrm{w}-\lambda)t}\, \|x\|\, dt = M(\lambda - \mathrm{w})^{-1}\|x\|, \end{aligned}$$

which shows that $\omega \mapsto R(\lambda, A(\omega)) \in L^\infty(\Omega, \mathcal{L}_s(E))$.
(*ii*) ⇒ (*i*) For all $x \in E$, we have

$$T_\omega(t)x = \lim_{n\to\infty} \left[\frac{n}{t} R\left(\frac{n}{t}, A(\omega)\right)\right]^n x.$$

Hence $\omega \mapsto T_\omega(t)x$ is measurable as the limit of measurable functions and since $\|T_\omega(t)x\| \le M\, \mathrm{e}^{\mathrm{w}t}\, \|x\|$, we obtain $\omega \mapsto T_\omega(t)x \in L^\infty(\Omega, \mathcal{L}_s(E))$. □

The previous results did not consider multiplication operators. This shall be done in the following two propositions.

Proposition 2.3.11. *Let $(\mathcal{A}, D(\mathcal{A}))$ be a densely defined closed multiplication operator with densely defined closed fiber operators $(A(\omega), D(A(\omega)))_{\omega\in\Omega}$. Assume that there exists a nullset $\mathcal{N} \subset \Omega$, such that for all $\omega \in \Omega \setminus \mathcal{N}$, the operator $A(\omega)$ is the generator of a C_0-semigroup $(T_\omega(t))_{t\ge 0}$, satisfying $\|T_\omega(t)\| \le M\, \mathrm{e}^{\mathrm{w}t}$, where $M \ge 1$ and $\mathrm{w} \in \mathbb{R}$ are some constants independent of t and ω. If for every $t \ge 0$, the map $\omega \mapsto \chi_{\Omega\setminus\mathcal{N}} T_\omega(t)$ is in $L^\infty(\Omega, \mathcal{L}_s(E))$, then $\mathcal{A}$ is the generator of the multiplication semigroup given by $(\mathcal{T}(t)f)(\omega) = \chi_{\Omega\setminus\mathcal{N}} T_\omega(t) f(\omega)$ for almost all $\omega \in \Omega$. Further $\|\mathcal{T}(t)\| \le M\, \mathrm{e}^{\mathrm{w}t}$.*

Proof. Since $\omega \mapsto \chi_{\Omega\setminus\mathcal{N}} T_\omega(t) \in L^\infty(\Omega, \mathcal{L}_s(E))$,

$$(\mathcal{T}(t)f)(\omega) = \chi_{\Omega\setminus\mathcal{N}} T_\omega(t) f(\omega)$$

defines a bounded linear operator from $L^p(\Omega, E)$ into itself for all $t \ge 0$. By Proposition 2.2.14 we immediately get the norm estimate

$$\|\mathcal{T}(t)\| = \|\chi_{\Omega\setminus\mathcal{N}} T_{(\cdot)}(t)\|_{L^\infty(\Omega, \mathcal{L}_s(E))} \le \sup_{\omega\in\Omega\setminus\mathcal{N}} \|T_\omega(t)\| \le M\, \mathrm{e}^{\mathrm{w}t}. \tag{2.13}$$

We show that $(\mathcal{T}(t))_{t\ge 0}$ is a C_0-semigroup with generator $\mathcal{A}$. Let $f \in L^p(\Omega, E)$. Observe, that

$$(\mathcal{T}(0)f)(\omega) = \chi_{\Omega\setminus\mathcal{N}} T_\omega(0) f(\omega) = \chi_{\Omega\setminus\mathcal{N}} f(\omega),$$

hence $\mathcal{T}(0)f = f$, which shows that $\mathcal{T}(0) = Id$. Further

$$\begin{aligned}(\mathcal{T}(t)\mathcal{T}(s)f)(\omega) &= \chi_{\Omega\setminus\mathcal{N}}T_\omega(t)(\mathcal{T}(s)f)(\omega) = \chi_{\Omega\setminus\mathcal{N}}T_\omega(t)T_\omega(s)f(\omega)\\ &= \chi_{\Omega\setminus\mathcal{N}}T_\omega(s+t)f)\omega) = (\mathcal{T}(t+s)f)(\omega),\end{aligned}$$

which shows $\mathcal{T}(t+s) = \mathcal{T}(t)\mathcal{T}(s)$.
Finally, as $t \to 0$, $(\mathcal{T}(t)f)(\omega) = \chi_{\Omega\setminus\mathcal{N}}T_\omega(t)f(\omega) \to f(\omega)$ almost everywhere, and $\|(\mathcal{T}(t)f)(\omega) - f(\omega)\|_E \le \|T_\omega(t)f(\omega)\|_E + \|f(\omega)\|_E \le C\|f(\omega)\|_E$, almost everywhere, where C is some constant. Hence, by the dominated convergence theorem

$$\begin{aligned}\|\mathcal{T}(t)f - f\|_{L^p(\Omega,E)} &= \left(\int_\Omega \|\mathcal{T}(t)f)(\omega) - f(\omega)\|_E^p\right)^{1/p}\\ &= \left(\int_{\Omega\setminus\mathcal{N}} \|T_\omega(t)f(\omega) - f(\omega)\|_E^p\right)^{1/p} \to 0,\end{aligned}$$

as $t \to 0$. Hence for every $f \in L^p(\Omega, E)$, the map $t \mapsto \mathcal{T}(t)f$ is continuous in 0, and by the functional equation, it is continuous in $t \ge 0$.
The generator $\mathcal{G}$ of $\mathcal{T}$ is defined by

$$\begin{aligned}D(\mathcal{G}) &= \left\{f \in L^p(\Omega, E) : \lim_{t\downarrow 0}\frac{\mathcal{T}(t)f - f}{t} \text{ exists in } L^p(\Omega, E)\right\}\\ \mathcal{G}f &= \lim_{t\downarrow 0}\frac{\mathcal{T}(t)f - f}{t}.\end{aligned}$$

We claim that $\mathcal{A} = \mathcal{G}$.
Let $f \in D(\mathcal{A})$. In order to obtain that $f \in D(G)$ and $Gf = \mathcal{A}f$, it suffices to show that $T(t)f - f = \int_0^t T(s)\mathcal{A}f ds$. But for almost every $\omega \in \Omega \setminus \mathcal{N}$, we have $f(\omega) \in D(A(\omega))$ and therefore

$$\begin{aligned}(T(t)f)(\omega) - f(\omega) &= T_\omega(t)f(\omega) - f(\omega) = \int_0^t T_\omega(s)A(\omega)f(\omega)\,ds\\ &= \int_0^t (T(s)\mathcal{A}f)(\omega)\,ds.\end{aligned}$$

Thus, $f \in D(G)$ and $Gf = \mathcal{A}f$. It remains to show that $D(G) \subset D(A)$.
The estimate $\|T_\omega(t)\| \le M\,\mathrm{e}^{\mathrm{w}t}$ implies that $\lambda \in \rho(A(\omega))$ for all $\lambda > \mathrm{w}$ and almost all $\omega \in \Omega$. By Lemma 2.3.10, $\omega \mapsto R(\lambda, A(\omega)) \in L^\infty(\Omega, \mathcal{L}_s(E))$, hence by Proposition 2.3.7, we obtain $\lambda \in \rho(\mathcal{A})$. On the other hand $\mathcal{G}$ is the generator of a semigroup satisfying (2.13), which implies $\lambda \in \rho(\mathcal{G})$. Then from the surjectivity of the operators, there exist for every $f \in D(\mathcal{G})$, $g \in L^p(\Omega, E)$ and $h \in D(\mathcal{A})$, such that

$$f = R(\lambda, \mathcal{G})g = R(\lambda, \mathcal{G})(\lambda - \mathcal{A})h = R(\lambda, \mathcal{G})(\lambda - \mathcal{G})h = h \in D(\mathcal{A}).$$

Hence $\mathcal{G} = \mathcal{A}$. □

The converse result reads as follows.

Proposition 2.3.12. *Assume that* $(\mathcal{A}, D(\mathcal{A}))$ *is a densely defined closed multiplication operator with densely defined closed fiber operators* $(A(\omega), D(A(\omega)))_{\omega\in\Omega}$. *If* $\mathcal{A}$ *is the generator of a* C_0*-semigroup* $(\mathcal{T}(t))_{t\geq 0}$*, satisfying* $\|\mathcal{T}(t)\| \leq M\,\mathrm{e}^{\mathrm{w}t}$*, then* $(\mathcal{T}(t))_{t\geq 0}$ *is a multiplication semigroup given by some* $T_{(\cdot)}(t) \in L^\infty(\Omega, \mathcal{L}_s(E))$ *for all* $t \geq 0$. *Further for almost all* $\omega \in \Omega$, $(T_\omega(t))_{t\geq 0}$ *is a* C_0*-semigroup with generator* $A(\omega)$ *and satisfies* $\|T_\omega(t)\| \leq M\,\mathrm{e}^{\mathrm{w}t}$.

Proof. Since $\mathcal{A}$ is the generator of a strongly continuous semigroup $\mathcal{T}$ satisfying $\|\mathcal{T}(t)\| \leq M\,\mathrm{e}^{\mathrm{w}t}$, for all $\lambda > \mathrm{w}$, $\lambda \in \rho(\mathcal{A})$, and for all $f \in L^p(\Omega, E)$, $\lim_{\lambda\to\infty} \lambda R(\lambda, \mathcal{A})f = f$. Hence by Proposition 2.3.7, $R(\lambda, \mathcal{A})$ is a bounded multiplication operator given by $R(\lambda, \mathcal{A}) = \mathcal{M}_{R(\lambda, A(\cdot))}$, and therefore satisfies $R(\lambda, \mathcal{A})(\varphi f) = \varphi R(\lambda, \mathcal{A})f$ for all $\varphi \in L^\infty(\Omega)$ and all $f \in L^p(\Omega, E)$ by Theorem 2.2.17. Hence for every $t \geq 0$,

$$\mathcal{T}(t)(\varphi f) = \lim_{n\to\infty} \left[\frac{n}{t} R\left(\frac{n}{t}, \mathcal{A}\right)\right]^n (\varphi f) = \varphi \lim_{n\to\infty} \left[\frac{n}{t} R\left(\frac{n}{t}, \mathcal{A}\right)\right]^n f = \varphi \mathcal{T}(t) f,$$

which implies again by Theorem 2.2.17, that $\mathcal{T}(t)$ is a bounded multiplication operator. Then for every $t \geq 0$ there exists $\omega \mapsto \mathcal{T}_\omega(t) \in L^\infty(\Omega, \mathcal{L}_s(E))$.
Since Ω is a σ-finite measure space, there exist Ω_n disjoint and of finite measure, such that $\Omega = \bigcup_{n\in\mathbb{N}} \Omega_n$. From the separability of E we get $\{x_m : m \in \mathbb{N}\}$ dense in E. Now define $f_{n,m} := \chi_{\Omega_n} \cdot x_m$. Then

$$(\mathcal{T}(t) f_{n,m})(\omega) = T_\omega(t)(\chi_{\Omega_n} \cdot x_m)(\omega) = \chi_{\Omega_n}(\omega) T_\omega(t) x_m,$$

for almost all $\omega \in \Omega$. Therefore there exists a nullset $\mathcal{N} \subset \Omega$, such that for all $\omega \notin \mathcal{N}$ there exists $n \in \mathbb{N}$ such that $\omega \in \Omega_n$ and for all $m \in \mathbb{N}$

$$\begin{aligned}
&T_\omega(t)x_m = (\mathcal{T}(t) f_{n,m})(\omega)\\
&T_\omega(0)x_m = x_m\\
&T_\omega(t+s)x_m = T_\omega(t)T_\omega(s)(\omega)x_m\\
&T_\omega(t)x_m \to x_m \quad (t \to 0).
\end{aligned}$$

By the density of $\{x_m : m \in \mathbb{N}\}$ in E and the continuity of the $T_\omega(t)$, we conclude that $(T_\omega(t))_{t\geq 0}$ is a C_0-semigroup for all $\omega \in \Omega \setminus \mathcal{N}$.
Further we have

$$(R(\lambda, \mathcal{A}) f_{n,m})(\omega) = R(\lambda, A(\omega))(\chi_{\Omega_n} \cdot x_m)(\omega) = \chi_{\Omega_n}(\omega) R(\lambda, A(\omega)) x_m.$$

Hence there exists a nullset $\tilde{\mathcal{N}} \supset \mathcal{N}$, such that for all $\omega \notin \tilde{\mathcal{N}}$ there exists $n \in \mathbb{N}$ such that $\omega \in \Omega_n$ and for all $m \in \mathbb{N}$

$$\begin{aligned}
R(\lambda, A(\omega))x_m &= (R(\lambda, \mathcal{A}) f_{n,m})(\omega)\\
&= \int_0^\infty \mathrm{e}^{-\lambda t} (\mathcal{T}(t)(f_{n,m})(\omega)\, dt\\
&= \int_0^\infty \mathrm{e}^{-\lambda t}\, T_\omega(t) x_m\, dt.
\end{aligned}$$

By the density of $\{x_m : m \in \mathbb{N}\}$ in E and the continuity of the $R(\lambda, A(\omega))$ and the dominated convergence theorem, we deduce

$$R(\lambda, A(\omega))x = \int_0^\infty \mathrm{e}^{-\lambda t}\, T_\omega(t)x\, dt$$

for all $x \in E$, hence $A(\omega)$ is the generator of $(T_\omega(t))_{t\geq 0}$.
Moreover, for almost all $\omega \in \Omega$, $\|T_\omega(t)\| \leq \|T_{(\cdot)}(t)\|_\infty = \|\mathcal{T}(t)\| \leq M\,\mathrm{e}^{\mathrm{w}t}$. $\square$

We conclude this section with some remarks. Theorem 2.3.6 will play an important role in the sequel, but it requires many assumptions.
However, if $\mathcal{A}$ is the generator of a C_0-semigroup with growth bound w, i.e. $\|T(t)\| \leq M\,\mathrm{e}^{\mathrm{w}t}$ for some constants $M \geq 1$ and $\mathrm{w} \in \mathbb{R}$, then $\lambda \in \rho(\mathcal{A})$ for all $\lambda > \mathrm{w}$ and $\lambda R(\lambda, \mathcal{A})f \to f$ as $\lambda \to \infty$ holds for every f.
By the following lemma, in this situation, it is even sufficient for the assumptions of Theorem 2.3.6, if $R(\lambda, A)$ is a bounded multiplication operator for one $\lambda > \mathrm{w}$.

Lemma 2.3.13. *Assume, that $R(\lambda_0, A)$ is a bounded multiplication operator for one $\lambda_0 \in \rho(A)$. Then $R(\lambda, A)$ is a bounded multiplication operator for all $\lambda \in \rho_0(A)$, where $\rho_0(A)$ is the connected component of $\rho(A)$ containing λ_0.*

Proof. For all $\lambda \in \mathbb{C}$ such that $|\lambda - \lambda_0| < \|R(\lambda_0, A)\|^{-1}$, we have $\lambda \in \rho(A)$ and the power series expansion $R(\lambda, A) = \sum_{n=0}^{\infty}(\lambda_0 - \lambda)^n R(\lambda_0, A)^{n+1}$. It is clear, that $\sum_{n=0}^{m}(\lambda_0 - \lambda)^n R(\lambda_0, A)^{n+1}$ for each $m \in \mathbb{N}$ is again a bounded multiplication operator, and since the set $\mathcal{M}$ of bounded multiplication operators is closed in $\mathcal{L}(L^p(\Omega, E))$, see Remark 2.2.16, we get that $R(\lambda, A)$ is a bounded multiplication operator.
Now we take a $\lambda_1 \in B(\lambda_0, \|R(\lambda_0, A)\|^{-1})$ and consequently obtain for each λ such that $|\lambda - \lambda_1| < \|R(\lambda_1, A)\|^{-1}$ that $R(\lambda, A)$ is a bounded multiplication operator.
By this method we can exhaust the entire connected component. $\square$

With the above considerations we immediately get the following result.

Corollary 2.3.14. Let $\mathcal{A}$ be the generator of a strongly continuous semigroup on $L^p(\Omega, E)$ with growth bound w and assume that $R(\lambda, \mathcal{A})$ is a bounded operator valued multiplication operator for one λ with $\operatorname{Re}\lambda > \mathrm{w}$. Then $\mathcal{A}$ is an operator valued multiplication operator.

We conclude this section by summing up the above results in one theorem.

Theorem 2.3.15. *Let $(\mathcal{A}, D(\mathcal{A}))$ be the generator of a strongly continuous semigroup $(\mathcal{T}(t))_{t\geq 0}$ on $L^p(\Omega, E)$, satisfying $\|\mathcal{T}(t)\| \leq M\,\mathrm{e}^{\mathrm{w}t}$. Then the following assertions are equivalent.*

(i) *$(\mathcal{T}(t))_{t\geq 0}$ is a multiplication semigroup.*

(ii) *The resolvent* $R(\lambda, \mathcal{A})$ *is a bounded operator valued multiplication operator for all* $\operatorname{Re}\lambda > \mathrm{w}$.

(iii) *The resolvent* $R(\lambda, \mathcal{A})$ *is a bounded operator valued multiplication operator for one* $\lambda > \mathrm{w}$.

(iv) $\mathcal{A}$ *is an unbounded operator valued multiplication operator with fiber operators* $(A(\omega), D(A(\omega))_{\omega\in\Omega}$. *Moreover, for almost all* $\omega \in \Omega$, $\lambda \in \rho(A(\omega))$ *whenever* $\operatorname{Re}\lambda > \mathrm{w}$, $R(\lambda, \mathcal{A}) = \mathcal{M}_{R(\lambda, A(\cdot))}$ *and* $(A(\omega), D(A(\omega))$ *is the generator of a* C_0*-semigroup* $(T_\omega(t))_{t\geq 0}$ *such that* $\mathcal{T}(t) = \mathcal{M}_{T_{(\cdot)}(t)}$ *for all* $t \geq 0$.

2.4 Local Forms and Multiplication Operators

On a Hilbert space H we consider a coercive closed form a with domain V, which is dense and continuously embedded in H. In order to use the results of the previous sections, we assume, that H is also a Banach lattice. Hence by [Scha], Theorem IV.6.7, $H = L^2(\Omega)$ for some measure space (Ω, Σ, μ). We assume additionally, that Ω is σ-finite.
We will also consider a more general setting, where the underlying Hilbert space is $\mathcal{H} = L^2(\Omega, H)$, the space of H-valued square integrable functions. Here we only suppose that H is a Hilbert space, but assume no order structure, such that $\mathcal{H}$ is not necessarily a lattice. Also in this case we assume the form domain $\mathcal{V}$ to be dense and continuously embedded in $\mathcal{H}$.

2.4.1 Locality and Support

We have seen in Section 1.3, that the operator associated to a coercive continuous sesquilinear form is the generator of a C_0-semigroup on H. By the Beurling-Deny criteria a necessary and sufficient condition for the positivity of the semigroup is the fact that the form domain V is a sublattice of H and that $a(u^+, u^-) \leq 0$ for all $u \in V$.
In most examples the form even satisfies $a(u^+, u^-) = 0$ for all $u \in V$. We want to discuss this property further.
Here we denote for $u, v \in L^2(\Omega)$ by $u \cdot v$ the pointwise product, which is measurable.

Lemma 2.4.1. *Let the form domain* V *of a coercive continuous sesquilinear form* a *be a sublattice of* $H = L^2(\Omega)$. *Then the following assertions are equivalent.*

(i) $a(u^+, u^-) = 0$ *for all* $u \in V$

(ii) $u \cdot v = 0$ *(a.e.) implies* $a(u, v) = 0$ *for all* $u, v \in V$.

Proof. *(i)* $\Rightarrow$ *(ii)* Since $u \cdot v = 0$ implies $|u| \cdot |v| = 0$, we can assume $u, v \geq 0$. Let $w = u - v$, then $u = w^+$ and $v = w^-$, because $u \cdot v = 0$. Hence from (i) follows $a(u,v) = a(w^+, w^-) = 0$.
(ii) $\Rightarrow$ *(i)* By the fact that $u^+ \cdot u^- = 0$ and (ii), we get $a(u^+, u^-) = 0$. □

Remark 2.4.2. Since V is a sublattice of H, $u \cdot v = 0$ is equivalent to $u \perp v$, i.e. u is lattice orthogonal to v which is again equivalent to $\inf\{|u|, |v|\} = 0$, where we take the infimum pointwise.

Now we want to give a name to this property in scalar and vector valued L^2-spaces.

Definition 2.4.3. (i) In $H = L^2(\Omega)$ a coercive continuous sesquilinear form a with domain V is called **local** if $u \cdot v = 0$ implies $a(u,v) = 0$ for all $u, v \in V$.

(ii) In $\mathcal{H} = L^2(\Omega, H)$ a coercive continuous sesquilinear form a with domain $\mathcal{V}$ is called **local** if $\|u\|_H \cdot \|v\|_H = 0$ implies $a(u,v) = 0$ for all $u, v \in \mathcal{V}$, where $\|u\|_H : \omega \mapsto \|u(\omega)\|_H \in L^2(\Omega)$.

If $H = L^2(X)$, where X is a locally compact measure space, in the literature, see for example [MR], a form is often called local if $\operatorname{supp} u \cap \operatorname{supp} v = \emptyset$ implies $a(u,v) = 0$, where the support of a measurable function is defined as in Definition 2.2.5.
At a first sight, one might think that on a locally compact space X the two definitions are equivalent. But our definition is stronger in general, since one has $\operatorname{supp} u \cap \operatorname{supp} v = \emptyset$ implies $|u| \cdot |v| = 0$ (a.e. on X). Indeed, $|u| \cdot |v| = 0$ a.e. on $O_u \cup O_v$, and $\operatorname{supp} u \cap \operatorname{supp} v = \emptyset$ implies $\operatorname{supp} u \subset O_v$, hence $|u| \cdot |v| = 0$ a.e. on $X = O_u \cup \operatorname{supp} u \subset O_u \cup O_v$.
Conversely, $|u| \cdot |v| = 0$ does not imply $\operatorname{supp} u \cap \operatorname{supp} v = \emptyset$, as is shown in the following example.

Example 2.4.4. On the interval $(0,1)$ we take the Lebesgue measure λ. Let $\mathbb{Q} \cap (0,1) = \{q_n; n \in \mathbb{N}\}$ and let $0 < \varepsilon_n < 1$, such that $\sum_n \varepsilon_n \leq \frac{1}{4}$, and for $A_n = (q_n - \varepsilon_n, q_n + \varepsilon_n)$ we have $A = \bigcup_n A_n \subset (0,1)$. Then we get for the measure $\lambda(A) \leq \sum_n \lambda(A_n) = 2 \sum \varepsilon_n \leq \frac{1}{2}$ and for $A^C = (0,1) \setminus A$, $\lambda(A^C) \geq \frac{1}{2}$. Let $f = \chi_A$, $g = \chi_{A^C}$, then $f \cdot g = 0$ on $(0,1)$.
However, $O_f = \{x : \exists U \in \mathcal{U}(x) : f_{|U} = 0 \text{ a.e.}\} = \emptyset$, because for all $U \subset (0,1)$ open, there exist $q_n \in U$ and $0 < \delta < \varepsilon_n$ such that $(q_n - \delta, q_n + \delta) \subset U$, but $f_{|(q_n - \delta, q_n + \delta)} \equiv 1$ and $\lambda((q_n - \delta, q_n + \delta)) = 2\delta > 0$. Hence $\operatorname{supp} f = (0,1)$, and $\operatorname{supp} f \cap \operatorname{supp} g = \operatorname{supp} g$.
We prove $\operatorname{supp} g \neq \emptyset$ by contradiction. Assume $\operatorname{supp} g = \emptyset$, then $O_g = (0,1)$, hence $g = 0$ a.e. on $(0,1)$, which contradicts, that $g = 1$ on A^C and $\lambda(A^C) \geq \frac{1}{2}$.

Analogously we call an operator **local** on $L^p(\Omega)$ if $Tf = 0$ a.e. on the set $\{f = 0\}$. We have seen in Section 2.2.1, that for a bounded operator T on $L^p(X)$, where X is a locally compact space, this definition is equivalent to $\operatorname{supp} Tf \subset \operatorname{supp} f$ for all $f \in C_c(X)$.

2.4.2 Characterization of Multiplication Operators

Here we want to characterize multiplication operators on $H = L^2(\Omega)$ associated to a coercive continuous sesquilinear form. We suppose the form domain V to be dense and continuously embedded in H. Additionally we require V to be a sublattice of H.

Theorem 2.4.5. *Let A be an operator on $L^2(\Omega)$, which is associated to a form a with domain V. The following assertions are equivalent.*

(i) A is a positive multiplication operator, i.e. there exists $m : \Omega \to [0,\infty)$ measurable, such that $(Au)(\omega) = m(\omega)u(\omega)$ almost everywhere and for all $u \in D(A) = \{u \in L^2 : mu \in L^2\}$

(ii) V has normal cone and a is local.

Proof. $(i) \Rightarrow (ii)$ Let A be a positive multiplication operator. In particular A is selfadjoint and the form is given by

$$\begin{aligned} V = D(A^{1/2}) &= \{u \in L^2 : m^{1/2}u \in L^2\} \\ a(u,v) &= (m^{1/2}u, m^{1/2}v)_H = \int_\Omega m(\omega)u(\omega)\overline{v(\omega)}\, d\mu \end{aligned}$$

Therefore V has normal cone, because for $u, v, w \in V$, the condition $v \leq u \leq w$ implies $|u(\omega)| \leq |v(\omega)| \vee |w(\omega)|$, for almost all $\omega \in \Omega$, hence

$$\begin{aligned} \|u\|_V &= \left(\int_\Omega m(\omega)|u(\omega)|^2 d\mu\right)^{1/2} \\ &\leq \left(\int_\Omega m(\omega)(|v(\omega)| \vee |w(\omega)|)^2 d\mu\right)^{1/2} \\ &\leq \left(2\left(\left(\int_\Omega m(\omega)|v(\omega)|^2 d\mu\right) \vee \left(\int_\Omega m(\omega)|w(\omega)|^2 d\mu\right)\right)\right)^{1/2} \\ &\leq \sqrt{2}\left(\left(\int_\Omega m(\omega)|v(\omega)|^2 d\mu\right)^{1/2} \vee \left(\int_\Omega m(\omega)|w(\omega)|^2 d\mu\right)^{1/2}\right) \\ &= \sqrt{2}(\|v\|_V \vee \|w\|_V). \end{aligned}$$

The form a is obviously local.
$(ii) \Rightarrow (i)$ First we show that V is an ideal in H. Since V is a sublattice, by Remark 2.1.4 it suffices to show, that for $u \in H,\ v \in V$, the condition $0 \leq u \leq v$ implies $u \in V$. Let $v \in V$ and $0 \leq u \leq v$. Since V is dense in H, there exists a sequence $u_n \in V$ converging to u in H. We can, without loss of generality, assume that $0 \leq u_n \leq v$, otherwise set $w_n := (u_n \wedge v) \vee 0$, then $0 \leq w_n \leq v$, $w_n \in V$, since V is a lattice and $w_n \to u$ in H, since the lattice operations are continuous in the Banach lattice H. Order bounded sets in V are norm bounded

because V has normal cone. Therefore $0 \leq u_n \leq v$ implies $\sup_{n\in\mathbb{N}} \|u_n\|_V < \infty$, from which we obtain since V is reflexive, a subsequence $u_{n_k} \rightharpoonup w$ converging weakly to w in V. And therefore u_n is converging weakly to w in H, since the embedding $V \hookrightarrow H$ is continuous. From this and the convergence of u_n to u in H we obtain $u = w \in V$, hence V is an ideal in H.
Since $-A$ is the generator of a C_0-semigroup, using Theorem 2.3.15 we only have to show, that the resolvent of A, $R(\lambda, A) = (\lambda - A)^{-1}$ is a bounded multiplication operator for one $\lambda <$ w, where w is the growth bound of the semigroup. By Theorem 2.2.3 this holds true if and only if $R(\lambda, A)u = 0$ almost everywhere on the set $\{\omega \in \Omega : u(\omega) = 0\}$, for all $u \in L^2(\Omega)$. Let $u \in L^2(\Omega)$ be arbitrary and define

$$K_u := \{v \in L^2(\Omega) : v(\omega) = 0 \text{ a.e. on } \{\omega \in \Omega : u(\omega) = 0\}\}.$$

Then K_u is a closed convex set in H and the orthogonal projection P_u onto K_u is given by

$$P_u v = v\chi_{\{\omega : u(\omega) \neq 0\}}.$$

Then $|(P_u v)(\cdot)| \leq |v(\cdot)|$ and since V is an ideal in H, we obtain

$$u \in V \Rightarrow P_u u \in V. \tag{2.14}$$

On the other hand

$$\begin{aligned} a(u, u - P_u u) &= a(u, u\chi_{\{u=0\}}) \\ &= a(u\chi_{\{u=0\}}, u\chi_{\{u=0\}}) + a(u\chi_{\{u\neq 0\}}, u\chi_{\{u=0\}}) \\ &\geq 0, \end{aligned} \tag{2.15}$$

by the positivity and locality of the form a. Since (2.14) and (2.15) give the condition (iii) of Theorem 1.3.5, we conclude that $\lambda R(\lambda, A) K_u \subset K_u$ for all $\lambda < 0$. In particular $u \in K_u$ and the multiplication with the scalar λ does not interfere with the property of belonging to the subspace K_u, hence $R(\lambda, A)u \in K_u$, i.e.

$$R(\lambda, A)u = 0 \text{ a.e. on the set } \{\omega \in \Omega : u(\omega) = 0\},$$

for all $u \in L^2(\Omega)$, since u was arbitrary. Consequently by Theorem 2.2.3, $R(\lambda, A)$ is a bounded multiplication operator, and hence A is an unbounded multiplication operator. □

2.4.3 Local Forms and Operator Valued Multiplication Operators

In this section we generalize the previous result to the space of vector valued square integrable functions. Hence the underlying Hilbert space is

$$\mathcal{H} = L^2(\Omega, H) := \{f : \Omega \to H \text{ measurable} : \int_\Omega \|f(\omega)\|_H^2 d\mu < \infty\},$$

where (Ω, Σ, μ) is an arbitrary σ-finite measure space and H a separable Hilbert space with norm and scalar product denoted by $\|.\|_H$ and $(.,.)_H$ respectively.
We generalize the notion of ideal in the following way.

Definition 2.4.6. A subspace $\mathcal{V}$ of $\mathcal{H}$ is called an **ideal** if for $u, v \in \mathcal{H}$ we have that $v \in \mathcal{V}$ and $\|u(\omega)\|_H \leq \|v(\omega)\|_H$ a.e. on Ω implies $u \in \mathcal{V}$.

Let $\mathcal{V}$ be a Hilbert space which is dense and continuously embedded in $\mathcal{H}$. Let a be a continuous coercive sesquilinear form with domain $\mathcal{V}$, and $(A, D(A))$ the associated operator.
Recall that we say that the sesquilinear form a is local, if $\|u(\omega)\|_H \cdot \|v(\omega)\|_H = 0$ for a.e. $\omega \in \Omega$ implies $a(u, v) = 0$.
We want to give conditions on the form such that the associated operator is an unbounded operator valued multiplication operator. Since the associated operator is densely defined and closed, we wish to get densely defined closed fiber operators, i.e. a family $(A(\omega))_{\omega\in\Omega}$ of densely defined closed operators on H, such that

$$\begin{aligned} D(A) &= \{f \in \mathcal{H} : f(\omega) \in D(A(\omega)) \text{ a.e. on } \Omega \text{ and } \omega \mapsto A(\omega)f(\omega) \in \mathcal{H}\} \\ Af(\omega) &= A(\omega)f(\omega) \text{ a.e.} \end{aligned}$$

The following example shows, that for a multiplication operator on $\mathcal{H}$, which is associated to a form, the form domain need not be an ideal in $\mathcal{H}$.

Example 2.4.7. Let $H = L^2(\mathbb{R}^n)$, hence $\mathcal{H} = L^2(\Omega, L^2(\mathbb{R}^n))$. Define the form a on $\mathcal{V} := L^2(\Omega, H^1(\mathbb{R}^n))$ by $a(u, v) = \int_\Omega \int_{\mathbb{R}^n} \nabla u(\omega) \nabla v(\omega) \, d\lambda \, d\mu$. Then the associated operator is given by $(Au)(\omega) = \Delta(u(\omega))$ a.e. on Ω on the domain $D(A) = \{u \in \mathcal{H} : u(\omega) \in D(\Delta) \text{ a.e. and } \Delta u(\omega) \in \mathcal{H}\}$. Hence A is a multiplication operator, but $\mathcal{V}$ is not an ideal in $\mathcal{H}$.

However, we still have the one implication.

Theorem 2.4.8. *Assume that the form domain $\mathcal{V}$ is an ideal in $\mathcal{H}$ and that the form a is local, then the associated operator A is a multiplication operator.*

Proof. Let $f \in L^2(\Omega, H)$ be arbitrary and define

$$K_f := \{g \in L^2(\Omega, H) : \|g(\omega)\|_H = 0 \text{ a.e. on } \{\omega \in \Omega : \|f(\omega)\|_H = 0\}\}.$$

Then K_f is a closed convex set in $\mathcal{H}$ and the orthogonal projection P_f onto K_f is given by

$$P_f g = g\chi_{\{\omega : \|f(\omega)\|_H \neq 0\}}.$$

Then $\|(P_f g)(\cdot)\|_H \leq \|g(\cdot)\|_H$ and since $\mathcal{V}$ is an ideal in $\mathcal{H}$, we obtain

$$u \in \mathcal{V} \Rightarrow P_f u \in \mathcal{V}. \tag{2.16}$$

On the other hand

$$\begin{aligned} a(u, u - P_f u) &= a(u, u\chi_{\{f=0\}}) \\ &= a(u\chi_{\{f=0\}}, u\chi_{\{f=0\}}) + a(u\chi_{\{f\neq 0\}}, u\chi_{\{f=0\}}) \\ &\geq 0, \end{aligned} \tag{2.17}$$

by the positivity and locality of the form a. Since (2.16) and (2.17) give the condition (iii) of Theorem 1.3.5, we conclude that $\lambda R(\lambda, A) K_f \subset K_f$ for all $\lambda < 0$. In particular $f \in K_f$ and again the multiplication with the scalar λ does not interfere with the property of belonging to the subspace K_f, hence $R(\lambda, A) f \in K_f$, i.e.

$$R(\lambda, A) f = 0 \text{ a.e. on the set } \{\omega \in \Omega : \|f(\omega)\|_H = 0\},$$

for all $f \in L^2(\Omega, H)$, since f was arbitrary. Consequently by Theorem 2.2.17 $R(\lambda, A)$ is a bounded operator valued multiplication operator for all $\lambda < 0$. Therefore by Theorem 2.3.15 the operator $-A$, and hence A is an unbounded operator valued multiplication operator. □

Chapter 3

Non-autonomous Cauchy Problems

In this chapter we study non-autonomous Cauchy problems, which are associated with a family of linear operators depending on the time parameter t. Thus, for a Hilbert space H we consider initial value problems of the type

$$\begin{cases} u'(t) + A(t)u(t) &= f(t), \quad t \geq 0 \\ u(0) &= x \qquad \in H \end{cases}$$

and seek solutions in the function space $L^2(0, T; H)$. Moreover, we assume, that the linear operators $A(t)$, which depend on the time parameter t, are associated with sesquilinear forms.

After recalling a result on existence and uniqueness of a solution, we examine further its properties. We are interested in positive and sub-Markovian solutions as well as regularity results. A different approach to non-autonomous Cauchy problems uses semigroup theory and evolution families. Here we use a stronger notion of well-posedness, but we will apply the result to the original problem. On this basis we obtain Beurling-Deny criteria as a consequence of invariance of closed convex sets characterized for generalized sesquilinear forms.

3.1 Solutions of Non-autonomous Cauchy Problems

As we only consider those problems, in which the operators are associated with sesquilinear forms, we first treat the variational formulation. Then we examine its relation to the abstract Cauchy problem. An ingenious representation theorem of linear functionals in terms of some kind of quadratic forms by J.L. Lions, see [Li], Chapitre III, then enables us to deduce well-posedness.

3.1.1 The Non-autonomous Variational Problem

In the following we use form methods as introduced in section 1.3. We consider two separable Hilbert spaces V and H, denoting the scalar product and norm in V by $((\ ,\ ))$ and $\|.\|$, and in H by $(\ ,\ )$ and $|.|$ respectively.
We suppose that V is a dense subspace of H with continuous embedding, i.e. $|u| \leq c\|u\|$, for all $u \in V$ and a constant c. If V' denotes the dual of V we obtain by identifying H with its dual H'

$$V \overset{d}{\hookrightarrow} H \overset{d}{\hookrightarrow} V'.$$

The duality pairing between V' and V is also denoted by $(\ ,\ )$, since for $u \in V$ and $h \in H \subset V'$ one has $h(u) = (h, u)$, the scalar product of h and u in H.
If we denote $\mathcal{V} := L^2(0, T; V)$ and $\mathcal{H} := L^2(0, T; H)$, then $\mathcal{V}' = L^2(0, T; V')$ and $\mathcal{V} \overset{d}{\hookrightarrow} \mathcal{H} \overset{d}{\hookrightarrow} \mathcal{V}'$ holds. Let X be a Banach space, in particular X stands for one of the spaces V, H or V'.

Definition 3.1.1. We say, that a function $u \in L^2(0, T; X)$ is **differentiable in the sense of distributions** if there exists a function $u' \in L^2(0, T; X)$ such that

$$\int_0^T u(t)\varphi'(t)\, dt = -\int_0^T u'(t)\varphi(t)\, dt$$

holds for all $\varphi \in \mathcal{D}(0, T) = \{\varphi \in C^\infty(\mathbb{R}) : \operatorname{supp}\varphi \subseteq [0, T]\}$. The function $u' \in L^2(0, T; X)$ is called the **derivative in the sense of distributions**.

Functions in $L^2(0, T; V)$ with the property, that $u' \in L^2(0, T; V')$, will play an important role in the following, so that it makes sense to introduce the following notation.

Definition 3.1.2. We denote by $W = W(0, T; V, V')$ the space of functions $u \in L^2(0, T, V)$ such that $u' \in L^2(0, T; V')$, i. e.

$$W = \{u \in L^2(0, T; V) : u' \in L^2(0, T; V')\}.$$

These spaces are thoroughly studied in ([DL5], XVIII §1.2.) and we will just quote the properties we need, referring to [DL5] and [Ta], Section 5.5 for the proofs.

Proposition 3.1.3. *The space $W = W(0, T; V, V')$ equipped with the norm*

$$\|u\|_W = \left(\|u\|^2_{L^2(0,T,V)} + \|u'\|^2_{L^2(0,T,V')}\right)^{\frac{1}{2}} = \left(\int_0^T [\|u(t)\|^2 + \|u'(t)\|^2_{V'}]dt\right)^{\frac{1}{2}}$$

is a Hilbert space.

Every $u \in W = W(0,T;V,V')$ is almost everywhere equal to a unique continuous function of $[0,T]$ in H. Further, we have a continuous embedding

$$W(0,T;V,V') \hookrightarrow C([0,T];H),$$

the space $C([0,T];H)$ being equipped with the norm of uniform convergence.
Consequently, for a function $u \in W(0,T;V,V')$ we may speak of the traces $u(0)$, $u(T) \in H$. Moreover, the mapping $W(0,T;V,V') \to H : u \mapsto u(0)$ is surjective.
If $u,\ v \in W = W(0,T;V,V')$, then $(u(t),v(t))$ is absolutely continuous and the following equality holds:

$$\frac{d}{dt}(u(t),v(t)) = (u'(t),v(t)) + (u(t),v'(t)).$$

In particular, for $u \in W = W(0,T;V,V')$, $v \in V$, we obtain that the equality $\frac{d}{dt}((u(t),v) = (u'(t),v)$ holds, where the derivatives are taken in the sense of distributions.

Remark 3.1.4. Note that the above proposition holds for $[0,T]$ replaced by any interval $[a,b]$ with $a < b \in \mathbb{R}$.

Now we return to the spaces $V \overset{d}{\hookrightarrow} H \overset{d}{\hookrightarrow} V'$. Assume that we are given a family of sesquilinear forms on H with domain V, denoted $a(t;u,v)$, depending on the time parameter $t \in [0,T]$, with T finite. We suppose the following:

$$\left.\begin{array}{c} \text{for } u,v \in V, \text{ the function } t \to a(t;u,v) \text{ is measurable, and} \\ |a(t;u,v)| \le M\|u\|\|v\|, \\ M \text{ being a constant independent of } t,\ u, \text{ and } v. \end{array}\right\} \quad (3.1)$$

Lemma 3.1.5. *Under these assumptions and for $u,\ v : [0,T] \to V$ measurable we have*

$$t \to a(t;u(t),v(t))$$

is measurable.

Proof. It suffices to show, that there exists a sequence of measurable functions $f_n(t)$ with $|f_n(t) - a(t;u(t),v(t))| \to 0 \quad (n \to \infty)$, for almost all $t \in [0,T]$.
We have $u,\ v$ measurable, therefore there exist sequences of step functions

$$u_n(t) = \sum_i \mu_i^{(n)} \chi_{A_i^{(n)}}(t), \qquad \text{for almost all } t \in [0,T]$$

and

$$v_n(t) = \sum_i \nu_i^{(n)} \chi_{A_i^{(n)}}(t), \qquad \text{for almost all } t \in [0,T]$$

such that $\|u_n(t) - u(t)\| \to 0$ and $\|v_n(t) - v(t)\| \to 0$ for $n \to \infty$ and almost all $t \in [0, T]$. Here we achieve the same measurable sets $A_i^{(n)}$ for u_n and v_n after an adequate consideration of intersections.
Let

$$\begin{aligned} f_n(t) &= a(t; u_n(t), v_n(t)) = a(t; \sum_i \mu_i^{(n)} \chi_{A_i^{(n)}}(t), \sum_i \nu_i^{(n)} \chi_{A_i^{(n)}}(t)) \\ &= \sum_i a(t; \mu_i^{(n)}, \nu_i^{(n)}) \chi_{A_i^{(n)}}(t) \end{aligned}$$

which is measurable, since each term is measurable and the sum is finite.
And we have

$$\begin{aligned} &|f_n(t) - a(t; u(t), v(t))| \\ &= |a(t; u_n(t), v_n(t)) - a(t; u(t), v(t))| \\ &= |a(t; u_n(t) - u(t), v_n(t))| + |a(t; u(t), v_n(t) - v(t))| \\ &\leq M\|u_n(t) - u(t)\| \, \|v_n(t)\| + M\|u(t)\| \, \|v_n(t) - v(t))\| \\ &\to 0 \quad (\text{as } n \to \infty), \end{aligned}$$

which completes the proof. □

Remark 3.1.6. Due to the inequality $|a(t; u, v)| \leq M\|u\| \, \|v\|$ with M independent of t, we obtain for $u, \ v \in L^2(0, T; V)$ and for all $t \in [0, T]$, the inequalities $|a(t; u(t), v(t))| \leq M\|u(t)\| \, \|v(t)\|$. Therefore we have $a(t; u(t), v(t)) \in L^1(0, T)$ and if $v \in V$ we get $a(t; u(t), v) \in L^2(0, T)$.

For given $u_0 \in H$ and $f \in L^2(0, T; V')$ we consider the following problem.

Problem 3.1.7. *Find a function $u \in L^2(0, T; V)$, with*

$$\int_0^T \{a(t; u(t), \varphi(t)) - (u(t), \varphi'(t))\} dt = \int_0^T (f(t), \varphi(t)) dt + (u_0, \varphi(0)), \tag{3.2}$$

for all functions φ satisfying

$$\varphi \in L^2(0, T; V), \quad \varphi' \in L^2(0, T; V'), \quad \varphi(T) = 0. \tag{3.3}$$

Definition 3.1.8. We say that a solution $u \in L^2(0, T; V)$ of Problem 3.1.7 **depends continuously on the given data**, if there exists a constant $C > 0$ such that

$$\|u\|_{L^2(0,T;V)} \leq C \left(\|u_0\|_H^2 + \|f\|^2_{L^2(0,T;V')} \right)^{1/2}.$$

Problem 3.1.7 is called **well-posed**, if for all $u_0 \in H$ and $f \in L^2(0, T; V')$ there exists a unique $u \in L^2(0, T; V)$ such that (3.2) is satisfied, and which depends continuously on the given data.

3.1.2 The Associated Abstract Cauchy Problem

Under the hypotheses (3.1), we can associate to $a(t; u, v)$ for each $t \in [0, T]$ a linear operator $\mathcal{A}(t) \in \mathcal{L}(V, V')$ defined by

$$(\mathcal{A}(t)u, v) = a(t; u, v),$$

for all $u,\ v \in V$.
Then for a function u given in $L^2(0, T; V)$ the function

$$t \mapsto \mathcal{A}(t)u(t)$$

is measurable with values in V'. Indeed, the map $t \mapsto \mathcal{A}(t)u(t)$ is weakly measurable, because $(\mathcal{A}(t)u(t), v) = a(t; u(t), v)$ is measurable, and has its values in the separable space V', hence it is measurable by a theorem due to B. J. Pettis, see [HP], Theorem 3.5.3, or [Pe], or [ABHN], Theorem 1.1.1. Furthermore, $\|\mathcal{A}(t)u(t)\|_{V'} \leq M\|u(t)\|_V$, from which we obtain $\mathcal{A}(t)u(t) \in L^2(0, T; V')$.

Proposition 3.1.9. *A solution of Problem 3.1.7 satisfies*

$$u' \in L^2(0, T; V')$$

where u' is the derivative in the sense of distributions. Moreover, $u(0) = u_0$.

Proof. We have to show, that there exists a function $w \in L^2(0, T; V')$ such that

$$\int_0^T w(t)\psi(t)dt = -\int_0^T u(t)\psi'(t)dt, \tag{3.4}$$

for all $\psi \in \mathcal{D}(0, T)$.
Note that u has its values in $V \subset V'$ and w has its values in V', so that (3.4) is meant to hold in V', i.e. for all $v \in V$

$$\left(\int_0^T w(t)\psi(t)dt, v\right) = -\left(\int_0^T u(t)\psi'(t)dt, v\right).$$

In the following we will use, that if $\psi \in \mathcal{D}(0, T)$, then so is the complex conjugate $\overline{\psi}$, and for $v \in V$ the function $\varphi(t) = \overline{\psi(t)}v$ satisfies (3.3) and $\varphi(0) = 0$. Moreover, $\varphi'(t) = \overline{\Psi'(t)}v$.

We set $w(t) = -\mathcal{A}(t)u(t) + f(t)$, then $w \in L^2(0,T;V')$ and

$$
\begin{aligned}
&\left(\int_0^T w(t)\psi(t)dt, v\right) \\
&= \int_0^T (w(t)\psi(t), v)dt = \int_0^T (w(t), \overline{\psi(t)}v)dt \\
&= \int_0^T (-\mathcal{A}(t)u(t) + f(t), \overline{\psi(t)}v)dt \\
&= \int_0^T -(\mathcal{A}(t)u(t), \overline{\psi(t)}v)dt + \int_0^T (f(t), \overline{\psi(t)}v)dt \\
&= -\int_0^T a(t; u(t), \overline{\psi(t)}v)dt + \int_0^T (f(t), \overline{\psi(t)}v)dt \\
&= -\int_0^T (u(t), \overline{\psi'(t)}v)dt \qquad \text{(by (3.2), since } u \text{ is a solution of Problem 3.1.7)} \\
&= -\int_0^T (u(t)\psi'(t), v)dt = -\left(\int_0^T u(t)\psi'(t)dt, v\right).
\end{aligned}
$$

Moreover, for all φ satisfying (3.3), one obtains from (3.2)

$$
\begin{aligned}
(u_0, \varphi(0)) &= \int_0^T \{a(t; u(t), \varphi(t)) - (u(t), \varphi'(t))\}dt - \int_0^T (f(t), \varphi(t))dt \\
&= \int_0^T \{(\mathcal{A}(t)u(t) - f(t), \varphi(t))dt - \int_0^T (u(t), \varphi'(t))\}dt \\
&= -\int_0^T (u'(t), \varphi(t))\}dt - \int_0^T (u(t), \varphi'(t))\}dt \\
&= -\int_0^T \frac{d}{dt}(u(t), \varphi(t))\}dt \\
&= (u(0), \varphi(0)).
\end{aligned}
$$

Observe that $\{\varphi(0) : \varphi \text{ satisfies (3.3)}\}$ is dense in H. Thus, $u(0) = u_0$. □

Proposition 3.1.10. *Problem 3.1.7 can equivalently be formulated as follows.*

Problem 3.1.11. *For given initial value $u_0 \in H$ and $f \in L^2(0,T;V')$ find a function $u \in W = W(0,T;V,V')$, satisfying $u(0) = u_0$ and*

$$\int_0^T \{(u'(t), \varphi(t)) + a(t; u(t), \varphi(t))\}dt = \int_0^T (f(t), \varphi(t))dt, \tag{3.5}$$

for all $\varphi \in L^2(0,T;V)$.

or

Problem 3.1.12. *For given $u_0 \in H$ and $f \in L^2(0,T;V')$ find $u \in L^2(0,T;V)$, satisfying*

$$u' \in L^2(0,T;V')$$
$$u'(t) + \mathcal{A}(t)u(t) = f(t) \quad \textit{a.e. in } V'$$
$$u(0) = u_0.$$

Proof. Let u be a solution of Problem 3.1.7. Then by Proposition 3.1.9 it satisfies $u \in W = W(0,T;V,V') \subset C([0,T];H)$ and $u(0) = u_0$. Moreover, (3.2) is equivalent to (3.5). As the set of all functions φ satisfying (3.3) is dense in $L^2(0,T;V)$, the function u is a solution of Problem 3.1.11. Conversely, a solution of Problem 3.1.11 obviously solves Problem 3.1.7.
For the equivalence of Problem 3.1.11 and Problem 3.1.12, observe that (3.5) holds in particular for all functions, which can be written as $\varphi = \psi \otimes v$, with $\psi \in \mathcal{D}(0,T)$ and $v \in V$, and we obtain this equation in vectorial form

$$u'(t) + \mathcal{A}(t)u(t) = f(t) \quad \text{a.e. in } V'.$$

Vice versa, the functions of the form $f = \sum_i \psi_i \otimes v_i$, where the sum is finite, are dense in the space $L^2(0,T;V)$. □

3.1.3 The representation theorem by Lions

The representation theorem by Riesz-Fréchet plays an important role for well-posedness of autonomous variational Cauchy problems. In [Li], J. L. Lions introduces a more general representation theorem, which serves as an important tool to show well-posedness for non-autonomous variational Cauchy-Problems. We want to present his methods here.
Let F be a Hilbert space. If u, $v \in F$, we denote by $(u,v)_F$ the scalar product of u and v and set $||u||_F = (u,u)_F^{1/2}$.
Let Φ be a subspace of F. We suppose that Φ is provided with a scalar product $(((\varphi,\psi)))$, for φ, $\psi \in \Phi$, such that Φ becomes a pre-Hilbert space.
For the norm $|||\varphi||| = (((\varphi,\varphi)))^{\frac{1}{2}}$ the space Φ is complete or not (the more interesting case corresponding to Φ being non-complete). We suppose that the function $\varphi \to \varphi$ from Φ into F is continuous, i.e.

$$||\varphi||_F \leq c_1|||\varphi|||, \tag{3.6}$$

for all $\varphi \in \Phi$, where c_1 is a constant.
Note that the space Φ is not necessarily dense in F.
The previous conditions are for example fulfilled if Φ is a vector-subspace of F, being neither closed nor dense in F, with the induced pre-Hilbert structure.
We assume to have a sesquilinear form $E(u,\varphi)$ on $F \times \Phi$ satisfying the following hypotheses:

$$\text{for all } \varphi \in \Phi, \text{ the form } u \to E(u,\varphi) \text{ is continuous on } F; \tag{3.7}$$

$$\left.\begin{array}{l}\text{there exists a constant } \alpha > 0 \text{ such that} \\ |E(\varphi,\varphi)| \geq \alpha |||\varphi|||^2, \text{ for all } \varphi \in \Phi.\end{array}\right\} \tag{3.8}$$

Note the fact, that the semilinear form $\varphi \to E(u,\varphi)$ need not be continuous on Φ for fixed u in F.
We have the following existence result.

Theorem 3.1.13. *We assume the hypotheses (3.6), (3.7) and (3.8). If $\varphi \to L(\varphi)$ is a continuous semilinear form on Φ, there exists an element u in F satisfying*

$$E(u,\varphi) = L(\varphi) \tag{3.9}$$

for all $\varphi \in \Phi$.

Proof. Since (3.7) holds, we can write

$$E(u,\varphi) = (u, K\varphi)_F, \tag{3.10}$$

which defines a linear operator $\varphi \to K\varphi$ from Φ into F. The mapping K from Φ into F is one-to-one; actually, if $K\varphi = 0$, then $(\varphi, K\varphi)_F = E(\varphi,\varphi) = 0$, and by (3.8) we get $\varphi = 0$, since $|||\varphi|||$ is a norm.
Let $K\Phi = \mathcal{A}$, then the inverse R_0 of K is continuous from $\mathcal{A}$ (with the topology induced by F) into Φ.
To see this, we let $K\varphi = a$, then $\varphi = R_0 a$, and with (3.8) we have

$$\alpha |||R_0 a|||^2 \leq |E(\varphi,\varphi)| = |(\varphi, K\varphi)_F| \leq ||\varphi||_F\, ||K\varphi||_F \leq c_1 |||\varphi|||\; ||K\varphi||_F,$$

from which we obtain

$$|||R_0 a||| \leq \left(\frac{c_1}{\alpha}\right) ||a||_F,$$

which shows that R_0 is continuous.
Therefore we can extend R_0 to $\bar{R}_0$, a continuous linear mapping from $\bar{\mathcal{A}} = \mathcal{B}$ (closure of $\mathcal{A}$ in F) into $\hat{\Phi}$ (completion of Φ with respect to $|||.|||$).
The semilinear form $\varphi \to L(\varphi)$ can by continuity be extended to $\hat{\Phi}$, so that

$$L(\varphi) = (((\xi_L, \varphi))),$$

with $\xi_L \in \hat{\Phi}$, and the equation (3.9) is equivalent to

$$(u, K\varphi)_F = (((\xi_L, \varphi))),$$

for all $\varphi \in \Phi$, or to

$$(u, a)_F = (((\xi_L, R_0 a))) = (((\xi_L, \bar{R}_0 a))), \quad \forall a \in \mathcal{A}. \tag{3.11}$$

A solution of (3.11) can be found immediately. Let for example P be the orthogonal projection (in F) onto $\mathcal{B}$, then $R = \bar{R}_0 P \in \mathcal{L}(F; \hat{\Phi})$. We denote by $R^* \in \mathcal{L}(\hat{\Phi}; F)$ its adjoint and (3.11) is equivalent to

$$(u, a)_F = (((\xi_L, Ra))) = (R^* \xi_L, a)_F, \quad \forall a \in \mathcal{A},$$

from which follows that one solution of the problem is

$$u = R^* \xi_L. \tag{3.12}$$

□

Remark 3.1.14. In general, there is no uniqueness of solutions; the necessary and sufficient condition for uniqueness is $\mathcal{A}$ being dense in F.

Remark 3.1.15. Let $|||L|||$ be the norm of L, i.e.

$$|||L||| = \sup_{\substack{\varphi \in \Phi \\ |||\varphi||| \leq 1}} |L(\varphi)|.$$

Then the Solution (3.12) of Equation (3.9) satisfies

$$\|u\|_F \leq \left(\frac{c_1}{\alpha}\right) |||L|||^2. \tag{3.13}$$

3.1.4 Well-Posedness

Recall that we say that Problem 3.1.7 well-posed, if there exists a unique solution which depends continuously on the given data. Under one additional assumption on the form $a(t; u, v)$, Problem 3.1.7 is well-posed. The proof of the following theorem is taken from [Li] and [Ta].

Theorem 3.1.16. *We assume that $a(t; u, v)$ satisfies (3.1), as well as the following hypothesis: there exists $\lambda \in \mathbb{R}$ and $\alpha > 0$, such that*

$$\operatorname{Re} a(t; u, u) + \lambda |u|^2 \geq \alpha \|u\|^2, \quad \forall u \in V. \tag{3.14}$$

Then Problem 3.1.7 is well-posed.

Proof. 1) *Preliminary reduction.*
One can always assume (3.14) to hold for $\lambda = 0$. Indeed, if we set $u = \exp(kt)w$, k a real number to be determined, the Problem 3.1.7 is equivalent to finding a function w, which is zero for $t < 0$ and satisfies

$$a(t; w(t), v) + k(w(t), v) + \frac{d}{dt}(w(t), v) = (\exp(-kt)f, v) + (u_0, v)\delta;$$

This is a problem equivalent to Problem 3.1.7 , but with $a(t; u, v)$ replaced by $a(t, u, v) + k(u, v)$, from which the result follows.
2) *Existence.*
We use the representation Theorem 3.1.13 in the following situation:

$$F = L^2(0, T; V), \quad \|u\|_F = (\int_0^T \|u(t)\|^2 dt)^{\frac{1}{2}},$$

where Φ is the space of functions φ satisfying (3.3) with norm

$$|||\varphi||| = (\|\varphi\|_F^2 + |\varphi(0)|^2)^{\frac{1}{2}}.$$

We set

$$E(u,\varphi) = \int_0^T \{a(t;u(t),\varphi(t)) - (u(t),\varphi'(t))\}dt;$$

and

$$L(\varphi) = \int_0^T (f(t),\varphi(t))dt + (u_0,\varphi(0)).$$

Let us now verify, that we are in the situation of Theorem 3.1.13.
For φ fix in Φ, the form $u \to E(u,\varphi)$ is continuous on F. Then

$$\begin{aligned}\operatorname{Re} E(\varphi,\varphi) &= \int_0^T \operatorname{Re} a(t;\varphi(t),\varphi(t))dt - \frac{1}{2}\int_0^T \frac{d}{dt}|\varphi(t)|^2 dt \\ &= \int_0^T \operatorname{Re} a(t;\varphi(t),\varphi(t))dt + \left(\frac{1}{2}\right)|\varphi(0)|^2.\end{aligned}$$

According to 1), we conclude

$$\operatorname{Re} E(\varphi,\varphi) \geq \alpha \int_0^T \|\varphi(t)\|^2 dt + \frac{1}{2}|\varphi(0)|^2 \geq \inf(\alpha,\frac{1}{2})|||\varphi|||^2.$$

Finally the semilinear form $\varphi \to L(\varphi)$ is continuous on Φ for the norm $|||\varphi|||$.
Therefore, by applying Theorem 3.1.13, there exists u in F with $E(u,\varphi) = L(\varphi)$ for all $\varphi \in \Phi$, i.e. a solution of Problem 3.1.7.

3) *Uniqueness.*

We only need to proof that $u = 0$ if $u_0 = 0$ and $f = 0$. By Proposition 3.1.3, $|u(t)|^2$ is absolutely continuous and

$$\begin{aligned}\frac{d}{dt}|u(t)|^2 + 2\operatorname{Re} a(t;u(t),u(t)) &= 2\operatorname{Re}(u'(t),u(t)) + 2\operatorname{Re}(\mathcal{A}(t)u(t),u(t)) \\ &= 2\operatorname{Re}(u'(t) + \mathcal{A}(t)u(t),u(t)) = 0,\end{aligned}$$

so that $\frac{d}{dt}|u(t)|^2 < 0$ and hence, $|u(t)|$ is a decreasing function. Since $|u(0)| = 0$, we have $u(t) \equiv 0$.

4) *Continuous Dependence on Data*

The solution u of $E(u,\varphi) = L(\varphi)$ depends continuously on L (see Remark 3.1.15), such that the mapping $\{f,u_0\} \to u$ is continuous from $L^2(-\infty,T;V') \times H$ into $L^2(-\infty,T;V)$. One obtains from (3.13) that

$$\|u\|_F \leq \frac{1}{\inf(\alpha,\frac{1}{2})}\left(\int_0^T |f(t)|^2 dt + |u_0|^2\right)^{\frac{1}{2}},$$

supposing that (3.14) holds for $\lambda = 0$. □

Recall that by Proposition 3.1.3, we have that a solution is an element of the space $C([0,T];H)$, and the following lemma shows, that in this space we still have continuous dependence on the given data.

Lemma 3.1.17. *Let $(u_0, f) \in H \times L^2(0,T;V')$ and let u be the corresponding solution of Problem 3.1.7. Then for all $t \in [0,T]$ the following estimate holds.*

$$\frac{1}{2}|u(t)|_H^2 + \frac{\alpha}{2}\int_0^t \|u(\tau)\|_V^2 \, d\tau \le \frac{1}{2}|u_0|_H^2 + \frac{1}{2\alpha}\int_0^T \|f(\tau)\|_{V'}^2 \, d\tau.$$

In particular, there exists a constant $C \in \mathbb{R}$, such that

$$\|u\|_{C([0,T];H)} = \sup_{t\in[0,T]} |u(t)|_H \le C(|u_0|_H + \|f\|_{L^2(0,T;V')}).$$

Proof. Since u is a solution of Problem 3.1.7, it satisfies (3.5), from which we obtain by replacing φ by u and T by τ, the *energy equality*

$$\int_0^\tau \{(u'(t), u(t)) + a(t; u(t), u(t))\}dt = \int_0^\tau (f(t), u(t))dt.$$

From (3.14), for $\lambda = 0$, we obtain

$$\begin{aligned}
&\frac{1}{2}(u(\tau), u(\tau)) - \frac{1}{2}(u(0), u(0)) + \int_0^\tau \alpha \|u(t)\|_V^2 dt \\
&\le \int_0^\tau \frac{1}{2}\frac{d}{dt}(u(t), u(t)) + \int_0^\tau \operatorname{Re} a(t; u(t), u(t))\, dt \\
&= \int_0^\tau \operatorname{Re}\{(u'(t), u(t)) + a(t; u(t), u(t))\}\, dt \\
&= \int_0^\tau \operatorname{Re}(f(t), u(t))\, dt \\
&\le \int_0^\tau |(f(t), u(t))|\, dt \\
&\le \int_0^\tau \|f(t)\|_{V'} \|u(t)\|_V\, dt \\
&\le \frac{1}{2\alpha}\int_0^\tau \|f(t)\|_{V'}^2\, dt + \frac{\alpha}{2}\int_0^\tau \|u(t)\|_V^2\, dt,
\end{aligned}$$

which implies

$$\begin{aligned}
\frac{1}{2}(u(\tau), u(\tau)) + \frac{\alpha}{2}\int_0^\tau \|u(t)\|_V^2 dt &\le \frac{1}{2}(u(0), u(0)) + \frac{1}{2\alpha}\int_0^\tau \|f(t)\|_{V'}^2\, dt \\
&\le \frac{1}{2}(u_0, u_0) + \frac{1}{2\alpha}\int_0^T \|f(t)\|_{V'}^2\, dt,
\end{aligned}$$

and the claim follows. ◻

3.2 Positive and Sub-Markovian Solutions

Now that we have well-posedness for the non-autonomous variational Cauchy problem, we are interested in the properties of the solution. Here we want to find sufficient conditions on the form such that the solution is positive or sub-Markovian.
In order to talk about positivity, we need the setting of a lattice. It is clear, that if V is a lattice, then $L^2(0,T;V)$ is a lattice. But we do not know an answer to the following question.

Question 3.2.1. If V is a lattice, do we have $W = W(0,T;V,V')$ is a lattice?

However, there is some sense to lattice operation, if we consider the following function spaces and spaces of distributions,

$$V = H_0^1(\Omega), \quad H = L^2(\Omega), \quad V' = H^{-1}(\Omega),$$

where $\Omega \subset \mathbb{R}^n$ is an open set.
We denote by $\|.\|$ the norm in $L^2(0,T;X)$, where X denotes one of the spaces V, H or V', and by $|.|$ the norm in $H = L^2(\Omega)$.
The results of this section have their roots in discussions with D. Daners, while he stayed at Ulm.

3.2.1 Lattice Operations

On the spaces of lattice valued functions, lattice operations have a sense. Here we study their properties in a particular setting.
Let $\mathcal{D}([0,T],V)$ be the space of all functions $\varphi : [0,T] \to V$, which are infinitely differentiable. Then $\mathcal{D}([0,T],V)$ is dense in $W = W(0,T;V,V')$, see [DL5], XVIII §1, Lemma 1.
Since $\mathcal{D}(\Omega)$ is dense in $V = H_0^1(\Omega)$ and $\mathcal{D}([0,T],\mathcal{D}(\Omega)) = \mathcal{D}([0,T]\times\Omega)$ we obtain the following density.

Proposition 3.2.2. $\mathcal{D}([0,T]\times\Omega)$ *is dense in* $W = W(0,T;V,V')$.

As V and H are lattices, we can consider pointwise the positive part of a function $u \in L^2(0,T;V)$, respectively $u \in L^2(0,T;H)$. We denote by u^+ the function $u^+ : (0,T) \to V$, respectively $u^+ : (0,T) \to H$, defined by $u^+(t) = u(t)^+$.
Since $H = L^2(\Omega)$ and $L^2(0,T;L^2(\Omega))$ is isometrically isomorphic to $L^2((0,T)\times\Omega)$ which is a Banach lattice, for a function $u \in L^2(0,T;L^2(\Omega))$, u^+ coincides with the positive part in $L^2((0,T)\times\Omega)$. Therefore, we can immediately deduce that $u^+ \in L^2(0,T;L^2(\Omega)$ and also get that $u_n \to u$ in $L^2(0,T;L^2(\Omega))$ implies $u_n^+ \to u^+$ in $L^2(0,T;L^2(\Omega))$.
The following lemma gives us the same properties in $L^2(0,T;H_0^1(\Omega))$.

Lemma 3.2.3. *(i) Let* $u \in L^2(0,T;H_0^1(\Omega))$*, then* $u^+ \in L^2(0,T;H_0^1(\Omega))$*.*

(ii) Let $u_n \to u$ in $L^2(0,T;H_0^1(\Omega))$, then $u_n^+ \to u^+$ in $L^2(0,T;H_0^1(\Omega))$.

Proof. (i) Since for almost every $t \in [0,T]$, $u(t) \in H_0^1(\Omega)$, which is a lattice, we have $u(t)^+ \in H_0^1(\Omega)$ and since $(u^+)' = u'\chi_{\{u\geq 0\}}$, we get $\|u(t)^+\|_{H_0^1} \leq \|u(t)\|_{H_0^1}$. We have $t \mapsto u(t)$ is measurable, and by the continuity of the lattice operations in $H_0^1(\Omega)$ we also get $t \mapsto u(t)^+$ is measurable and moreover obtain the estimate $\int_0^T \|u(t)^+\|^2_{H_0^1} dt \leq \int_0^T \|u(t)\|^2_{H_0^1} \leq \infty$, from which we deduce that $u^+ \in L^2(0,T;H_0^1(\Omega))$.
(ii) By part (i), the Hilbert space $L^2(0,T;H_0^1(\Omega))$ is a vector lattice, in which the equality $\|u\|_{L^2(0,T;H_0^1(\Omega))} = \|\,|u|\,\|_{L^2(0,T;H_0^1(\Omega))}$ holds. Therefore it is sufficient to show, that the lattice operations are weakly continuous. Indeed, if $u_n \to u$ implies $|u_n| \rightharpoonup |u|$, one gets strong convergence in the Hilbert space, because $\|\,|u_n|\,\| = \|u_n\| \to \|u\| = \|\,|u|\,\|$, and thus for all lattice operations.
Now as $u_n \to u$ in $L^2(0,T;H_0^1(\Omega))$ implies $u_n \to u$ in $L^2(0,T;L^2(\Omega))$, we obtain by the above considerations $u_n^+ \to u^+$ in $L^2(0,T;L^2(\Omega))$. Furthermore, $u_n^+ \rightharpoonup u^+$ weakly in $L^2(0,T;H_0^1(\Omega))$. Indeed for every $\varphi \in \mathcal{D}(0,T;\mathcal{D}(\Omega)) \cong \mathcal{D}((0,T)\times\Omega)$,

$$\begin{aligned}
(u_n^+,\varphi)_{L^2(0,T;H_0^1(\Omega))} &= \int_0^T (u_n(t)^+,\varphi(t))_{H_0^1(\Omega)} \\
&= \int_0^T (u_n(t)^+,\varphi(t))_{L^2(\Omega)} + \int_0^T ((u_n(t)^+)',(\varphi(t))')_{L^2(\Omega)} \\
&= \int_0^T (u_n(t)^+,\varphi(t))_{L^2(\Omega)} - \int_0^T (u_n(t)^+,(\varphi(t))'')_{L^2(\Omega)} \\
&= (u_n^+,\varphi)_{L^2(0,T;L^2(\Omega))} - (u_n^+,(\varphi(.))'')_{L^2(0,T;L^2(\Omega))} \\
&\to (u^+,\varphi)_{L^2(0,T;L^2(\Omega))} - (u^+,(\varphi(.))'')_{L^2(0,T;L^2(\Omega))} \\
&= (u^+,\varphi)_{L^2(0,T;H_0^1(\Omega))}.
\end{aligned}$$

Note that with the same argument, we get that all lattice operations are weakly continuous. □

Since we have chosen V and H to be a real valued function space, we can approximate the positive part u^+ of a function $u \in L^2(0,T;V)$ or $L^2(0,T;H)$ with the help of the following function.

Definition 3.2.4. For $\varepsilon > 0$ we set

$$j_\varepsilon(r) = \begin{cases} (r^2+\varepsilon^2)^{\frac{1}{2}} - \varepsilon & r > 0 \\ 0 & r \leq 0. \end{cases}$$

Then $j_\varepsilon \in C^\infty(\mathbb{R})$ and $j_\varepsilon(r)$ converges pointwise to $j_0(r) = r^+$ as $\varepsilon \to 0$.

Theorem 3.2.5. *For $u \in W = W(0,T;V,V')$ we have*

$$|u(T)^+|^2 - |u(0)^+|^2 = 2\int_0^T (u^+(t),u'(t))dt.$$

Proof. First let $u \in \mathcal{D}([0,T] \times \Omega)$, then we obtain for the composition with j_ε that $j_\varepsilon \circ u \in \mathcal{D}([0,T] \times \Omega) \subset C^1([0,T]; H)$. For $v \in C^1([0,T]; H)$ we have

$$\frac{d}{dt}(v(t), v(t)) = 2(v(t), v'(t)),$$

omitting the real part, since we assumed H to be real valued. Applying this to $j_\varepsilon \circ u$ we get

$$\frac{d}{dt}|j_\varepsilon \circ u(t)|^2_{L^2(\Omega)} = 2\int_\Omega j_\varepsilon \circ u(t) \cdot j'_\varepsilon \circ u(t) \cdot u'(t) dx.$$

Integrating on both sides from 0 to T leads to

$$|j_\varepsilon(u(T))|^2_{L^2(\Omega)} - |j_\varepsilon(u(0))|^2_{L^2(\Omega)} = 2\int_0^T (j_\varepsilon \circ u(t), j'_\varepsilon \circ u(t) \cdot u'(t))_{L^2(\Omega)} dt.$$

Now we let ε tend to 0. The first term $|j_\varepsilon(u(T))|^2_{L^2(\Omega)}$ converges to $|u(T)^+|^2_{L^2(\Omega)}$ by Lebesgue's dominated convergence theorem, since $j_\varepsilon(u(T))$ converges pointwise to $u(T)^+$ and $|j_\varepsilon(r)| \le |r|$ and here $r = u(T)$ is square integrable. Analogously we obtain for the second term $|j_\varepsilon(u(0))|^2_{L^2(\Omega)} \to |u(0)^+|^2$. For the third term we observe that $j_\varepsilon \circ u(t) \to u^+(t)$ and $j'_\varepsilon(r) \to \chi_{\{r>0\}}$ pointwise and again Lebesgue's theorem yields $\int_0^T (j_\varepsilon \circ u(t), j'_\varepsilon \circ u(t) \cdot u'(t))_{L^2(\Omega)} dt \to \int_0^T (u^+(t), u'(t)) dt$. So that we have established the result

$$|u(T)^+|^2 - |u(0)^+|^2 = 2\int_0^T (u^+(t), u'(t)) dt$$

for $u \in \mathcal{D}([0,T] \times \Omega)$.
Consider now $u \in W = W(0,T; V, V')$. Then there exists a sequence of functions $u_n \subset \mathcal{D}([0,T] \times \Omega)$, such that

$$u_n \to u \quad \text{in } L^2(0,T; H_0^1(\Omega))$$

and

$$u'_n \to u' \quad \text{in } L^2(0,T; H^{-1}(\Omega)).$$

From the first property we obtain by Lemma 3.2.3, that also $u_n^+ \to u^+$ in $L^2(0,T; H_0^1(\Omega))$. For these $u_n \in \mathcal{D}([0,T] \times \Omega)$, we have

$$|u_n(T)^+|^2 - |u_n(0)^+|^2 = 2\int_0^T (u_n^+(t), u'_n(t)) dt.$$

We now let n tend to infinity. Observe that $u_n \to u$ in $W = W(0,T; V, V')$, which is continuously embedded in $C([0,T]; H)$. Therefore we obtain $u_n(T) \to u(T)$ and $u_n(0) \to u(0)$ in H, and since $H = L^2(\Omega)$ is a Banach lattice with continuous lattice operations, we have $|u_n(T)^+| \to |u(T)^+|$ and $|u_n(0)^+| \to |u(0)^+|$.

Further $(u_n^+(t), u_n'(t))$ denotes the scalar product in H as well as the duality pairing between V and V', if $u_n'(t)$ is viewed as an element of V'. Hence the expression $\int_0^T (u_n^+(t), u_n'(t))dt$ denotes the duality pairing between $L^2(0,T;H_0^1(\Omega))$ and $L^2(0,T;H^{-1}(\Omega))$. Therefore we get $\int_0^T (u_n^+(t), u_n'(t))dt \to \int_0^T (u^+(t), u'(t))dt$. This completes the proof. □

Remark 3.2.6. The same results can be obtained for any $s \in [0,T]$ instead of T, if we just consider the interval $[0,s]$ instead of $[0,T]$. Therefore we have $\forall s \in [0,T]$

$$|u(s)^+|^2 - |u(0)^+|^2 = 2\int_0^s (u^+(t), u'(t))dt.$$

And since $u^- = (-u)^+$ we also get

$$|u(s)^-|^2 - |u(0)^-|^2 = -2\int_0^s (u^-(t), u'(t))dt.$$

3.2.2 Positivity

Now we want to examine positivity of the non-autonomous Cauchy problem. We will give sufficient conditions on the form in order to have a positive solution, whenever the given data $u_0 \in H$ and $f \in L^2(0,T;V')$ are positive. This reminds of the Beurling-Deny criteria, which have been studied for the autonomous case in Section 1.3. We will make the same assumptions on the forms $a(t;u,v)$ for almost all $t \in [0,T]$, and show that they are sufficient for positivity.
However, this does not extend the Beurling-Deny criteria to the the general non-autonomous case, since these results are restricted to the case where $V = H_0^1(\Omega)$ and $H = L^2(\Omega)$ are real valued function spaces and $V' = H^{-1}(\Omega)$.

Theorem 3.2.7. *Assume that for each $u \in V = H_0^1(\Omega)$*

$$a(t;u^+,u^-) \leq 0,$$

for almost all $t \in [0,T]$, then the solution of the non-autonomous Cauchy-Problem is positive.

Proof. We have to show that for $H \ni u_0 \geq 0$ and $L^2(0,T,V') \ni f \geq 0$ the solution $u \in W = W(0,T,V,V')$ is positive. Note that we have $u(0) = u_0 \geq 0$ and therefore $u^-(0) = 0$.
Since u is a solution, it satisfies by Proposition 3.1.10

$$\int_0^T \{(u'(t), \varphi(t)) + a(t;u(t), \varphi(t))\}dt = \int_0^T (f(t), \varphi(t))dt, \tag{3.15}$$

for all $\varphi \in L^2(0,T;V)$. If we take $\varphi(t) = u^-(t)$ we obtain

$$\int_0^T \{(u'(t), u^-(t)) + a(t;u(t), u^-(t))\}dt = \int_0^T (f(t), u^-(t))dt,$$

which together with Remark 3.2.6 and also here T replaced by s leads to

$$\begin{aligned}
|u^-(s)|^2 &= |u^-(s)|^2 - |u^-(0)|^2 \\
&= -2\int_0^s (u'(t), u^-(t))\,dt \\
&= +2\int_0^s a(t; u(t), u^-(t))\}\,dt - 2\int_0^s (f(t), u^-(t))\,dt \\
&= +2\int_0^s a(t; u^+(t), u^-(t))\,dt \\
&\quad -2\int_0^s a(t; u^-(t), u^-(t))\,dt - 2\int_0^s (f(t), u^-(t))\,dt \\
&\leq 0,
\end{aligned}$$

because the first term is negative by assumption, the second is negative because the form is positive and the last by the positivity of f. Thus, $|u^-(s)|^2 \leq 0$, i.e. $u^-(s) = 0$, for all $s \in [0, T]$, which says that u is positive. □

Example 3.2.8. Let $\mathcal{A}$ be an elliptic Operator, i.e. the form $a(t; u, v)$ is given by

$$a(t; u, v) = \int_\Omega \Big\{ \sum_{i,j=1}^n a_{ij}(t)\, D_i u\, D_j v + \sum_{i=1}^n b_i(t)\, u\, D_i v - \sum_{i=1}^n c_i(t)\, D_i u\, v - d\, u\, v \Big\}\, dx,$$

for all $u,\ v \in V = H_0^1$, with functions $a_{ij}, b_i, c_i, d \in L^\infty((0,T) \times \Omega)$. Further we assume

$$\sum_{i,j=1}^n a_{ij}(t,x)\xi_i\xi_j \geq \alpha \sum_{i=1}^n \xi_i^2,$$

where $\alpha > 0$ is a constant independent of x and t, for all $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$, and almost every $(t, x) \in [0, T] \times \Omega$, in order to obtain $a(t; u, u) \geq \alpha \|u\|_V^2$.
Now $a(t; u, v)$ satisfies the conditions, such that the Cauchy-Problem is well-posed. Additionally $a(t; u^+, u^-) \leq 0$ for for each $u \in V = H_0^1(\Omega)$ and almost all $t \in [0, T]$.
Hence the previous theorem implies positivity of the solution.

3.2.3 Sub-Markovian Solutions

Positivity is linked to the first Beurling-Deny criterion, so it is natural to investigate the property of the second criterion as well.

Definition 3.2.9. The Cauchy-Problem is called **homogeneous**, if $f \equiv 0$. We say, that a homogeneous Cauchy-Problem has **sub-Markovian** solutions, if the solution u satisfies $u \leq 1$, whenever $u_0 \leq 1$.

Remark 3.2.10. Remember that we are restricted to the case where $V = H_0^1(\Omega)$. Therefore we have for each $u \in V$, that the minimum between u and the constant $\mathbf{1}$ function $u \wedge \mathbf{1} \in V$.

Theorem 3.2.11. *We assume the Cauchy problem to have positive solutions, and additionally that $a(t; u, u \wedge \mathbf{1}) \leq a(t; u, u)$ for all $u \in V = H_0^1(\Omega)$ and for almost all $t \in [0, T]$. Then the solution of the non-autonomous Cauchy problem is sub-Markovian.*

Proof. We have to show, that $f \equiv 0$ and $H \ni u_0 \leq 1$ implies that the solution satisfies $W \ni u \leq 1$.
We decompose u in the following way,

$$u = (u \wedge \mathbf{1}) + (u - \mathbf{1})^+.$$

In order to show that $u \leq 1$, which is equivalent to $u = u \wedge \mathbf{1}$, we only have to verify that $(u-1)^+ \equiv 0$. Note that by the assumptions we have $(u-1)^+(0) = 0$. As in the proof of positivity we replace $\varphi(t)$ in equation (3.15) by $(u-1)^+(t)$ and obtain

$$\begin{aligned} &\int_0^T \{(u'(t), (u-1)^+(t)) + a(t; u(t), (u-1)^+(t))\}dt \\ &= \int_0^T (f(t), (u-1)^+(t))dt. \end{aligned}$$

Again this is valid for T replaced by each $s \in [0, T]$ and together with u replaced by $(u-1)$ in Remark 3.2.6 (note that $(u-1)' = u'$), we get

$$\begin{aligned} |(u-1)^+(s)|^2 &= |(u-1)^+(s)|^2 - |(u-1)^+(0)|^2 \\ &= 2\int_0^s (u'(t), (u-1)^+(t))dt \\ &= -2\int_0^s a(t; u(t), (u-1)^+(t))dt \\ &= -2\int_0^s a(t; u(t), u(t))dt + 2\int_0^s a(t; u(t), (u \wedge \mathbf{1})(t))dt \\ &\leq 0, \end{aligned}$$

because the second term is less than or equal to $2\int_0^s a(t; u(t), u(t))dt$ by the assumption. Therefore we have $(u-1)^+(s) = 0$, for all $s \in [0, T]$, and thus $u \leq 1$. ☐

Example 3.2.12. Let $\mathcal{A}$ be again an elliptic Operator, i.e. the form $a(t; u, v)$ is given by

$$a(t; u, v) = \int_\Omega \Big\{ \sum_{i,j=1}^n a_{ij}(t)\, D_i u\, D_j v + \sum_{i=1}^n b_i(t)\, u\, D_i v - \sum_{i=1}^n c_i(t)\, D_i u\, v - d\, u\, v \Big\}\, dx,$$

for all $u,\ v \in V = H_0^1$, with functions $a_{ij}, b_i, c_i, d \in L^\infty((0,T)\times\Omega)$. Further we assume

$$\sum_{i,j=1}^{n} a_{ij}(t,x)\xi_i\xi_j \geq \alpha \sum_{i=1}^{n} \xi_i^2,$$

where $\alpha > 0$ is a constant independent of x and t, for all $\xi = (\xi_1,\dots,\xi_n) \in \mathbb{R}^n$, and almost every $(t,x) \in [0,T]\times\Omega$, in order to obtain $a(t;u,u) \geq \alpha\|u\|_V^2$.

Now $a(t;u,v)$ satisfies the conditions, such that the Cauchy-Problem is well-posed. We have seen before that the inequality $a(t;u^+,u^-) \leq 0$ is satisfied for for each $u \in V = H_0^1(\Omega)$ and almost all $t \in [0,T]$.

Additionally, we assume

$$\sum_{i=1}^{n} D_i b_i - d \leq 0.$$

But since the derivatives $D_i b_i$ need not exist as functions, this expression must be interpreted in a generalized sense, i.e.

$$\int_\Omega (dv - \sum_{i=1}^{n} b_i D_i v)dx \leq 0,$$

for all $v \geq 0$ and $v \in C_0^1(\Omega)$. And since b_i and d are bounded, this inequality extends to all non-negative $v \in W_0^{1,1}(\Omega)$.

Then we have the inequality $a(t,u,(u-1)^+) \geq a(t,(u-1)^+,(u-1)^+)$, indeed, since $D_i(u-1)^+ = D_i u \chi_{\{u>1\}}$,

$$
\begin{aligned}
& a(t,u,(u-1)^+) - a(t,(u-1)^+,(u-1)^+) \\
= & \int_\Omega \Big\{ \sum_{i,j=1}^n a_{ij}(t)\, D_i u\, D_j u \chi_{\{u>1\}} + \sum_{i=1}^n b_i(t)\, u\, D_i u \chi_{\{u>1\}} \\
& - \sum_{i=1}^n c_i(t)\, D_i u\, (u-1)^+ - d\, u\, (u-1)^+ \Big\}\, dx \\
& - \int_\Omega \Big\{ \sum_{i,j=1}^n a_{ij}(t)\, D_i u \chi_{\{u>1\}}\, D_j u \chi_{\{u>1\}} - \sum_{i=1}^n b_i(t)\, (u-1)^+\, D_i u \chi_{\{u>1\}} \\
& + \sum_{i=1}^n c_i(t)\, D_i u \chi_{\{u>1\}}\, (u-1)^+ + d\, (u-1)^+\, (u-1)^+ \Big\}\, dx \\
= & \int_\Omega \Big\{ \sum_{i=1}^n b_i(t)\, [u-(u-1)^+]\, D_i u \chi_{\{u>1\}} - d\, [u-(u-1)^+]\, (u-1)^+ \Big\}\, dx \\
= & \int_\Omega \Big\{ \sum_{i=1}^n b_i(t)\, D_i([u-(u-1)^+](u-1)^+) - d\, [u-(u-1)^+]\, (u-1)^+ \Big\}\, dx \\
= & \int_\Omega \Big\{ \sum_{i=1}^n b_i(t)\, D_i v - d\, v \Big\}\, dx \\
\geq & \; 0
\end{aligned}
$$

since

$$
\begin{aligned}
& D_i([u-(u-1)^+](u-1)^+) \\
& = (D_i[u-(u-1)^+])(u-1)^+ + [u-(u-1)^+]D_i(u-1)^+) \\
& = (D_i u - D_i u \chi_{\{u>1\}})(u-1)^+ + [u-(u-1)^+]D_i(u-1)^+) \\
& = (D_i u \chi_{\{u\leq 1\}})(u-1)^+ + [u-(u-1)^+]D_i u \chi_{\{u>1\}} \\
& = [u-(u-1)^+]D_i u \chi_{\{u>1\}}
\end{aligned}
$$

and

$$v = ([u-(u-1)^+](u-1)^+) \geq 0 \text{ and } v \in W_0^{1,1}(\Omega),$$

because $u \in H_0^1(\Omega)$ implies $u \wedge \mathbf{1} \in H_0^1(\Omega)$, and thus $(u-1)^+ = u - u\wedge\mathbf{1} \in H_0^1(\Omega)$ and $u-(u-1)^+ = u\wedge\mathbf{1} \in H_0^1(\Omega)$ and the product of these two functions is then in $W_0^{1,1}(\Omega)$.

Therefore, an elliptic operator satisfying

$$\sum_{i=1}^n D_i b_i - d \leq 0,$$

has sub-Markovian solutions.

3.3 Maximal Regularity

A further property of solutions which is worth to be investigated is the regularity, this is differentiability in an appropriate sense and space. There have been some results even for non-autonomous Cauchy problems, see e.g. [HM], but under more regularity assumptions on the operator $A(t)$ with respect to t.
Since we have established existence and uniqueness only under measurability assumptions, we seek a regularity result in this situation as well.

3.3.1 Idea of the Proof

We are still in the setting of a function space $L^2(0,T;X)$, where X can be replaced by any of the Hilbert spaces V, H and V' respectively. We do no longer assume an order structure on any of these spaces.

Definition 3.3.1. Let $(A(t))_{t\geq 0}$ be a family of closed densely defined operators on X. We say that the associated non-autonomous Cauchy problem

$$\begin{cases} u'(t) + A(t)u(t) &= f(t) \qquad t > 0 \\ u(0) &= 0 \end{cases}$$

has **maximal regularity** in $L^2(0,T;X)$, if for every $f \in L^2(0,T;X)$, there exists a unique solution u satisfying $u \in D(\mathcal{A}) \cap H^1(0,T;X)$, where $\mathcal{A}$ is the multiplication operator associated with $(A(t))_{t\geq 0}$ in $L^2(0,T;X)$.

Observe, that the representation theorem by Lions implies not only well-posedness of Problem 3.1.12, but also gives maximal regularity in $L^2(0,T;V')$.
However, the given space is $L^2(0,T;H)$, and in general we do not even know exactly the space V'. Hence our interest is maximal regularity in $L^2(0,T;H)$. Therefore we need to interpret $A(t)$ as operator in H with domain $D(A(t))$, i.e.

$$\begin{aligned} D(A(t)) &= \{u \in V : \exists\, h \in H : a(t;u,v) = (h,v)\ \forall v \in V\} \\ A(t)u &= h \end{aligned}$$

Then we have $D(A(t)) \overset{d}{\hookrightarrow} V \overset{d}{\hookrightarrow} H \overset{d}{\hookrightarrow} V'$.
The idea is now to find Hilbert spaces $\tilde{V}$ and $\tilde{V}'$, which are isomorphic to $D(A(t))$ and H respectively. For a family of operators $B(t)$ associated with continuous elliptic forms on $\tilde{V}$, we have maximal regularity in $L^2(0,T;\tilde{V}')$. If then $B(t)$ coincides with $A(t)$ with respect to the isomorphisms, we can deduce maximal regularity for the Cauchy problem associated with the family $(A(t))_{t\geq 0}$ in the space $L^2(0,T;H)$.
We encounter the following difficulties. For the application of Lions' theorem, we need a fixed Hilbert space $\tilde{V}$, but $D(A(t))$ depends on the time parameter t. Hence we will have time dependent isomorphisms $i(t) : D(A(t)) \to \tilde{V}$. This also implies, that we have to assume, that the spaces $D(A(t))$ are all isomorphic.

Denote by j the isomorphism from H to $\tilde{V}'$. Let $(B(t))_{t\geq 0}$ be a family of operators, such that the relation $A(t) = j^{-1}B(t)i(t)$ holds. If $B(t)$ is associated with a continuous elliptic form, then the Cauchy problem associated with the family $(B(t))_{t\geq 0}$ has maximal regularity in $L^2(0,T:\tilde{V}')$, i.e.

$$\forall f \in L^2(0,T;\tilde{V}')\ \exists!\, u \in L^2(0,T;\tilde{V}) \cap H^1(0,T;\tilde{V}') :$$

$$\begin{aligned} u'(t) + B(t)u(t) &= f(t) \\ u(0) &= 0. \end{aligned}$$

In order to get to the space $L^2(0,T;H)$, we apply j^{-1} in each line and obtain

$$\begin{aligned} j^{-1}u'(t) + j^{-1}B(t)i(t)(i(t))^{-1}u(t) &= j^{-1}f(t) \\ j^{-1}u(0) &= 0, \end{aligned}$$

which is equivalent to

$$\begin{aligned} j^{-1}u'(t) + A(t)(i(t))^{-1}u(t) &= j^{-1}f(t) \\ j^{-1}u(0) &= 0, \end{aligned}$$

since $A(t) = j^{-1}B(t)i(t)$. Hence we obtain maximal regularity for the Cauchy problem associated with the family $(A(t))_{t\geq 0}$ if

$$((i(t))^{-1}u(t))' = j^{-1}u'(t) \text{ and } (i(0))^{-1}u(t) = j^{-1}u(0).$$

Both equations hold if and only if $i(t) \equiv j$, hence $D(A(t)) \equiv D$.
The other problem is to verify that $B(t)$, t fixed in the following, is associated with a continuous coercive form. We have supposed, that $B(t) = jA(t)j^{-1} : \tilde{V} \to \tilde{V}'$. Hence the candidate for the form is

$$b(u,v) = < B(t)u, v >_{\tilde{V},\tilde{V}'} = < jA(t)j^{-1}u, v >_{\tilde{V},\tilde{V}'} .$$

As $A(t) : D \to H$, $j : D \to \tilde{V}$ and $j : H \to \tilde{V}'$ are isomorphisms, we obtain equivalence between the norms

$$\|.\|_D \sim \|A(t).\|_H,\ \|.\|_D \sim \|j.\|_{\tilde{V}} \text{ and } \|.\|_H \sim \|j.\|_{\tilde{V}'}.$$

Therefore

$$\begin{aligned} |b(u,v)| &= |< jA(t)j^{-1}u, v >_{\tilde{V},\tilde{V}'} | \leq \|jA(t)j^{-1}u\|_{\tilde{V}'}\|v\|_{\tilde{V}} \\ &\leq C_1\|A(t)j^{-1}u\|_H\|v\|_{\tilde{V}} \leq C_2\|j^{-1}u\|_D\|v\|_{\tilde{V}} \\ &\leq C_3\|u\|_{\tilde{V}}\|v\|_{\tilde{V}}, \end{aligned}$$

which shows that the form b is continuous.
On the other hand it is not obvious, if b is a coercive form. This inconvenience does not occur in many autonomous cases.

3.3.2 Maximal Regularity for Autonomous Forms

In the autonomous case, the form does not depend on a time parameter t. Hence in the usual setting $V \overset{d}{\hookrightarrow} H \overset{d}{\hookrightarrow} V'$, we consider a continuous elliptic sesquilinear form $a : V \times V \to \mathbb{C}$, where

$$|a(u,v)| \leq M\|u\|_V \, \|v\|_V$$

and

$$\operatorname{Re} a(u,u) \geq \delta\|u\|_V,$$

with constants $M \geq 1$ and $\delta > 0$. The associated operator A on H is defined by

$$\begin{aligned} D(A) &= \{u \in V : \exists\, h \in H : a(u,v) = (h,v)_H \; \forall\, v \in V\} \\ Au &= h. \end{aligned}$$

Since the operator A is associated with a continuous coercive form we can define on H the operator $A^{1/2}$ by

$$A^{1/2} = (A^{-1/2})^{-1} \text{ where } A^{-1/2} = \frac{1}{2\pi i} \int_\Gamma \lambda^{-1/2}(A-\lambda)^{-1}\, d\lambda,$$

see [Ta], Chapter 2 for more details.
If a is symmetric, i.e. $a(u,v) = \overline{a(v,u)}$, then A is selfadjoint and one has $D(A^{1/2}) = V$. The question, weather this equality holds for all operators associated with a continuous coercive form, is known as Kato's famous square root problem posed in [Ka1]. Recently answers were given. First A. McIntosh gave a counterexample in [McI], hence the answer is no in general. A survey on the square root problem is given in [AT], where a positive answer under more assumptions can be found. Finally, in [AHLMT] it was shown that $D(A^{1/2}) = V$ for elliptic second order differential operators in divergence form on $\mathbb{R}^n$.
Hence we still cover enough operators, even if we restrict our observations to the case where $D(A^{1/2}) = V$.
The result of maximal regularity is well known for generators of analytic semigroups, thus in particular for autonomous forms. However, the method to proof it with Lions' representation theorem given below seems to be new.

Theorem 3.3.2. *Let $V \overset{d}{\hookrightarrow} H \overset{d}{\hookrightarrow} V'$ and let a be a continuous coercive form with domain V, such that the associated operator A in H satisfies $D(A^{1/2}) = V$. Then the Cauchy problem*

$$\begin{cases} u' + Au &= f \\ u(0) &= 0 \end{cases}$$

has maximal regularity in $L^2(0,T;H)$.

Proof. After changing to equivalent norms, the operator $A^{1/2}$ is an isometric isomorphism between all spaces as illustrated below.

$$D(A) \overset{d}{\hookrightarrow} V \overset{d}{\hookrightarrow} H \overset{d}{\hookrightarrow} V'$$

From Lions' theorem we deduce maximal regularity for the Cauchy problem associated with A in $L^2(0,T;V')$. Hence

$$\forall f \in L^2(0,T;V')\ \exists!\, u \in L^2(0,T;V) \cap H^1(0,T;V') :$$

$$\begin{aligned} u'(t) + Au(t) &= f(t) \\ u(0) &= 0. \end{aligned}$$

Applying $A^{-1/2}$ on the equations, we obtain

$$\begin{aligned} A^{-1/2}u'(t) + A^{-1/2}AA^{1/2}A^{-1/2}u(t) &= A^{-1/2}f(t) \\ A^{-1/2}u(0) &= 0. \end{aligned}$$

As $A^{-1/2}$ is an isomorphism from V' to A as well as from V to $D(A)$, and since $A^{-1/2}u' = (A^{-1/2}u(t))'$, this implies

$$\forall f \in L^2(0,T;H)\ \exists!\, u \in L^2(0,T;D(A)) \cap H^1(0,T;H) :$$

$$\begin{aligned} u'(t) + Au(t) &= f(t) \\ u(0) &= 0, \end{aligned}$$

which is maximal regularity for the Cauchy problem associated with the operator A on the space $L^2(0,T;H)$. □

3.3.3 Remark on Non-Autonomous Forms

We conclude this section with a remark on the non-autonomous setting. From the proof of Theorem 3.3.2 we immediately deduce the following result.

Theorem 3.3.3. *Let $V \overset{d}{\hookrightarrow} H \overset{d}{\hookrightarrow} V'$ and let $a(t;u,v)$ be a family of sesquilinear forms satisfying (3.1) and (3.14). Assume that the associated operators satisfy $D(A(t)) \equiv D$ and that there exists an isomorphism Λ from D to V as well as from H to V', which commutes with $A(t)$ for every $t \geq 0$. Then the associated Cauchy problem has maximal regularity in $L^2(0,T;H)$.*

For the proof we only need to replace in the proof of Theorem 3.3.2 the operators $A^{1/2}$ by Λ, A by $A(t)$ and the space $D(A)$ by D.

3.4 Semigroup Methods

So far, we have studied the solutions of the variational non-autonomous Cauchy problems as they are given by Lions' theorem. However, one can also apply semigroup methods to non-autonomous Cauchy problems. These have their roots in [Ev] and [Ho], and a nice overview is given in the survey article [Na2], see also [EN], Section VI.9. Note that in this context, the notion of well-posedness is more restrictive.

First we want to give the background on strongly continuous evolution families and the associated semigroups. An excursion over the generator property of a surjective dissipative operator leads us to an evolution semigroup associated with our problem in $L^2(0,T;V')$. Finally, we will get a result for our original problem by a restriction to the space $L^2(0,T;H)$, and conclude this section with some invariance considerations.

3.4.1 Evolution Families and Semigroups

The ideas for this section are mainly taken from [Schn] and [Ni1]. However, we will replace the Banach space $C_0(\mathbb{R};X)$, where X is a Banach space, by the spaces $L^p(0,T;X)$, $1 \leq p < \infty$, $0 < T < \infty$.

On the Banach space X we consider the non-autonomous Cauchy problem

$$(NCP)_{s,x} \qquad \begin{cases} \dot{u}(t) &= A(t)u(t), \quad t \geq s, \; t,s \in [0,T] \\ u(s) &= x. \end{cases}$$

If the Cauchy problem is considered for several initial values, then we denote it by $(NCP)_s$, and by (NCP), if the Cauchy problem is considered for all initial times $s \in [0,T]$.

Recall that a function $u \in W^{1,p}(0,T;X)$ is continuous after changing its values on a set of measure zero, hence in the sequel we always mean its continuous representative, in particular for $u \in W^{1,p}(0,T;X)$ and $s \in [0,T]$ the expression $u(s) = x$ has a sense.

In the following, $A(.)$ denotes the multiplication operator with fiber operators $(A(t))_{t\in[0,T]}$ on $L^p(0,T;X)$. Recall that the domain $D(A(.))$ is given as the set of all functions $u \in L^p(0,T)$ such that $u(t) \in D(A(t))$ for almost all $t \in [0,T]$ and $t \mapsto A(t)u(t) \in L^p(0,T;X)$. Furthermore, let $A_s(.)$ denote the multiplication operator with fiber operators $(A(t))_{t\in[s,T]}$, then $D(A_s(.)) := L^p(s,T) \cap D(A(.))$.

Definition 3.4.1. Let Y_t, $t \in [0,T]$ be dense subspaces of X. The Cauchy problem (NCP) is called **well-posed on the spaces Y_t, $t \in [0,T]$**, if for each $s \in [0,T]$ and $x \in Y_s$ there exists a unique function $u \in W^{1,p}(s,T;X) \cap D(A_s(.))$, satisfying $(NCP)_{s,x}$, with $u(t) = u(t;s,x) \in Y_t$ for $t \geq s$. Moreover, the solutions depend continuously on the initial data, i.e. if $[0,T] \ni s_n \to s \in [0,T]$

and additionally $Y_{s_n} \ni x_n \to x \in Y_s$ with respect to the topology of X, then $\hat{u}(t; s_n, x_n) \to \hat{u}(t; s, x)$ uniformly in $t \in [0, T]$, where

$$\hat{u}(t; s, x) := \begin{cases} u(t; s, x) & \text{if } t \geq s,\ t, s \in [0, T], \\ x & \text{if } t < s,\ t, s \in [0, T]. \end{cases}$$

The Cauchy problem (NCP) is called **well-posed**, if there exist dense subspaces Y_t, $t \in [0, T]$, of X, such that (NCP) is well-posed on the spaces Y_t, $t \in [0, T]$.

For well-posed autonomous Cauchy problems, the solutions are given as the orbits of strongly continuous semigroups. An analogous result for non-autonomous Cauchy problems on spaces of continuous functions was shown in [Ni1], Proposition 3.10. Below, we adapt his proof to the spaces $L^p(0, T; X)$, $1 \leq p < \infty$. We set $I := \{(t, s) \in [0, T]^2 : t \geq s\}$.

Proposition 3.4.2. *Assume that (NCP) is well-posed on spaces Y_t for $t \in [0, T]$. Then there is a unique family of operators $U(t, s) \in \mathcal{L}(X)$, $(t, s) \in I$, satisfying*

(E1) $U(s, s) = Id$, $U(t, s) = U(t, r)U(r, s)$ for $t \geq r \geq s$;

(E2) the mapping $I \ni (t, s) \mapsto U(t, s)$ is strongly continuous;

(E3) $U(t, s)Y_s \subseteq Y_t$, the mapping $t \mapsto U(t, s)x$, $t \geq s$, belongs to the space $W^{1,p}(s, T; X) \cap D(A_s(.))$ for $x \in Y_s$, and $\frac{\partial}{\partial t}U(t, s)x = A(t)U(t, s)x$.

(E4) For a solution u of (NCP), one has $\frac{U(.+h,.)u(.)-u(.)}{h} \to A(.)u(.)$ in $L^1(0, T; X)$ as $h \to 0$.

Conversely, if there is a unique family $(U(t, s))_{(t,s)\in I} \subseteq \mathcal{L}(X)$ satisfying (E1) - (E4) for dense subspaces Y_t, $t \in [0, T]$, then (NCP) is well-posed on Y_t.

Proof. Let (NCP) be well-posed on the spaces Y_t for $t \in [0, T]$ and denote by $u(t; s, x)$ the solution of $(NCP)_{s,x}$. For all $s \in [0, T]$ and $x \in Y_s$ we define $U(t, s)x = u(t; s, x)$ for $t \in [s, T]$. This yields an operator $U(t, s) : Y_s \to Y_t$, which is obviously linear. By the continuous dependence, we can extend this mapping uniquely to a bounded operator on X. The fact that $u(t; s, x)$ is the solution of $(NCP)_{s,x}$ implies all the algebraic properties in (E1). To show strong continuity, we first prove that

$$\|U(t, s)\| \leq K. \tag{3.16}$$

If this were false, there would exist a sequence s_n in $[0, T]$ such that the norm estimate $\sup_t \|U(t, s_n)\| \geq 2n$ holds. Since the spaces Y_s are dense in X and the operators $U(t, s_n)$ are bounded, we would obtain $x_n \in Y_{s_n}$ with $\|x_n\| \leq \frac{1}{n}$ and $\sup_t \|U(t, s)x_n\| \geq 1$. Since $[0, T]$ is compact, there exists a continuous subsequence of (s_n), again denoted by $s_n \to s$. Then $x_n \to 0$ while

$$\sup_t \|u(t; s_n, x_n)\| = \sup_t \|U(t, s_n)x_n\| \not\to 0,$$

which contradicts the continuous dependence. Thus (3.16) holds and we only have to show the strong continuity on a dense subspace of X. Let now $(t_n, s_n) \to (t, s)$ and $x \in Y_s$. Choose a sequence $x_n \to x$ with $x_n \in Y_{s_n}$. We can now estimate

$$\begin{aligned}
&\|U(t_n, s_n)x - U(t, s)x\| \\
&\quad \leq \|U(t_n, s_n)(x - x_n)\| + \|U(t_n, s_n)x_n - \hat{u}(t_n; s, x)\| + \|\hat{u}(t_n; s, x) - U(t, s)x\| \\
&\quad \leq K\|x - x_n\| + \|\hat{u}(t_n; s_n, x_n) - \hat{u}(t_n; s, x)\| + \|\hat{u}(t_n; s, x) - u(t; s, x)\| \\
&\quad \leq K\|x - x_n\| + \sup_{\tau\in[0,T]} \|\hat{u}(\tau; s_n, x_n) - \hat{u}(\tau; s, x)\| + \|\hat{u}(t_n; s, x) - u(t; s, x)\|.
\end{aligned}$$

For $n \to \infty$, the first term vanishes, since $x_n \to x$, the second by the continuous dependence, and finally the third, since the extended solutions $\hat{u}$ are continuous. Moreover, for $x \in Y_s$ the function

$$U(., s)x$$

clearly belongs to the space $W^{1,p}(s, T; X) \cap D(A_s(.))$ and satisfies the equality $\frac{\partial}{\partial t}U(t, s)x = A(t)U(t, s)x$. Finally, for a solution $\frac{u(.+h)-u(.)}{h} \to u'(.) = A(.)f(.)$ in $L^p(0, T; X)$, because $u \in W^{1,p}(0, T; X)$. Thus,

$$\begin{aligned}
&\frac{U(. + h, .)u(.) - u(.)}{h} - A(.)u(.) \\
&\quad = \frac{U(. + h, .)u(.) - u(.)}{h} - \frac{u(. + h) - u(.)}{h} + \frac{u(. + h) - u(.)}{h} - A(.)u(.) \\
&\quad = \frac{u(. + h; ., u(.)) - u(. + h)}{h} + \frac{u(. + h) - u(.)}{h} - A(.)u(.) \\
&\quad = \frac{u(. + h) - u(.)}{h} - u'(.) \to 0
\end{aligned}$$

in $L^p(0, T; X)$ as $h \to 0$, because for all $sin[0, T]$ the solutions of $(NCP)_{s,u(s)}$ are unique.

Suppose now the existence of a unique family $(U(t, s))_{(t,s)\in I} \subseteq \mathcal{L}(X)$ satisfying (E1) - (E4) for dense subspaces Y_t with $t \in [0, T]$. For $s \in [0, T]$ and $x \in Y_s$, define $u(t) = u(t; s, x) := U(t, s)x$. Then obviously, u is a solution of $(NCP)_{s,x}$ and depends continuously on the given data. We still have to show uniqueness. For that we prove below that for a given solution $u(.) := u(.; s, x)$ and fixed $t \in [s, T]$, the function $[s, t] \ni r \mapsto U(t, r)u(r) \in W^{1,p}(s, t; X)$ and the derivative in the sense of distributions $[U(t, .)u(.)]' = 0$. Then $[s, t] \ni r \mapsto U(t, r)u(r)$ is constant and $u(t) = U(t, t)u(t) = U(t, s)u(s) = U(t, s)x$, which shows that the solution is uniquely given by $U(t, s)x$.

Indeed, $r \mapsto U(t, r)u(r)$ is continuous, since the solution u is continuous and

$U(t,r)$ is strongly continuous. Hence,

$$\begin{aligned}
&U(t,r)u(r) - U(t,s)u(s) \\
&= \lim_{h\to 0}\frac{1}{h}\int_r^{r+h} U(t,\tau)u(\tau)\,d\tau - \lim_{h\to 0}\frac{1}{h}\int_s^{s+h} U(t,\tau)u(\tau)\,d\tau \\
&= \lim_{h\to 0}\frac{1}{h}\left[\int_{s+h}^{r+h} U(t,\tau)u(\tau)\,d\tau - \int_s^{r} U(t,\tau)u(\tau)\,d\tau\right] \\
&= \lim_{h\to 0}\int_s^r \frac{1}{h}\left[U(t,\tau+h)u(\tau+h) - U(t,\tau)u(\tau)\right]\,d\tau.
\end{aligned}$$

Observe that

$$\begin{aligned}
&\frac{1}{h}\left[U(t,\tau+h)u(\tau+h) - U(t,\tau)u(\tau)\right] \\
&= \frac{1}{h}\left[U(t,\tau+h)[u(\tau+h)-u(\tau)]\right] + \frac{1}{h}\left[[U(t,\tau+h)-U(t,\tau)]u(\tau)\right] \\
&= U(t,\tau+h)\frac{u(\tau+h)-u(\tau)}{h} + U(t,\tau+h)\frac{[U(\tau,\tau)-U(\tau+h,\tau)]u(\tau)}{h} \\
&\to U(t,\tau)A(\tau)u(\tau) - U(t,\tau)A(\tau)u(\tau) = 0,
\end{aligned}$$

in $L^p(0,T;X)$ as $h \to 0$, because $u(.)$ is a solutions of (NCP) and (E4) is satisfied. Hence, the convergence holds in particular in $L^1(0,T;X)$, which proves the claim. □

Proposition 3.4.2 motivates the following definition.

Definition 3.4.3. A family $\mathcal{U} = (U(t,s))_{(t,s)\in I} \subseteq \mathcal{L}(X)$ satisfying $(E1)$ is called an **evolution family**. It is **strongly continuous** if $(E2)$ holds. We say that $\mathcal{U}$ **solves** (NCP) (on spaces Y_t) if $(E3)$ is satisfied.

Let $\mathcal{U}$ be a strongly continuous evolution family. As $[0,T]$ is compact, the principle of uniform boundedness yields $\sup_{(t,s)\in I}\|U(t,s)\| < \infty$. Furthermore, for $f \in L^p(0,T;X)$ and $t \geq 0$ set

$$(T(t)f)(s) := \begin{cases} U(s,s-t)f(s-t), & \text{if } s-t \in [0,T], \\ 0 & \text{if } s-t \notin [0,T]. \end{cases} \tag{3.17}$$

Then $(T(t))_{t\geq 0}$ is a C_0-semigroup on $L^p(0,T;X)$, see [Schn], Proposition 1.9.

Definition 3.4.4. We call a strongly continuous semigroup $T = (T(t))_{t\geq 0}$ on $L^p(0,T;X)$ an **evolution semigroup**, if there exists a strongly continuous evolution family $\mathcal{U} = (U(t,s))_{(t,s)\in I}$ such that (3.17) holds.

We remark that an evolution semigroup uniquely determines the underlying evolution family.

For the spaces $L^p([a,b];X)$, $1 \leq p < \infty$, evolution semigroups were characterized in [Ne], Theorem 4.12. This was also done for more general Banach function spaces in [Schn], Theorem 2.6, which we cite here for $L^p(0,T;X)$, in order to keep the same notations. We set $C_0((0,T];X) := \{f \in C([0,T];X) : f(0) = 0\}$.

Theorem 3.4.5. *Let X be a Banach space and $\mathcal{T} = (\mathcal{T}(t))_{t\geq 0}$ a C_0-semigroup on $L^p(0,T;X)$ with generator $(G, D(G))$. Then the following assertions are equivalent.*

(i) $\mathcal{T}$ is an evolution semigroup.

(ii) (a) For all $f \in D(G)$ and $\varphi \in C^1([0,T])$ with support in $(0,T]$ we have that $\varphi f \in D(G)$ and

$$G(\varphi f) = -\varphi' f + \varphi G(f).$$

(b) There is $\lambda \in \rho(G)$ such that $R(\lambda, G) : L^p(0,T;X) \to C_0((0,T];X)$ is continuous with dense image.

Remark 3.4.6. Note that since T is finite, $R(\lambda, G) : L^p(0,T;X) \to C_0((0,T];X)$ automatically implies, that the mapping is continuous by the closed graph theorem. Indeed, if $u_n \to u$ in $L^p(0,T;X)$ and $R(\lambda, G)u_n \to v$ in $C_0((0,T];X)$, then in particular $R(\lambda, G)u_n \to v$ in $L^p(0,T;X)$, which yields $v = R(\lambda, G)u$. Thus, $R(\lambda, G) : L^p(0,T;X) \to C_0((0,T];X)$ is a closed operator between two Banach spaces and therefore continuous.

Our definition of well-posedness is weaker that the one given in [Ni1], because there the solutions are assumed to be continuously differentiable. Nevertheless, we obtain an analogous characterization of well-posedness in terms of evolution semigroups.
Note that an evolution semigroup $\mathcal{T} = (\mathcal{T}(t))_{t\geq 0}$ on $L^p(0,T;X)$ leaves the spaces $L^p(\tau,T;X)$ for $\tau > 0$ invariant and the restriction $\mathcal{T}_\tau = (\mathcal{T}(t)_{|L^p(\tau,T;X)})_{t\geq 0}$ is a strongly continuous evolution semigroup.

Theorem 3.4.7. *Let X be a Banach space and $(A(t), D(A(t)))_{t\in[0,T]}$ a family of linear operators on X. The following assertions are equivalent.*

(i) The non-autonomous Cauchy problem (NCP) for $(A(t))_{t\in[0,T]}$ is well-posed.

(ii) There exists a unique evolution semigroup $\mathcal{T} = (\mathcal{T}(t))_{t\geq 0}$ with generator $(G, D(G))$ on $L^p(0,T;X)$ and for every $\tau \in [0,T]$ there exists an invariant core $D_\tau \subseteq W^{1,p}(\tau,T;X) \cap D(A_\tau(.))$ containing the solutions of $(NCP)_\tau$, such that

$$Gf + f' = A(.)f$$

for $f \in D_\tau$ and the spaces

$$Y_s := \{y \in X : \exists f \in D_0 \text{ with } f(s) = y\}$$

are dense in X.

Proof. $(i) \Rightarrow (ii)$ If (NCP) is well-posed, then there exists a unique evolution family solving (NCP), and thus a unique evolution semigroup. We still need to find invariant cores with the claimed properties. But (i) implies (NCP) is well-posed for $(A(t))_{t\in[\tau,T]}$, for every $\tau \in [0,T]$ and the following proof is analog for $\tau \neq 0$. Thus, we only need to find an invariant core for $(T(t))_{t\geq 0}$ with the claimed properties. It is obtained using an idea from [LMR], Proposition 2.9.
Let $\varepsilon > 0$ and for $s < 0$ define $Y_s := Y_0$ and $U(t,s) := U(t,0)$, where $t \geq 0$. Then consider $s \in [-\varepsilon, T]$, $y \in Y_s$ and a function $\alpha \in C^\infty(\mathbb{R})$ – the space of smooth functions – with $\alpha(t) = 0$ for $t \leq s$. Then the function $f \in L^p(0,T;X)$ defined by

$$f(t) := \begin{cases} \alpha(t)U(t,s)y & \text{if } t > s, \\ 0 & \text{otherwise,} \end{cases}$$

is contained in $W^{1,p}(0,T;X) \cap D(A(.))$. Moreover, we obtain that $f \in D(G)$ and $Gf = -f' + A(.)f$. Indeed,

$$T(t)f(r) = \begin{cases} \alpha(r-t)U(r,s)y & \text{if } r-t > s, \\ 0 & \text{otherwise,} \end{cases} \tag{3.18}$$

which implies $(Gf)(r) = \frac{d}{dt}T(t)f(r)_{|t=0} = -\alpha'(r)U(r,s)y$. Since $U(t,s)$ solves (NCP), we have

$$\frac{d}{dr}f(r) = \alpha'(r)U(r,s)y + \alpha(r)\frac{\partial}{\partial r}U(r,s)y = \alpha'(r)U(r,s)y + \alpha(r)A(r)U(r,s)y,$$

for almost every $r > s$ and therefore $Gf = -f' + A(.)f$.
Obviously the space

$$D := \operatorname{span}\{\alpha(t)U(t,s)y : s \in [-\varepsilon, T], y \in Y_s, \alpha \in C^\infty(\mathbb{R}), \alpha(t) = 0 \text{ for } t \leq s\}$$

contains the solutions of $(NCP)_0$. It remains to show, that the space D is invariant under $(T(t))_{t\geq 0}$ and dense in $L^p(0,T;X)$. From equation (3.18) we immediately obtain the invariance. As in [LMR], Proposition 2.9, we obtain, that D is dense in $C([0,T];X)$, from which one easily gets that the spaces given by $\{y \in X : \exists f \in D \text{ with } f(s) = y\}$ are dense in X. Moreover, since $[0,T]$ is a finite measure space $C([0,T];X)$ is dense and continuously embedded in $L^p(0,T;X)$, hence D is dense in $L^p(0,T;X)$. Finally, the uniqueness of the evolution semigroup follows by differentiating $s \mapsto S(t-s)T(s)f$ for two such semigroups and $f \in D$.
$(ii) \Rightarrow (i)$ Conversely suppose that there exists a unique evolution semigroup and thus a unique strongly continuous evolution family $(U(t,s))_{(t,s)\in I}$. Moreover, assume that there exist invariant cores $D_\tau \subset W^{1,p}(0,T;X) \cap D(A(.))$ for all $\tau \in [0,T]$, containing the solutions of $(NCP)_\tau$ and such that $Gf + f' = A(.)f$ for $f \in D_\tau$ and $Y_s := \{y \in X : \exists f \in D_0 \text{ with } f(s) = y\}$ dense in X. Then for

$f \in D_\tau \subset D(G) \cap W^{1,p}(\tau, T; X) \cap D(A_\tau(.))$, one has

$$\begin{aligned} \frac{U(.+h,.)f(.) - f(.)}{h} &= \frac{T(h)f(.+h) - f(.)}{h} \\ &= T(h)\frac{f(.+h) - f(.)}{h} - \frac{T(h)f(.) - f(.)}{h} \\ &\to f'(.) + Gf(.) = A(.)f(.) \end{aligned} \tag{3.19}$$

in $L^p(\tau, T; X)$ as $h \to 0$. In particular, for a solution u of $(NCP)_\tau$ one has $u \in D_\tau$ and (3.19) implies (E4) of Proposition 3.4.2. Thus it suffices to show, that the evolution family $(U(t,s))_{(t,s)\in I}$ solves (NCP) on the spaces Y_s.
Let $s \in [0,T]$ and $y \in Y_s$. Then by the definition of Y_s, there exists an $f \in D$ such that $f(s) = y$. For $t \geq s$ one obtains $U(t,s)y = U(t,s)f(s) = [T(t-s)f](t) \in Y_t$, because $T(t)f \in D$ for all $t \in [0,T]$. Moreover, by the same argument, one gets that $t \mapsto U(t,s)y = [T(t-s)f](t) \in D(A_s(.))$. It remains to show that $t \mapsto U(t,s)y \in W^{1,p}(s,T;X)$ and $\frac{\partial}{\partial t}U(t,s)y = A(t)U(t,s)y$, which is equivalent to $U(t,s)y = y + \int_s^t A(r)U(r,s)y\,dr$. Since $t \mapsto U(t,s)y$ is continuous, we obtain

$$\begin{aligned} U(t,s)y - U(s,s)y &= \lim_{h\to 0}\left[\frac{1}{h}\int_t^{t+h} U(r,s)f(s)\,dr - \frac{1}{h}\int_s^{s+h} U(r,s)f(s)\,dr\right] \\ &= \lim_{h\to 0}\frac{1}{h}\left[\int_{s+h}^{t+h} U(r,s)f(s)\,dr - \int_s^t U(r,s)f(s)\,dr\right] \\ &= \lim_{h\to 0}\int_s^t \frac{U(r+h,s)f(s) - U(r,s)f(s)}{h}\,dr. \end{aligned}$$

The claim now follows, if $\frac{U(.+h,s)f(s)-U(.,s)f(s)}{h} \to A(.)U(.,s)y$ in $L^1(s,T;X)$ as $h \to 0$. Note that

$$\frac{U(.+h,s)f(s) - U(.,s)f(s)}{h} = \frac{U(.+h,.)U(.,s)f(s) - U(.,s)f(s)}{h}$$

and by (3.19), it suffices to show, that $U(.,s)f(s) \in D_s$. Now observe that for $s \in [0,T]$ and $f \in D(G)$ the function $t \mapsto U(t,s)f(s) = T(t-s)f(t) \in D \subset D(G)$ and one has the equality $G[T(.-s)f(.)] = [T(.-s)Gf](.)$ in $L^p(0,T;X)$. Indeed,

$$\begin{aligned} &\frac{T(h)T(.-s)f(.) - T(.-s)f(.)}{h} - T(.-s)Gf(.) \\ &= \frac{T(.-s+h)f(.) - T(.-s)f(.)}{h} - T(.-s)\frac{T(h)f(.) - f(.)}{h} \\ &\quad + T(.-s)\frac{T(h)f(.) - f(.)}{h} - T(.-s)Gf(.) \\ &= T(.-s)\left[\frac{T(h)f(.) - f(.)}{h} - Gf(.)\right] \to 0 \end{aligned}$$

in $L^p(0,T;X)$, because T is a bounded semigroup and $f \in D(G)$. Moreover, for $s \in [0,T]$ and $f \in D \subset W^{1,p}(0,T;X) \cap D(G)$ one obtains that the function $t \mapsto U(t,s)f(s) = T(t-s)f(t) \in W^{1,p}(0,T;X)$, because

$$\begin{aligned}
&\frac{T(.+h-s)f(.+h) - T(.-s)f(.)}{h} \\
&= T(.+h-s)\frac{f(.+h)-f(.)}{h} + T(.-s)\frac{T(h)f(.)-f(.)}{h} \\
&\to T(.-s)f'(.) + T(.-s)Gf(.)
\end{aligned}$$

in $L^p(0,T;X)$ as $h \to 0$. Thus

$$\begin{aligned}
\frac{U(.+h,s)f(s) - U(.,s)f(s)}{h} &= \frac{U(.+h,.)U(.,s)f(s) - U(.,s)f(s)}{h} \\
&\to [U(.,s)f(s)]' + G[U(.,s)f(s)]
\end{aligned}$$

in $L^p(0,T;X)$, which implies $[U(.,s)f(s)]' = [U(.,s)f(s)]' + G[U(.,s)f(s)]$. Hence, $U(.,s)f(s) \in \operatorname{Ker} G \subset \operatorname{Ker}(-\frac{d}{dt} - A(.))$, which is contained in the solutions of $(NCP)_s$ and thus in D_s, which proves the claim. □

In view of Theorem 3.4.7, the non-autonomous Cauchy problem (NCP) is well-posed if the operator $(-\frac{d}{dt} + A(.), W^{1,p}(0,T;X) \cap D(A(.))$ is the generator of an evolution semigroup on $L^p(0,T;X)$. In the application we have in mind, we will have, that this operator is surjective and dissipative, which motivates the study of such operators.

3.4.2 Generator Property of a Surjective and Dissipative Operator

In order to get that a surjective and dissipative operator on a Hilbert space is the generator of a strongly continuous semigroup, we need some properties of surjective operators. This study was encouraged by Exercises 250 and 272 in [Ma].

Lemma 3.4.8. *Let E and F be Banach spaces. The set of open surjective mappings in $\mathcal{L}(E,F)$ is open in $\mathcal{L}(E,F)$.*

Proof. Let $T \in \mathcal{L}(E,F)$ be open and surjective, i.e. there exists $\delta_T > 0$ such that for every $y \in F$ with $\|y\| < 1$ there exists an $x \in E$ with $\|x\| < \frac{1}{\delta_T}$ and $Tx = y$. Let $\varepsilon > 0$ and $S \in \mathcal{L}(E,F)$ such that $\|T - S\| < \varepsilon$. We have to show, that S is open and surjective.
Let $y \in F$ with $\|y\| < 1$. Then there exists $x_1 \in E$ such that $\|x_1\| < \frac{1}{\delta_T}$ and $Tx_1 = y$. Then $\|(T-S)x_1\| \le \|T-S\|\,\|x_1\| = \varepsilon\frac{1}{\delta_T} < \frac{1}{2}$, if we choose $\varepsilon < \frac{1}{2}\delta_T$. Consequently, there is $x_2 \in E$ with $\|x_2\| < \frac{1}{2\delta_T}$ and $Tx_2 = (T-S)x_1$ and

$\|(T-S)x_2\| \leq \varepsilon\frac{1}{2\delta_T} < \frac{1}{4}$. If we continue this procedure, we obtain a sequence $(x_r)_{r\in\mathbb{N}} \subset E$ such that $\|x_{r+1}\| < \frac{1}{2^r\delta_T}$ and $Tx_{r+1} = (T-S)x_r$.
Observe that $\sum_{r=1}^{\infty} x_r$ converges in E. Let $x = \sum_{r=1}^{\infty} x_r$ and $\delta_S = \frac{\delta_T}{2}$. Then $\|x\| \leq \sum_{r=1}^{\infty}\|x_r\| < \sum_{r=1}^{\infty}\frac{1}{2^r\delta_T} = \frac{2}{\delta_T} = \frac{1}{\delta_S}$ and

$$Sx = S\sum_{r=1}^{\infty} x_r = \sum_{r=1}^{\infty} Sx_r = \sum_{r=1}^{\infty}(Tx_r - Tx_{r+1}) = Tx_1 = y,$$

because $\lim_{r\to\infty} Tx_r = 0$, as $\lim_{r\to\infty} x_r = 0$ and T is continuous. Thus we have, S is open and surjective. □

Corollary 3.4.9. Let E and F be Banach spaces. The set of surjective mappings in $\mathcal{L}(E,F)$ is open in $\mathcal{L}(E,F)$.

Proof. By the open mapping theorem, every surjective operator $T \in \mathcal{L}(E,F)$ is open. □

Corollary 3.4.10. On a Banach space E, for every operator $T \in \mathcal{L}(E)$, the deficiency spectrum

$$\sigma_d := \{\lambda \in \mathbb{C} : \lambda - T \text{ is not surjective}\}$$

is closed.

The following lemma shows, that the closure of a surjective and dissipative operator is the generator of a strongly continuous semigroup.

Lemma 3.4.11. *Let E be a Banach space and $A : E \supset D(A) \to E$ a densely defined, dissipative and surjective operator. Then the closure $\overline{A}$ of A is the generator of a contraction semigroup.*

Proof. Since A is densely defined and dissipative, it follows that A is closable and its closure $\overline{A}$ is also densely defined and dissipative. As $\overline{A}$ is closed, $(D(\overline{A}), \|\ \|_{\overline{A}})$ is a Banach space and

$$\overline{A} : (D(\overline{A}), \|\ \|_{\overline{A}}) \to E$$

is continuous and surjective, because A is surjective. Hence, by Corollary 3.4.9, there exists $\lambda > 0$ such that $\lambda - \overline{A} : (D(\overline{A}), \|\ \|_{\overline{A}}) \to E$ is continuous and surjective. Then Theorem 1.2.9 of Lumer and Phillips implies that $\overline{A}$ is the generator of a contraction semigroup. □

Now we only have to show, that surjective and dissipative operator on a Hilbert space, or more generally on a Banach space, is closed. For that we need the following result, see [ABHN], Proposition 4.3.6.

Lemma 3.4.12. *Let E be a Banach space and A the generator of a bounded C_0-semigroup $(T(t))_{t\geq 0}$ on X. Then* $\operatorname{Ker} A^*$ *separates the points of* $\operatorname{Ker} A$.

Now we are ready to formulate the main result of this section.

Theorem 3.4.13. *Let E be a Banach space and $A : E \supset D(A) \to E$ a densely defined, dissipative and surjective operator. Then A is the generator of a strongly continuous contraction semigroup.*

Proof. By Proposition 3.4.11, we only have to show that A is closed. It suffices to show that $\overline{A}$ is injective. Indeed, for every $x \in D(\overline{A})$, there exists $\tilde{x} \in D(A)$ such that $A\tilde{x} = \overline{A}x$, because A is surjective. Hence $\overline{A}(x - \tilde{x}) = 0$ and the injectivity of $\overline{A}$ implies $x = \tilde{x} \in D(A)$. Thus $D(\overline{A}) \subseteq D(A)$, which gives equality of the domains and hence of the operators.
We show the injectivity by contradiction. Assume that $\overline{A}$ is not injective, then there exists $0 \neq x \in \operatorname{Ker} \overline{A}$. Recall that $\overline{A}$ is surjective, because A is so. By Lemma 3.4.11, $\overline{A}$ is the generator of a contraction semigroup on the Hilbert space H. Hence, we get from Lemma 3.4.12 that $\operatorname{Ker} \overline{A}^*$ separates the points of $\operatorname{Ker} \overline{A}$, i.e. for $0 \neq x \in \operatorname{Ker} \overline{A}$, there exists an $x^* \in \operatorname{Ker} \overline{A}^*$, such that $x^*(x) \neq 0$; in particular $x^* \neq 0$. Then for every $x \in D(\overline{A})$ we have

$$0 =< \overline{A}^* x^*, x >=< x^*, \overline{A}x >,$$

which contradicts the surjectivity of $\overline{A}$. Thus, $\overline{A}$ has to be injective. □

3.4.3 Application to the variational setting

We now return to the setting of Section 3.1. Hence, let V and H be two Hilbert spaces, where V is a dense subspace of H with continuous embedding. Moreover, let $a(t; ., .)$, $t \in [0, T]$ be a family of sesquilinear forms, satisfying (3.1) and (3.14), and denote by $(A(t))_{t\in[0,T]}$ the family of associated operators.
Then, as we saw in Section 3.1, for every $x \in H$ and every $f \in L^2(0, T; V')$ there exists a unique function $u \in L^2(0, T; V) \cap H^1(0, T; V') \subset C([0, T], H)$ satisfying the inhomogeneous non-autonomous Cauchy problem

$$(INCP) \qquad \begin{cases} \dot{u}(t) &= -A(t)u(t) + f(t), \quad t > 0 \\ u(0) &= x. \end{cases}$$

Note, that even here we could consider a variable initial time s, but obtain no continuous dependence. Therefore, for $f \equiv 0$, we obtain an evolution family $U(t, s)_{0\leq s\leq t\leq T}$ on H, solving (NCP), but which is not strongly continuous. However, we shall obtain this property, if we regard the problem in the larger space V'.

Proposition 3.4.14. *The operator $\mathcal{G} := -\frac{d}{dt} - A(\cdot)$ defined on the domain $D(\mathcal{G}) = \{f \in H^1(0, T; V') \cap L^2(0, T, V) : f(0) = 0\}$ is the generator of a strongly continuous semigroup $\mathcal{T}$ on the Hilbert space $L^2(0, T; V')$.*

Proof. By Theorem 3.4.13 above, it is sufficient to show, that the operator $(\mathcal{G}, D(\mathcal{G}))$ is a densely defined, dissipative and surjective operator.
First observe, that the space $C_0^1((0,T];V) := \{f \in C^1([0,T];V) : f(0) = 0\}$ is contained in $D(\mathcal{G})$ and dense in $L^2(0,T;V')$, hence the operator $\mathcal{G}$ is densely defined.
Now the multiplication operator $A(\cdot)$ is associated with the sesquilinear form $\mathfrak{a} : L^2(0,T;V) \times L^2(0,T;V) \to \mathbb{C}$ given by $\mathfrak{a}(f,g) := \int_0^T a(t; f(t), g(t))\, dt$, which is continuous and elliptic. With the usual reduction, we only have to consider the case, where $a(t,.,.)$ for almosta all $t \in [0,T]$ and hence $\mathfrak{a}$ are coercive. Hence $(-A(\cdot), L^2(0,T,V))$ is the generator of a contraction semigroup on $\mathcal{V}' := L^2(0,T;V')$, and in particular it satisfies

$$\mathrm{Re} < -A(\cdot)f, f' >_{\mathcal{V}',(\mathcal{V}')'} \leq 0,$$

for all $f \in D(A(\cdot)) = L^2(0,T;V)$ and arbitrary $f' \in J(f)$. In particular, $-A(\cdot)$ is dissipative. Moreover, with $D(-\frac{d}{dt}) := \{f \in H^1(0,T;V') : f(0) = 0\}$, the operator $(-\frac{d}{dt}, D(-\frac{d}{dt}))$ is the generator of the shift group in $L^2(0,T;V')$, which is contractive, and therefore also for all $f \in D(-\frac{d}{dt})$ and arbitrary $f' \in J(f)$ we have

$$\mathrm{Re} < -\frac{d}{dt} f, f' >_{\mathcal{V}',(\mathcal{V}')'} \leq 0.$$

Thus, for every $f \in D(\mathcal{G}) = D(A(\cdot)) \cap D(-\frac{d}{dt})$ and arbitrary $f' \in J(f)$ we have

$$\mathrm{Re} < \Big(-\frac{d}{dt} - A(\cdot) \Big) f, f' > = \mathrm{Re} < -A(\cdot) f, f' > + \mathrm{Re} < -\frac{d}{dt} f, f' > \leq 0,$$

hence $-\frac{d}{dt} - A(\cdot)$ is dissipative.
Finally, the surjectivity follows from Lions' theorem. For $x = 0$ and for all $f \in L^2(0,T;V')$, there exists a (unique) $u \in H^1(0,T;V') \cap L^2(0,T,V))$ such that $u(0) = 0$, i.e. $u \in D(\mathcal{G})$, and

$$\left(-\frac{d}{dt} - A(\cdot) \right) u = -f.$$

□

We wish to apply Theorem 3.4.7. Therefore, we need the following result.

Proposition 3.4.15. *The semigroup $\mathcal{T}$ on $L^2(0,T;V')$ generated by the operator $\mathcal{G}$ of the theorem above is an evolution semigroup.*

Proof. We use the characterization given in Theorem 3.4.5. First observe, that for $f \in D(\mathcal{G})$ and $\varphi \in C^1([0,T])$ with support in $(0,T]$, we have $\varphi f \in D(\mathcal{G})$ and

$$\begin{aligned} G(\varphi f) &= -\frac{d}{dt}(\varphi f) - A(\cdot)(\varphi f) \\ &= -(\frac{d}{dt}\varphi) f - \varphi(\frac{d}{dt} f) - \varphi(A(\cdot) f) \\ &= -(\frac{d}{dt}\varphi) f - \varphi(\mathcal{G} f), \end{aligned}$$

because $A(\cdot)$ is a multiplication operator, and by Lemma 2.3.2 we have that $A(\cdot)(\varphi f) = \varphi(A(\cdot)f)$.
We still have to verify property (b), i.e. there exists a $\lambda \in \rho(\mathcal{G})$ such that $R(\lambda, \mathcal{G}) : L^2(0,T;V') \to C_0((0,T];V')$ is continuous with dense image. Recall that $\mathcal{G}$ is a surjective operator, because there exists a unique solution. Moreover G^{-1} is a is a bounded operator on $L^2(0,T;V')$. Indeed, one has the estimate $\|\mathcal{G}^{-1}f\|_{L^2(0,T;V')} \leq C_1\|\mathcal{G}^{-1}f\|_{L^2(0,T;V)} \leq C_2\|f\|_{L^2(0,T;V')}$ by Lemma 3.1.17. In particular $\mathcal{G}$ is injective and $0 \in \rho(\mathcal{G})$. Moreover, $R(0,\mathcal{G}) : L^2(0,T;V') \to D(\mathcal{G})$ is surjective, and from $C_0^1((0,T];V) \subset D(\mathcal{G}) \subset C_0((0,T];V')$ we obtain that $R(0,\mathcal{G})$ has dense image in $C_0((0,T];V')$. Finally, again Lemma 3.1.17 yields that $R(0,\mathcal{G}) : L^2(0,T;V') \to C_0((0,T];V')$ is continuous. □

Now we obtain our desired result.

Theorem 3.4.16. *Let $(A(t))_{t\in[0,T]}$ be a family of operators associated with a family of sesquilinear forms satisfying* (3.1) *and* (3.14). *Then the homogeneous non-autonomous variational Cauchy problem on $L^2(0,T;V')$*

$$(NCP) \qquad \begin{cases} \dot{u}(t) &= A(t)u(t), \quad t \geq s, \ t,s \in [0,T] \\ u(s) &= x. \end{cases}$$

is well-posed in the sense of Definition 3.4.1.

Proof. We apply Theorem 3.4.7 and take the cores $D_\tau = D(\mathcal{G}) \cap L^p(\tau,T;V')$. For $s \in (0,T]$ one has $H \subset Y_s$, by Proposition 3.1.3, hence the spaces are dense in V' and we get well-posedness on the spaces $(Y_s)_{s\in(0,T]}$. Well-posedness in the case where $s = 0$ and again $H \subset Y_0$ is obtained by extending the problem to the negative real axis. □

Thus, we obtain a strongly continuous evolution family $(\mathcal{U}(t,s))_{(t,s)\in I} \subset \mathcal{L}(V')$ solving (NCP) in $L^2(0,T;V')$.
However, Theorem 3.1.16 provides solutions of the problem (NCP) in the space $C([0,T];H)$. Regarding again the problem for various initial times, this gives rise to a family of operators $(U(t,s))_{(t,s)\in I} \subset \mathcal{L}(H)$ solving (NCP) and satisfying

- $U(t,r)U(r,s) = U(t,s)$ and $U(s,s) = Id$.
- $[s,T] \ni t \mapsto U(t,s)x \in H$ is continuous for all $x \in H$ and $0 \leq s \leq T$.

Note that we obtain no strong continuity in the sense of (E2) in Proposition 3.4.2.
Now for initial values $x \in H$ and by uniqueness of solutions, one gets that $\mathcal{U}(t,s)x = U(t,s)x$, which implies

$$U(t,s) = \mathcal{U}(t,s)_{|H}, \qquad \text{for all } (t,s) \in I.$$

This observation leads to the following result.

Proposition 3.4.17. *$\mathcal{T}$ leaves the space $\mathcal{H} := L^2(0,T;H)$ invariant and the restriction $\mathcal{T}_{|\mathcal{H}}$ is a strongly continuous semigroup.*

Proof. Let $\mathcal{G}$ be the generator of $\mathcal{T}$. Then it is sufficient to show that the the part of $\mathcal{G}$ in $\mathcal{H}$, denoted $\mathcal{G}_{\mathcal{H}}$ is the generator of a strongly continuous semigroup on $\mathcal{H}$. This semigroup then coincides with the restriction $\mathcal{T}_{|\mathcal{H}}$ of $\mathcal{T}$ to the space $\mathcal{H} := L^2(0,T;H)$. First observe, that $D(\mathcal{G})$ is a dense subset of $\mathcal{H}$. Therefore, for each $\lambda > 0$, we can restrict the resolvent $R(\lambda, \mathcal{G})$ to the space $\mathcal{H}$ and obtain a linear operator $R(\lambda, \mathcal{G})_{|\mathcal{H}} : \mathcal{H} \to \mathcal{H}$, which obviously satisfies the resolvent equation and has dense image. Moreover, by the dissipativity of the operators $-A(.)$ and $-\frac{d}{dt}$ in $\mathcal{H}$, we obtain for each $f \in \mathcal{H}$ and $\lambda > 0$,

$$\begin{aligned}
|\lambda| \|R(\lambda,\mathcal{G})f\|_{\mathcal{H}}^2 &\leq \lambda \operatorname{Re}(R(\lambda,\mathcal{G})f, R(\lambda,\mathcal{G})f) \\
&\quad + \operatorname{Re}(A(.)R(\lambda,\mathcal{G})f, R(\lambda,\mathcal{G})f) + \operatorname{Re}(\frac{d}{dt}R(\lambda,\mathcal{G})f, R(\lambda,\mathcal{G})f) \\
&= \operatorname{Re}(f, R(\lambda,\mathcal{G})f) \\
&\leq \|f\|_{\mathcal{H}} \|R(\lambda,\mathcal{G})f\|_{\mathcal{H}},
\end{aligned}$$

because $\mathcal{G} = -\frac{d}{dt} - A(.)$ Therefore $\|\lambda R(\lambda,\mathcal{G})_{|\mathcal{H}}\| \leq 1$ and $(R(\lambda,\mathcal{G})_{|\mathcal{H}})_{\lambda>0}$ is the resolvent of a densely defined operator on $\mathcal{H}$, which coincides with $G_{\mathcal{H}}$ and is the generator of a strongly continuous semigroup. □

Remark 3.4.18. (i) Note that $R(\lambda,\mathcal{G})\mathcal{H} \subset D(\mathcal{G}_{\mathcal{H}}) \subset D(G) \subset C_0((0,T];H)$ with dense inclusions implies that $R(\lambda,\mathcal{G}_{\mathcal{H}}) : \mathcal{H} \to C_0((0,T];H)$ is continuous with dense image. But we do not know if condition (a) of Proposition 3.4.5 is satisfied for $\mathcal{G}_{\mathcal{H}}$. Therefore, we cannot expect the restriction $T_{|\mathcal{H}}$ to be an evolution semigroup.

(ii) Moreover, $R(\lambda,\mathcal{G}) : C_0((0,T];H) \to C_0((0,T];H)$ is continuous with dense image, which shows that the semigroup $\mathcal{T}$ leaves $C_0((0,T];H)$ invariant. However, the restriction need not be strongly continuous.

From Proposition 3.4.17, we get a representation

$$(\mathcal{T}_{|\mathcal{H}}(t)f)(s) = \mathcal{U}(s, s-t)_{|H} f(s-t),$$

and as a consequence we obtain the invariance result below.
For a closed convex set $K \subset H$, let $L^2(0,T;K)$ denote the space of functions in $L^2(0,T;H)$ which take their values almost everywhere in K. Observe that $L^2(0,T;K)$ is a closed convex subset of $L^2(0,T;H)$.

Proposition 3.4.19. *Let $K \subset H$ be a closed convex subset. Then one has $\mathcal{T}_{|\mathcal{H}}(t)L^2(0,T;K) \subset L^2(0,T;K)$ for all $t \geq 0$, if and only if $U(t,s)_{|H}K \subset K$ for all $s \in [0,T]$ and all $t \in [s,T]$.*

Proof. Let $\mathcal{T}_{|\mathcal{H}}(t)L^2(0,T;K) \subset L^2(0,T;K)$ for all $t \geq 0$, and $x \in K$. Then the continuous function $f(t) \equiv x$ belongs to $L^2(0,T;K)$, and we obtain

$$U(s,s-t)_{|H}x = (\mathcal{T}_{|\mathcal{H}}(t)f)(s) \in K,$$

for almost all $s \in [0,T]$ and all $t \in [0,T]$, such that $s-t \in [0,T]$. This implies $U(t,s)_{|H}K \subset K$ for all $s \in [0,T]$ and all $t \in [s,T]$, because $t \mapsto U(t,s)x$ is continuous.
Conversely, if $U(t,s)_{|H}K \subset K$ for all $s \in [0,T]$ and all $t \in [s,T]$, then for $f \in L^2(0,T;K)$, one has $f(t) \in K$ for almost all $t \in [0,T]$. Therefore

$$(\mathcal{T}_{|\mathcal{H}}(t)f)(s) = U(s,s-t)_{|H}f(s-t) \in K,$$

for all $t \in [0,T]$ and almost all $s \in [0,T]$, such that $s-t \in [0,T]$ and $f(s-t) \in K$, therefore $\mathcal{T}_{|\mathcal{H}}(t)f \in L^2(0,T;K)$. □

Therefore, an invariance of closed convex sets for the solution of the non-autonomous Cauchy problem is given by the invariance of closed convex sets under the restriction of the evolution semigroup. We shall study the latter in the following section.

3.5 Generalized Forms

In the previous section, we have seen the importance of invariance of closed convex sets under the semigroup, which is obtained as the restriction of an evolution semigroup. As in the autonomous case, we wish to characterize this property with form methods. Unfortunately, neither the evolution semigroup nor its restriction is generated by an operator, which is associated with a sesquilinear form.
This motivates the study of generalized forms, which were introduced by W. Stannat in [Sta]. Moreover, he gave a characterization when the associated C_0-semigroup is positivity preserving or sub-Markovian. After an introduction to this theory, we will generalize these Beurling-Deny criteria to the invariance of closed convex sets in the same way as in [Ou]. Finally, we will give an application to non-autonomous Cauchy problems.

3.5.1 Motivation

Before we give the abstract definition of a generalized form, we explain the motivation. For that we start with the autonomous setting. For later application, we use the same notation as in the preceding section. Let $\mathcal{V} \overset{d}{\hookrightarrow} \mathcal{H} \overset{d}{\hookrightarrow} \mathcal{V}'$ be the usual triple of Hilbert spaces. Define a sesquilinear form $\mathfrak{a}$ on $\mathcal{H}$ with domain $\mathcal{V}$, which is continuous and elliptic. Then there exists a unique operator $\mathcal{A} \in \mathcal{L}(\mathcal{V},\mathcal{V}')$, such that $< \mathcal{A}u, v >_{V',V} = \mathfrak{a}(u,v)$ for all $u,v \in \mathcal{V}$. Then $-\mathcal{A}$ and

$-\mathcal{A}_{\mathcal{H}}$ are generators of strongly continuous semigroups on $\mathcal{V}'$ and $\mathcal{H}$ respectively. Moreover, the operator $\mathcal{A}^* \in \mathcal{L}(\mathcal{V}, \mathcal{V}')$ associated with the continuous and elliptic sesquilinear form $\mathfrak{a}^*(u,v) := \overline{\mathfrak{a}(v,u)}$, is the adjoint operator of $\mathcal{A}$ and generates the adjoint semigroup. Conversely, if $\mathcal{A} \in \mathcal{L}(\mathcal{V}, \mathcal{V}')$, then $\mathcal{A}^* \in \mathcal{L}(\mathcal{V}, \mathcal{V}')$ and $\mathfrak{a}(u,v) :=< \mathcal{A}u, v >_{V',V} = \overline{< \mathcal{A}^*v, u >_{V',V}}$ defines a continuous sesquilinear form on $\mathcal{H}$ with domain $\mathcal{V}$.
Now let us replace $\mathcal{A}$ and $\mathcal{A}^*$ by unbounded densely defined operators

$$\mathcal{G} : \mathcal{V} \supset D(\mathcal{G}) \to \mathcal{V}', \quad \mathcal{G}^* : \mathcal{V} \supset D(\mathcal{G}^*) \to \mathcal{V}',$$

where $\mathcal{G}^*$ is the adjoint operator of $\mathcal{G}$, when $\mathcal{G}$ is regarded as an operator from $\mathcal{V}$ to $\mathcal{V}'$ and define a form by

$$\mathfrak{g}(u,v) := \begin{cases} < \mathcal{G}u, v >_{V',V} & \text{if } u \in D(\mathcal{G}), v \in \mathcal{V} \\ \overline{< \mathcal{G}^*v, u >_{V',V}} & \text{if } u \in \mathcal{V}, v \in D(\mathcal{G}^*), \end{cases}$$

which we call not yet a generalized form, unless the operator $\mathcal{G}$ is of a certain structure as in [Sta]. Assume that $-\mathcal{G}$ is the generator of a contraction semigroup $\mathcal{T} = (T(t))_{t\geq 0} \subset \mathcal{L}(V')$ on $\mathcal{V}'$. Then for every $t \geq 0$, $T(t) \in \mathcal{L}(V, V')$ and defines a strongly continuous semigroup of contractions $\mathcal{T}_{\mathcal{V}} = (T(t))_{t\geq 0} \subset \mathcal{L}(V, V')$, with the same generator $-\mathcal{G}$, because $D(\mathcal{G}) \subset \mathcal{V}$. Moreover, $-\mathcal{G}^*$ is the generator of the adjoint semigroup $\mathcal{T}_{\mathcal{V}}^* = (T(t)^*)_{t\geq 0} \subset \mathcal{L}(V, V')$. Then $(0, \infty) \subset \rho(-\mathcal{G}) = \rho(-G^*)$ and $\|\lambda R(\lambda, -\mathcal{G})\| = \|\lambda R(\lambda, -\mathcal{G})^*\| = \|\lambda R(\lambda, -\mathcal{G}^*)\| \leq 1$. From the definition of $\mathfrak{g}$, one obtains for all $u \in \mathcal{V}', v \in V$ and $\lambda > 0$ that

$$\begin{aligned} \mathfrak{g}_\lambda((R(\lambda, -\mathcal{G})u, v) &:= \mathfrak{g}(R(\lambda, -\mathcal{G})u, v) + \lambda < u, v >_{\mathcal{V}',\mathcal{V}} \\ &= < u, v >_{\mathcal{V}',\mathcal{V}} = < (\mathcal{G}^* + \lambda)R(\lambda, -\mathcal{G}^*)u, v >_{\mathcal{V}',\mathcal{V}} \\ &= \overline{\mathfrak{g}(v, R(\lambda, -\mathcal{G}^*)u)} + \lambda \overline{< v, R(\lambda, -\mathcal{G}^*)u >_{\mathcal{V}',\mathcal{V}}} \\ &= \overline{\mathfrak{g}_\lambda(v, R(\lambda, -\mathcal{G}^*)u)}. \end{aligned}$$

Finally, if the semigroup $\mathcal{T}$ leaves $\mathcal{H}$ invariant, and the restriction is a strongly continuous semigroup with generator $\mathcal{G}_{\mathcal{H}}$, the part of $\mathcal{G}$ in $\mathcal{H}$, then $(\mathcal{G}_{\mathcal{H}})^* = \mathcal{G}_{\mathcal{H}}^*$ and

$$\mathfrak{g}_\lambda((R(\lambda, -\mathcal{G})u, v) = (u, v)_{\mathcal{H}} = \overline{\mathfrak{g}_\lambda(v, R(\lambda, -\mathcal{G}^*)u)},$$

for all $u \in \mathcal{H}, v \in V$ and $\lambda > 0$.
In view of the previous section, we shall consider operators $\mathcal{G}$ given as the sum two operators, the one being associated with a continuous and elliptic sesquilinear form, the other belonging to a class of operators containing the derivative.

3.5.2 Definitions and Basic Properties

We adapt the notation from [Sta] to our setting. In particular, we consider the triple of complex Hilbert spaces $\mathcal{V} \overset{d}{\hookrightarrow} \mathcal{H} \overset{d}{\hookrightarrow} \mathcal{V}'$. Let $\mathfrak{a}$ be a sesquilinear form

on $\mathcal{H}$ with domain $\mathcal{V}$, which is continuous and elliptic, and $\mathcal{A} \in \mathcal{L}(\mathcal{V}, \mathcal{V}')$ the associated operator.
Furthermore, let $(\Lambda, D(\Lambda))$ be a linear operator on $\mathcal{H}$, satisfying the following assumptions:

(i) $(\Lambda, D(\Lambda))$ generates a semigroup of contractions $(S_t)_{t\geq 0}$ on $\mathcal{H}$.

(ii) $(S_t)_{t\geq 0}$ leaves the space $\mathcal{V}$ invariant and the restriction is a C_0-semigroup on $\mathcal{V}$.

Then the generator of the restricted semigroup is the part of Λ in $\mathcal{V}$, denoted $\Lambda_{\mathcal{V}}$. Note that in particular the operator $-\frac{d}{dt}$ on the space $\mathcal{H} = L^2(0,T,H)$ with domain $D(-\frac{d}{dt}) = \{f \in H^1(0,T,H) : f(0) = 0\}$ satisfies these assumptions, if $\mathcal{V} = L^2(0,T,V)$ and $V \overset{d}{\hookrightarrow} H$. In order to define a generalized form with respect to the operator $\mathcal{G} = \mathcal{A} - \Lambda$, we wish to interpret Λ as an operator from $\mathcal{V}$ to $\mathcal{V}'$. For that observe, that $D(\Lambda_{\mathcal{V}}) \subset D(\Lambda) \cap \mathcal{V} \subset \mathcal{V}$, thus $D(\Lambda) \cap \mathcal{V}$ is dense in $\mathcal{V}$. Moreover, we have the following property.

Lemma 3.5.1. *Let the operator* $(\Lambda, D(\Lambda))$ *on* $\mathcal{H}$ *satisfy (i) and (ii) above. Then* $\Lambda : D(\Lambda) \cap \mathcal{V} \to \mathcal{V}'$ *is closable as an operator from* $\mathcal{V}$ *to* $\mathcal{V}'$.

Proof. Let $(u_n)_{n\in\mathbb{N}} \subset D(\Lambda) \cap \mathcal{V}$ and $f \in \mathcal{V}'$ such that $\lim_{n\to\infty} u_n = 0$ in $\mathcal{V}$ and $\lim_{n\to\infty} \Lambda u_n = f$ in $\mathcal{V}'$. Note that Λ is the generator of a contraction semigroup and thus dissipative, hence

$$< \Lambda v - f, v >= \lim_{n\to\infty} < \Lambda v - \Lambda u_n, v - u_n >= \lim_{n\to\infty} (\Lambda(v - u_n), (v - u_n)) \leq 0,$$

for all $v \in D(\Lambda) \cap \mathcal{V}$. In particular $\gamma^2 < \Lambda v, v > \leq \gamma < f, v >$ for all $\gamma \in \mathbb{R}$, which implies $< f, v >= 0$ for all $v \in D(\Lambda) \cap \mathcal{V}$, and thus for all $v \in \mathcal{V}$, because $D(\Lambda) \cap \mathcal{V}$ is dense in $\mathcal{V}$. It follows that $f = 0$. □

Let $\mathcal{F}$ denote the domain of the closure and denote again by $\Lambda : \mathcal{F} \to \mathcal{V}'$ the closure. Then $\mathcal{F}$ provided with the graph norm $\|u\|^2_{\mathcal{F}} = \|u\|^2_{\mathcal{V}} + \|\Lambda u\|^2_{\mathcal{V}'}$ is a Hilbert space.

Remark 3.5.2. In the particular case, where the semigroup S_t can also be extended to a C_0 semigroup on $\mathcal{V}'$, then $\mathcal{F} = D(\Lambda, \mathcal{V}') \cap \mathcal{V}$, where $D(\Lambda, \mathcal{V}')$ denotes the domain of the extension of Λ to $\mathcal{V}'$, which generates the extended semigroup.

Moreover, the adjoint semigroup $(S_t^*)_{t\geq 0}$ of $(S_t)_{t\geq 0}$ can be extended to a C_0-semigroup on $\mathcal{V}'$. Its generator is the adjoint $(\Lambda^*_{\mathcal{V}}, D(\Lambda^*_{\mathcal{V}}))$ of $\Lambda_{\mathcal{V}}$, see [Sta], Lemma 2.4.
Let $\hat{\mathcal{F}} := D(\Lambda^*_{\mathcal{V}}) \cap \mathcal{V}$. Since $\Lambda^*_{\mathcal{V}} : D(\Lambda^*_{\mathcal{V}}) \to \mathcal{V}'$ is closed, $\Lambda^*_{\mathcal{V}} : \hat{\mathcal{F}} \to \mathcal{V}'$ is closed as an operator from $\mathcal{V}$ to $\mathcal{V}'$, and thus the space $\hat{\mathcal{F}}$ provided with the graph norm $\|u\|^2_{\hat{\mathcal{F}}} = \|u\|^2_{\mathcal{V}} + \|\Lambda^*_{\mathcal{V}} u\|^2_{\mathcal{V}'}$ is a Hilbert space.

Definition 3.5.3. Let

$$\mathcal{E}(u,v) := \begin{cases} \mathfrak{a}(u,v) - < \Lambda u, v >_{\mathcal{V}',\mathcal{V}} & \text{if } u \in \mathcal{F}, v \in \mathcal{V} \\ \mathfrak{a}(u,v) - \overline{< \Lambda^*_{\mathcal{V}} v, u >}_{\mathcal{V}',\mathcal{V}} & \text{if } u \in \mathcal{V}, v \in \hat{\mathcal{F}}. \end{cases}$$

We call $\mathcal{E}$ the **generalized sesquilinear form** associated with the form $(\mathfrak{a}, \mathcal{V})$ and the operator $(\Lambda, D(\Lambda))$. Further we set $\mathcal{E}_\lambda(u,v) := \mathcal{E}(u,v) + \lambda(u,v)_{\mathcal{H}}$.

By [Sta], Lemma 2.7, $< \Lambda u, v > = < \hat{\Lambda} v, u >$ for all $u \in \mathcal{F}, v \in \hat{\mathcal{F}}$, so that $\mathcal{E}$ is well defined.

Remark 3.5.4. Since Λ generates a contraction semigroup on $\mathcal{H}$, the operator Λ is dissipative, i.e. $\mathrm{Re}(\Lambda u, u)_{\mathcal{H}} \leq 0$, for all $u \in D(\Lambda, \mathcal{H})$. Further, for all $u \in \mathcal{F}$ there exists a sequence $u_n \in D(\Lambda, \mathcal{H}) \cap \mathcal{V}$ converging to u in $\mathcal{F}$. Then $\mathrm{Re} < \Lambda u, u > = \lim_{n\to\infty} < \Lambda u_n, u_n > = \lim_{n\to\infty} (\Lambda u_n, u_n) \leq 0$. Consequently we obtain for all $u \in \mathcal{F}$

$$\mathrm{Re}\, \mathfrak{a}(u,u) \leq \mathrm{Re}\, \mathfrak{a}(u,u) - \mathrm{Re} < \Lambda u, u > = \mathrm{Re}\, \mathcal{E}(u,u). \tag{3.20}$$

Lemma 3.5.5. *Let $u_n \to u$ in $\mathcal{F}$ and $v_n \to v$ in $\mathcal{V}$. Then $\mathcal{E}(u_n, v_n) \to \mathcal{E}(u,v)$.*

Proof. For $u \in \mathcal{F}$ and $v \in \mathcal{V}$ observe that

$$\begin{aligned} |\mathcal{E}(u,v)| &= |\mathfrak{a}(u,v) + < \Lambda u, v >_{\mathcal{V}',\mathcal{V}}| \\ &\leq |\mathfrak{a}(u,v)| + |< \Lambda u, v >_{\mathcal{V}',\mathcal{V}}| \\ &\leq M\|u\|_{\mathcal{V}}\|v\|_{\mathcal{V}} + \|\Lambda u\|_{\mathcal{V}'}\|v\|_{\mathcal{V}}, \end{aligned}$$

since the form $\mathfrak{a}$ is continuous. If $u_n \to u$ in $\mathcal{F}$, then the convergence holds in particular in $\mathcal{V}$ and thus u_n is bounded in $\mathcal{V}$. Moreover, $\Lambda u_n \to \Lambda u$ in $\mathcal{V}'$ and Λu_n is bounded in $\mathcal{V}'$. Finally,

$$\begin{aligned} &|\mathcal{E}(u_n, v_n) - \mathcal{E}(u,v)| \\ &\quad\leq |\mathcal{E}(u_n, v_n) - \mathcal{E}(u_n, v)| + |\mathcal{E}(u_n, v) - \mathcal{E}(u,v)| \\ &\quad= |\mathcal{E}(u_n, v_n - v)| + |\mathcal{E}(u_n - u, v)| \\ &\quad\leq M\|u_n\|_{\mathcal{V}}\|v_n - v\|_{\mathcal{V}} + \|\Lambda u_n\|_{\mathcal{V}'}\|v_n - v\|_{\mathcal{V}} \\ &\qquad + M\|u_n - u\|_{\mathcal{V}}\|v\|_{\mathcal{V}} + \|\Lambda(u_n - u)\|_{\mathcal{V}'}\|v\|_{\mathcal{V}} \\ &\quad\to 0, \end{aligned}$$

as $n \to \infty$ and the claim follows. □

Theorem 3.5.6. *The operator $(\Lambda - \mathcal{A}, \mathcal{F})$ is the generator of a strongly continuous semigroup $\mathcal{T} = (T(t))_{t\geq 0}$ of contractions on $\mathcal{V}'$.*

Proof. As usual, we may only consider the case, where the sesquilinear form $\mathfrak{a}$ is coercive. First observe that for all $u \in \mathcal{F}$ one has

$$\operatorname{Re} < (\Lambda - \mathcal{A})u, u >_{\mathcal{V}',\mathcal{V}} = \operatorname{Re} < \Lambda u, u >_{\mathcal{V}',\mathcal{V}} - \operatorname{Re} \mathfrak{a}(u,u) \leq 0,$$

thus, $\Lambda - \mathcal{A}$ is dissipative. By Theorem 3.4.13, it now suffices to show that the operator $\Lambda - \mathcal{A}$ is surjective. We claim that for all $f \in \mathcal{V}'$, there exists an element $u \in \mathcal{F}$ such that $(\Lambda - \mathcal{A})u = f$, which is equivalent to the equality $< (\Lambda - \mathcal{A})u, v >_{\mathcal{V}',\mathcal{V}} = < f, v >_{\mathcal{V}',\mathcal{V}}$ for all $v \in \mathcal{V}$. We will use Lions' Representation Theorem 3.1.13, in order to show that for all $f \in \mathcal{V}'$, there exists a $u \in \mathcal{V}$ such that $< -\mathcal{A}u, v >_{\mathcal{V}',\mathcal{V}} + \overline{< \Lambda^* v, u >_{\mathcal{V}',\mathcal{V}}} = < f, v >_{\mathcal{V}',\mathcal{V}}$ holds for all $v \in \mathcal{V}$. Let $F := \mathcal{V}$ and $\Phi := D(\Lambda_{\mathcal{V}}^*) \cap \mathcal{V}$ provided with the norm of $\mathcal{V}$, then $\Phi \subset F$ with continuous injection. On $F \times \Phi$ define a sesquilinear form by

$$E(u, \varphi) = < -\mathcal{A}u, \varphi >_{\mathcal{V}',\mathcal{V}} + \overline{< \Lambda_{\mathcal{V}}^* \varphi, u >_{\mathcal{V}',\mathcal{V}}}.$$

Then for all $\varphi \in \Phi$, the form $u \mapsto E(u, \varphi)$ is continuous on F. Moreover, there exists a constant $\alpha > 0$ such that $|E(\varphi, \varphi)| \geq \alpha \|\varphi\|_{\mathcal{V}}^2$. Indeed,

$$\begin{aligned} |E(\varphi,\varphi)| &\geq -\operatorname{Re} E(\varphi,\varphi) \\ &= \operatorname{Re} < \mathcal{A}\varphi, \varphi >_{\mathcal{V}',\mathcal{V}} - \operatorname{Re} < \Lambda_{\mathcal{V}}^* \varphi, \varphi >_{\mathcal{V}',\mathcal{V}} \\ &\geq \operatorname{Re} \mathfrak{a}(\varphi,\varphi) \geq \alpha \|\varphi\|_{\mathcal{V}}^2. \end{aligned}$$

Moreover, $L(\varphi) := < f, \varphi >_{\mathcal{V}',\mathcal{V}}$ defines a continuous semilinear form on Φ. Thus, all assumptions of Theorem 3.1.13 are satisfied, and we conclude, that there exists a $u \in F = \mathcal{V}$, such that

$$< -\mathcal{A}u, \varphi >_{\mathcal{V}',\mathcal{V}} + \overline{< \Lambda_{\mathcal{V}}^* \varphi, u >_{\mathcal{V}',\mathcal{V}}} = E(u, \varphi) = L(\varphi) = < f, \varphi >_{\mathcal{V}',\mathcal{V}} .$$

Now observe, that for $u \in \mathcal{V}$ one has $\mathcal{A}u + f \in \mathcal{V}'$, thus one gets from the equality $< \Lambda_{\mathcal{V}}^* \varphi, u >_{\mathcal{V}',\mathcal{V}} = \overline{< \mathcal{A}u + f, \varphi >_{\mathcal{V}',\mathcal{V}}}$ that $u \in D(\Lambda_{\mathcal{V}}^{**}) = D(\overline{\Lambda_{\mathcal{V}}}) = \mathcal{F}$ and $< \Lambda_{\mathcal{V}}^* \varphi, u >_{\mathcal{V}',\mathcal{V}} = < \Lambda_{\mathcal{V}} u, \varphi >_{\mathcal{V}',\mathcal{V}}$, which proofs the claim. ☐

Theorem 3.5.7. *The semigroup $\mathcal{T} = (T(t))_{t \geq 0}$ leaves the space $\mathcal{H}$ invariant and the restriction $\mathcal{T}_{|\mathcal{H}} = (T(t)_{|\mathcal{H}})_{t \geq 0}$ is a strongly continuous semigroup on $\mathcal{H}$.*

Proof. Let $\mathcal{G} := \Lambda - \mathcal{A}$ be the generator of $\mathcal{T}$. Then it is sufficient to show that the the part of $\mathcal{G}$ in $\mathcal{H}$, denoted $\mathcal{G}_{\mathcal{H}}$ is the generator of a strongly continuous semigroup on $\mathcal{H}$. This semigroup then coincides with the restriction $\mathcal{T}_{|\mathcal{H}}$ of $\mathcal{T}$ to the space $\mathcal{H}$. First observe, that $D(\mathcal{G}) = \mathcal{F}$ is a dense subset of $\mathcal{V}$ and thus of $\mathcal{H}$. Therefore, for each $\lambda > 0$, we can restrict the resolvent $R(\lambda, \mathcal{G})$ to the space $\mathcal{H}$ and obtain a linear operator $R(\lambda, \mathcal{G})_{|\mathcal{H}} : \mathcal{H} \to \mathcal{H}$, which obviously satisfies the resolvent equation and has dense image. Moreover, by the dissipativity of the operators Λ and $-\mathcal{A}$ as operators on $\mathcal{H}$, we obtain for each $f \in \mathcal{H}$ and $\lambda > 0$,

$$\begin{aligned}
|\lambda| \|R(\lambda,\mathcal{G})f\|_{\mathcal{H}}^2 &\leq \lambda \operatorname{Re}(R(\lambda,\mathcal{G})f, R(\lambda,\mathcal{G})f) \\
&\quad + \operatorname{Re}(\Lambda R(\lambda,\mathcal{G})f, R(\lambda,\mathcal{G})f) + \operatorname{Re}(-\mathcal{A}R(\lambda,\mathcal{G})f, R(\lambda,\mathcal{G})f) \\
&= \operatorname{Re}(f, R(\lambda,\mathcal{G})f) \\
&\leq \|f\|_{\mathcal{H}} \|R(\lambda,\mathcal{G})f\|_{\mathcal{H}},
\end{aligned}$$

because $\mathcal{G} = \Lambda - \mathcal{A}$. Therefore $\|\lambda R(\lambda,\mathcal{G})_{|\mathcal{H}}\| \leq 1$ and $(R(\lambda,\mathcal{G})_{|\mathcal{H}})_{\lambda>0}$ is the resolvent of a densely defined operator on $\mathcal{H}$, which coincides with $G_{\mathcal{H}}$ and is the generator of a strongly continuous semigroup. □

Definition 3.5.8. The semigroup $\mathcal{T}_{|\mathcal{H}} = (T(t)_{|\mathcal{H}})_{t\geq 0}$ is called the **semigroup associated with $\mathcal{E}$**.

Observe that $\mathcal{G}^* = (\Lambda - \mathcal{A})^* = \Lambda^* - \mathcal{A}^*$ because $\mathcal{A} \in \mathcal{L}(\mathcal{V}, \mathcal{V}')$. Thus,

$$\mathcal{E}(u,v) := \begin{cases} <\mathcal{G}u, v>_{V',V} & \text{if } u \in \mathcal{F} = D(\mathcal{G}), v \in \mathcal{V} \\ \overline{<\mathcal{G}^*v, u>_{V',V}} & \text{if } u \in \mathcal{V}, v \in \hat{\mathcal{F}} = D(\mathcal{G}^*), \end{cases}$$

Therefore, from the considerations in section 3.5.1, we obtain $(\mathcal{G}_{\mathcal{H}})^* = G_{\mathcal{H}}^*$ and the equation

$$\mathcal{E}_\lambda((R(\lambda, -\mathcal{G}_{\mathcal{H}})u, v) = (u,v)_{\mathcal{H}} = \overline{\mathcal{E}_\lambda(v, R(\lambda, -\mathcal{G}_{\mathcal{H}}^*)u)},$$

for all $\lambda > 0$.
For simplicity, we shall denote $R_\lambda := R(\lambda, -\mathcal{G}_{\mathcal{H}})$.

Remark 3.5.9. (i) $R_1(\mathcal{H})$ is a dense subset of $\mathcal{F}$ by [Sta], Remark 3.5.

(ii) By the resolvent equation $(\lambda - \mu)R_\lambda R_\mu = R_\mu - R_\lambda$ with $\mu = 1$ we get $\lambda R_\lambda R_1 - R_\lambda R_1 = R_1 - R_\lambda$ hence for $u = R_1 h$ we have the equation

$$u - \lambda R_\lambda u = R_\lambda h - R_\lambda u. \tag{3.21}$$

(iii) Since $\mathcal{G}_{\mathcal{H}}$ is the generator of a C_0-semigroup, we have $\lim_{\lambda\to\infty} \lambda R_\lambda u = u$ in $\mathcal{H}$ for all $u \in \mathcal{H}$ and by [Sta], Proposition 3.7 in $\mathcal{V}$ for all $u \in \mathcal{V}$.

3.5.3 Invariance of closed convex sets

We keep the notation as above and characterize in terms of the generalized form $\mathcal{E}$ when the associated semigroup $\mathcal{T}_{|\mathcal{H}}$ leaves closed convex sets invariant.
Let K be a closed convex set in $\mathcal{H}$ and denote by P the orthogonal projection from $\mathcal{H}$ onto K. Then Pu for $u \in \mathcal{H}$ is characterized by $\operatorname{Re}(u - Pu, k - Pu) \leq 0$ for all $k \in K$.

Theorem 3.5.10. *The following assertions are equivalent.*

(i) $\mathcal{T}(t)_{|\mathcal{H}} K \subset K$ *for all* $t \geq 0$;

(ii) $\lambda R_\lambda K \subset K$ *for all* $\lambda > 0$;

(iii) $u \in \mathcal{F} \Rightarrow Pu \in \mathcal{V}$ *and* $\operatorname{Re} \mathcal{E}(u, u - Pu) \geq 0$.

Proof. $(i) \Leftrightarrow (ii)$: This is obvious from the identities

$$R(\lambda, -\mathcal{G}_\mathcal{H}) = \int_0^\infty \mathrm{e}^{-\lambda t}\, \mathcal{T}_{|\mathcal{H}}(t)\, dt$$

and

$$\mathcal{T}_{|\mathcal{H}}(t) = \lim_{n \to \infty} \left[\frac{n}{t} R\left(\frac{n}{t}, -\mathcal{G}_\mathcal{H}\right)\right]^n.$$

$(ii) \Rightarrow (iii)$: First observe that one obtains the estimate

$$\operatorname{Re}((u - Pu) - \lambda R_\lambda (u - Pu), u - Pu)_\mathcal{H} \leq \operatorname{Re}(u - \lambda R_\lambda u, u - Pu)_\mathcal{H}, \qquad (3.22)$$

indeed, $\operatorname{Re}(Pu - \lambda R_\lambda Pu, u - Pu)_\mathcal{H} \geq 0$ because $\lambda R_\lambda Pu \in K$ by (ii).
Now consider $u = R_1 h$ for $h \in \mathcal{H}$. We want to show that $\lambda R_\lambda (u - Pu)$ is bounded in the Hilbert space $\mathcal{V}$. The operator λR_λ is a contraction in $\mathcal{H}$, hence $\|\lambda R_\lambda (u - Pu)\|_\mathcal{H} \leq \|(u - Pu)\|_\mathcal{H}$, and as $\|.\|_\mathcal{V}^2$ is equivalent to $\operatorname{Re} \mathfrak{a}(.,.) + (.,.)_\mathcal{H}$, it suffices to show, that $\operatorname{Re} \mathfrak{a}(\lambda R_\lambda (u - Pu), \lambda R_\lambda (u - Pu))$ is bounded.
With (3.20), (3.21) and (3.22) we get

$$\begin{aligned}
&\operatorname{Re} \mathfrak{a}(\lambda R_\lambda (u - Pu), \lambda R_\lambda (u - Pu)) \\
&\quad \leq \operatorname{Re} \mathcal{E}(\lambda R_\lambda (u - Pu), \lambda R_\lambda (u - Pu)) \\
&\quad = \lambda \operatorname{Re} \mathcal{E}_\lambda (R_\lambda (u - Pu), \lambda R_\lambda (u - Pu)) - \lambda^2 \operatorname{Re}(R_\lambda (u - Pu), \lambda R_\lambda (u - Pu))_\mathcal{H} \\
&\quad = \lambda \operatorname{Re}((u - Pu), \lambda R_\lambda (u - Pu))_\mathcal{H} - \lambda^2 \operatorname{Re}(R_\lambda (u - Pu), \lambda R_\lambda (u - Pu))_\mathcal{H} \\
&\quad = \lambda \operatorname{Re}((u - Pu) - \lambda R_\lambda (u - Pu), \lambda R_\lambda (u - Pu))_\mathcal{H} \\
&\quad = \lambda \operatorname{Re}((u - Pu) - \lambda R_\lambda (u - Pu), \lambda R_\lambda (u - Pu) - (u - Pu))_\mathcal{H} \\
&\qquad + \lambda \operatorname{Re}((u - Pu) - \lambda R_\lambda (u - Pu), (u - Pu))_\mathcal{H} \\
&\quad \leq \lambda \operatorname{Re}((u - Pu) - \lambda R_\lambda (u - Pu), (u - Pu))_\mathcal{H} \\
&\quad \leq \lambda \operatorname{Re}(u - \lambda R_\lambda u, u - Pu)_\mathcal{H} \\
&\quad = \lambda \operatorname{Re}(R_\lambda h - R_\lambda u, u - Pu)_\mathcal{H}, \qquad (3.23)
\end{aligned}$$

which is bounded, because the last term converges to $\operatorname{Re}(h - u, u - Pu)_\mathcal{H}$, and hence $\lambda R_\lambda (u - Pu)$ is bounded in $\mathcal{V}$. Since $\mathcal{V}$ is reflexive, there exists a subsequence $\lambda_n R_{\lambda_n} (u - Pu)$ converging weakly to $v \in \mathcal{V}$. But one has strong convergence $\lim_{\lambda \to \infty} \lambda R_\lambda (u - Pu) = (u - Pu)$ in $\mathcal{H}$. Therefore $u - Pu = v \in \mathcal{V}$ and as $u \in \mathcal{V}$ we get $Pu \in \mathcal{V}$.

Additionally we get for $\tilde{\mathfrak{a}}(u,v) := \frac{1}{2}(\mathfrak{a}(u,v)+\mathfrak{a}(v,u))$

$$\begin{aligned}
&\operatorname{Re}\mathfrak{a}((u-Pu),(u-Pu))\\
&= \lim_{n\to\infty} \tilde{\mathfrak{a}}((u-Pu),\lambda_n R_{\lambda_n}(u-Pu))\\
&\le \liminf_{n\to\infty} (\tilde{\mathfrak{a}}((u-Pu),(u-Pu)))^{1/2}(\tilde{\mathfrak{a}}(\lambda_n R_{\lambda_n}(u-Pu),\lambda_n R_{\lambda_n}(u-Pu)))^{1/2},
\end{aligned}$$

by Cauchy-Schwarz's inequality. Hence, with (3.23)

$$\begin{aligned}
&\operatorname{Re}\mathfrak{a}((u-Pu),(u-Pu))\\
&\le \liminf_{n\to\infty} \operatorname{Re}\mathfrak{a}(\lambda_n R_{\lambda_n}(u-Pu),\lambda_n R_{\lambda_n}(u-Pu))\\
&\le \limsup_{n\to\infty} \lambda_n \operatorname{Re}(R_{\lambda_n}h - R_{\lambda_n}u, u-Pu)_{\mathcal{H}}\\
&= \operatorname{Re}(h-u,u-Pu)_{\mathcal{H}} = \operatorname{Re}\mathcal{E}(u,u-Pu) \le |\mathcal{E}(u,u-Pu)|\\
&\le M\|u\|_{\mathcal{V}}\|u-Pu\|_{\mathcal{V}} + \|\Lambda u\|_{\mathcal{V}'}\|u-Pu\|_{\mathcal{V}}\\
&\le C(\|u\|_{\mathcal{V}} + \|\Lambda u\|_{\mathcal{V}'})\|u-Pu\|_{\mathcal{V}}\\
&= C\|u\|_{\mathcal{F}}\|u-Pu\|_{\mathcal{V}},
\end{aligned}$$

where the constant $C = \max\{M,1\}$. Moreover,

$$\begin{aligned}
\operatorname{Re}\mathcal{E}(u,u-Pu) &= \operatorname{Re}\mathcal{E}_1(R_1h,u-Pu) - \operatorname{Re}(u,u-Pu)\\
&= \operatorname{Re}(h-u,u-Pu) = \lim_{\lambda\to\infty}\lambda(R_\lambda(h-u),u-Pu)\\
&= \lim_{\lambda\to\infty}\lambda(u-\lambda R_\lambda u, u-Pu) \ge 0,
\end{aligned}$$

since $\lambda R_\lambda u \in K$.
For arbitrary $u \in \mathcal{F}$ let $(u_n)_{n\ge1} \subset R_1(\mathcal{H})$ such that $u_n \to u$ in $\mathcal{F}$, hence in particular in $\mathcal{V}$ and therefore also in $\mathcal{H}$. The orthogonal projection is continuous on $\mathcal{H}$ and we get $Pu_n \to Pu$ in $\mathcal{H}$.
By the above consideration

$$\mathfrak{a}(u_n - Pu_n, u_n - Pu_n) \le C\|u_n\|_{\mathcal{F}}\|u_n - Pu_n\|_{\mathcal{V}}$$

is bounded, hence $u_n - Pu_n$ is bounded in $\mathcal{V}$. Then there exists a subsequence, which is weakly convergent to v in $\mathcal{V}$, but $u_n - Pu_n \to u - Pu$ in $\mathcal{H}$. Therefore, $u - Pu = v \in \mathcal{V}$, and since $u \in \mathcal{V}$ we get $Pu \in \mathcal{V}$.
Finally we get $\mathcal{E}(u,u-Pu) = \lim_{n\to\infty}\mathcal{E}(u_n,u_n-Pu_n) \ge 0$, by Lemma 3.5.5.
(iii) $\Rightarrow$ *(ii)*: Assume that $f \in K$, then $\operatorname{Re}(f-Ph,h-Ph)_{\mathcal{H}} \le 0$ for all $h \in \mathcal{H}$. Now for $\lambda > 0$, let $u = R_\lambda f$. We want to show, that $u \in K$. By (iii) we have

$$\begin{aligned}
0 &\le \operatorname{Re}\mathcal{E}(u,u-Pu) = \operatorname{Re}\mathcal{E}(R_\lambda f,u-Pu)\\
&= \operatorname{Re}[\mathcal{E}_\lambda(R_\lambda f,u-Pu) - \lambda(R_\lambda f,u-Pu)_{\mathcal{H}}] = \operatorname{Re}(f-\lambda R_\lambda f,u-Pu)_{\mathcal{H}}\\
&= (1/\lambda)\operatorname{Re}(f-\lambda R_\lambda f,\lambda R_\lambda f - P\lambda R_\lambda f)_{\mathcal{H}}\\
&= (1/\lambda)\operatorname{Re}(f-P\lambda R_\lambda f,\lambda R_\lambda f - P\lambda R_\lambda f)_{\mathcal{H}}\\
&\quad -(1/\lambda)\operatorname{Re}(\lambda R_\lambda f - P\lambda R_\lambda f,\lambda R_\lambda f - P\lambda R_\lambda f)_{\mathcal{H}}\\
&\le -\frac{1}{\lambda}\|\lambda R_\lambda f - P\lambda R_\lambda f\|_{\mathcal{H}}^2.
\end{aligned}$$

Therefore $\lambda R_\lambda f = P\lambda R_\lambda f \in K$ and thus $u = R_\lambda f \in K$. □

3.5.4 The Beurling-Deny Criteria

In the following we assume, that the underlying Hilbert space $\mathcal{H}$ is also a Banach lattice, which is the case if and only if $\mathcal{H}$ is of L^2-type. Hence fix some σ-finite measure space $(E, \mathcal{B}, m)$, such that $\mathcal{H} := L^2(E, m)$ is separable.
For real valued functions $u, v : E \to \mathbb{R}$ we set

$$u \vee v := \sup(u, v), \ u \wedge v := \inf(u, v)$$

$$u^+ := u \vee 0 \text{ and } u^- := (-u)^+.$$

Definition 3.5.11. (i) A bounded linear operator B is called **positivity preserving** (resp. **sub-Markovian**) if $Bf \geq 0$ (resp. $0 \leq Bf \leq 1$) for all $f \in \mathcal{H}$ with $f \geq 0$ (resp. $0 \leq f \leq 1$).

(ii) A C_0-semigroup $(T(t))_{t\geq 0}$ is called **positivity preserving** (resp. **sub-Markovian**) if $T(t)$ is positivity preserving (resp. sub-Markovian) for all $t \geq 0$.

The Beurling-Deny Criteria characterize in terms of the generalized form, when the associated C_0-semigroup is positivity preserving (resp. sub-Markovian), and can be easily deduced from Theorem 3.5.10.

Proposition 3.5.12. *The following statements are equivalent:*

(i) $\mathcal{T}_{|\mathcal{H}} = (\mathcal{T}(t)_{|\mathcal{H}})_{t\geq 0}$ *is positivity preserving.*

(ii) λR_λ *is positivity preserving for all* $\lambda > 0$.

(iii) $u \in \mathcal{F}$ *implies* $(\operatorname{Re} u)^+ \in \mathcal{V}$ *and* $\operatorname{Re} \mathcal{E}(u, u - (\operatorname{Re} u)^+) \geq 0$.

Proof. Consider the set $K = \{u \in \mathcal{H} : u \geq 0\}$, which is closed and convex. Then a bounded operator B is positivity preserving, if and only if $BK \subset K$. The projection onto K is given by $Pu = (\operatorname{Re} u)^+$, since for all $v \geq 0$

$$\begin{aligned}\operatorname{Re}(u - (\operatorname{Re} u)^+, v - (\operatorname{Re} u)^+) &= (\operatorname{Re} u - (\operatorname{Re} u)^+, v - (\operatorname{Re} u)^+) \\ &= -((\operatorname{Re} u)^-, v) + ((\operatorname{Re} u)^-, (\operatorname{Re} u)^+) \leq 0.\end{aligned}$$

Now $\mathcal{T}_{|\mathcal{H}}$ is positivity preserving if and only if $\mathcal{T}(t)_{|\mathcal{H}} K \subset K$ for all $t \geq 0$. By Theorem 3.5.10 this is equivalent to $\lambda R_\lambda K \subset K$ for all $\lambda > 0$ and the fact that $u \in \mathcal{F}$ implies $(\operatorname{Re} u)^+ \in \mathcal{V}$ and $\operatorname{Re} \mathcal{E}(u, u - (\operatorname{Re} u)^+) \geq 0$. □

Proposition 3.5.13. *The following statements are equivalent:*

(i) $\mathcal{T}_{|\mathcal{H}} = (\mathcal{T}(t)_{|\mathcal{H}})_{t\geq 0}$ *is sub-Markovian.*

(ii) λR_λ *is sub-Markovian for all* $\lambda > 0$.

(iii) $u \in \mathcal{F} \Rightarrow (\operatorname{Re} u \wedge 1)^+ \in \mathcal{V}$ *and* $\operatorname{Re} \mathcal{E}(u, u - (\operatorname{Re} u \wedge 1)^+) \geq 0$.

Proof. Consider the set $K = \{u \in \mathcal{H} : 0 \leq u \leq 1\}$, which is closed and convex. Then a bounded operator B is sub-Markovian, if and only if $BK \subset K$. The projection onto K is given by $Pu = (\operatorname{Re} u \wedge 1)^+ = (\operatorname{Re} u)^+ \wedge 1$, since for all $0 \leq v \leq 1$

$$\begin{aligned} &\operatorname{Re}(u - (\operatorname{Re} u \wedge 1)^+, v - (\operatorname{Re} u \wedge 1)^+) \\ &= (\operatorname{Re} u - (\operatorname{Re} u \wedge 1)^+, v - (\operatorname{Re} u \wedge 1)^+) \\ &= \begin{cases} (\operatorname{Re} u - 1, v - 1) & \leq 0 \quad \operatorname{Re} u \geq 1 \\ (0, v - \operatorname{Re} u) & = 0 \quad 0 \leq \operatorname{Re} u \leq 1 \\ (\operatorname{Re} u, v) & \leq 0 \quad \operatorname{Re} u \leq 0 \end{cases} \end{aligned}$$

Now $\mathcal{T}_{|\mathcal{H}}$ is sub-Markovian if and only if $\mathcal{T}(t)_{|\mathcal{H}} K \subset K$ for all $t \geq 0$. By Theorem 3.5.10 this is equivalent to $\lambda R_\lambda K \subset K$ for all $\lambda > 0$ and the fact that $u \in \mathcal{F}$ implies $(\operatorname{Re} u \wedge 1)^+ \in \mathcal{V}$ and $\operatorname{Re} \mathcal{E}(u, u - (\operatorname{Re} u \wedge 1)^+) \geq 0$. □

In the case of a real Hilbert space the projection onto $K = \{u \in \mathcal{H} : u \geq 0\}$ is given by $Pu = u^+$. Then $\mathcal{E}(u, u - u^+) = \mathcal{E}(u, -(u^-)) = \mathcal{E}(-u, (-u)^+)$ and therefore $\mathcal{E}(u, u - u^+) \geq 0$, if and only if $\mathcal{E}(u, u^+) \geq 0$ and we obtain the Beurling-Deny criterion as in [Sta].

Remark 3.5.14. The third equivalence given in [Sta], Proposition 4.4, for $\mathcal{H}$ real:

$$u \in \hat{\mathcal{F}} \Rightarrow u^+ \in \mathcal{V} \text{ and } \mathcal{E}(u^+, u) \geq 0,$$

reflects the fact, that if λR_λ is positivity preserving, so is λR_λ^*. The proof uses the fact that for $Pu = u^+$ we have $Pu - u = P(-u)$, which cannot be generalized to arbitrary closed convex sets.

For applications one is interested in a formulation of sufficient conditions independently for $\mathcal{A}$ and Λ. This was done in [Sta], Lemma 4.5 and Proposition 4.7, in the real setting, which we cite here for completeness.

Lemma 3.5.15. *Let* $(\mathfrak{a}, \mathcal{V})$ *satisfy*

$$u \in \mathcal{V} \textit{ implies } u^+ \in \mathcal{V} \textit{ and } \mathfrak{a}(u, u^+) \geq 0,$$

and $(\Lambda u, u^+)_{\mathcal{H}} \leq 0$ *for all* $u \in D(\Lambda)$. *Then condition (iii) of Proposition 3.5.12 is satisfied.*

Lemma 3.5.16. *Let* $(\mathfrak{a}, \mathcal{V})$ *satisfy*

$$u \in \mathcal{V} \textit{ implies } u^+ \wedge 1 \in \mathcal{V} \textit{ and } \mathfrak{a}(u + u^+ \wedge 1, u - u^+ \wedge 1) \geq 0,$$

and $(\Lambda u, (u-1)^+) \leq 0$ *for all* $u \in D(\Lambda)$. *Then condition (iii) of Proposition 3.5.13 is satisfied.*

3.5.5 Application to non-autonomous Cauchy problems

The relation of generalized forms to non-autonomous variational Cauchy problems is given in [Sta], Example 4.9.(iii). In this section we recall the main steps and apply the Beurling-Deny criteria.

As in section 3.1, we are given a bounded interval $[0,T] \subset \mathbb{R}$ and a triple of Hilbert spaces $V \overset{d}{\hookrightarrow} H \overset{d}{\hookrightarrow} V'$. Let $a(t,.,.) : V \times V \to \mathbb{C}$, $t \in [0,T]$, be a family of sesquilinear forms satisfying (3.1) and (3.14).

Then for the spaces $\mathcal{V} := L^2(0,T;V)$ and $\mathcal{H} := L^2(0,T;H)$, we obtain for the dual space of $\mathcal{V}$ that $\mathcal{V}' = L^2(0,T;V')$ and get dense continuous embeddings $\mathcal{V} \overset{d}{\hookrightarrow} \mathcal{H} \overset{d}{\hookrightarrow} \mathcal{V}'$. Define for $u, v \in \mathcal{V}$

$$\mathfrak{a}(u,v) := \int_0^T a(t,u(t),v(t))\,dt.$$

Then $(\mathfrak{a}, \mathcal{V})$ is a continuous and elliptic sesquilinear form on $\mathcal{H}$.

Consider on $\mathcal{H}$ the shift semigroup $(S(t))_{t\geq 0}$ defined for $f \in \mathcal{H}$ by

$$S(t)h(s) := \begin{cases} h(s-t) & \text{if } s, s-t \in [0,T] \\ 0 & \text{otherwise.} \end{cases}$$

Then both $(S(t))_{t\geq 0}$ and the adjoint semigroup $(S^*(t))_{t\geq 0}$ can be extended to C_0-semigroups of contractions on $\mathcal{V}'$. Denote by $(-\frac{d}{dt}, D(-\frac{d}{dt}, \mathcal{V}'))$ and $(\frac{d}{dt}, D(\frac{d}{dt}, \mathcal{V}'))$ the generators, where

$$\begin{aligned} D\Big(-\frac{d}{dt}, \mathcal{V}'\Big) &= \{u \in \mathcal{V}' : \frac{d}{dt}u \in \mathcal{V}' \text{ and } u(0) = 0\} \qquad \text{and} \\ D\Big(\frac{d}{dt}, \mathcal{V}'\Big) &= \{u \in \mathcal{V}' : \frac{d}{dt}u \in \mathcal{V}' \text{ and } u(T) = 0\} \end{aligned}$$

Let $\mathcal{F} := D(-\frac{d}{dt}, \mathcal{V}') \cap \mathcal{V}$ and $\hat{\mathcal{F}} := D(\frac{d}{dt}, \mathcal{V}') \cap \mathcal{V}$. The time dependent generalized form corresponding to $(a(t,.,.), V)_{t\in[0,T]}$ is now given as follows.

$$\mathcal{E}(u,v) := \begin{cases} \mathfrak{a}(u,v) + < \frac{du}{dt}, v > & \text{if } u \in \mathcal{F}, v \in \mathcal{V}, \\ \mathfrak{a}(u,v) - < \frac{dv}{dt}, u > & \text{if } u \in \mathcal{V}, v \in \hat{\mathcal{F}}. \end{cases}$$

It is obvious, that the corresponding C_0-semigroup coincides with the restricted C_0-semigroup $\mathcal{T}_{|\mathcal{H}}$ obtained in Theorem 3.4.17.

Now assume that the underlying Hilbert space H is a real Banach lattice, then $H = L^2(\Omega, \mathbb{R})$ for some measure space Ω. Moreover, for all $u \in D(-\frac{d}{dt}, \mathcal{H})$, one has $(-\frac{d}{dt}u, u^+) = (-\frac{d}{dt}u^+, u^+) + (\frac{d}{dt}u^-, u^+) \leq 0$ and on the other hand, one gets the estimate $(-\frac{d}{dt}u, (u-1)^+) = (-\frac{d}{dt}(u \wedge 1), (u-1)^+) + (-\frac{d}{dt}(u-1)^+, (u-1)^+) \leq 0$. Therefore we obtain the following results as a consequence of Lemma 3.5.15 and Lemma 3.5.16.

Lemma 3.5.17. *Let* $(\mathfrak{a}, \mathcal{V})$ *satisfy*

$$u \in \mathcal{V} \text{ implies } u^+ \in \mathcal{V} \text{ and } \mathfrak{a}(u, u^+) \geq 0.$$

Then condition (iii) *of Proposition 3.5.12 is satisfied.*

Lemma 3.5.18. *Let* $(\mathfrak{a}, \mathcal{V})$ *satisfy*

$$u \in \mathcal{V} \text{ implies } u^+ \wedge 1 \in \mathcal{V} \text{ and } \mathfrak{a}(u + u^+ \wedge 1, u - u^+ \wedge 1) \geq 0.$$

Then condition (iii) *of Proposition 3.5.13 is satisfied.*

Thus we obtain a generalization of the results in Section 3.2 to arbitrary Hilbert spaces $V \overset{d}{\hookrightarrow} H = L^2(\Omega, \mathbb{R})$. Again, we assume that the sesquilinear forms $a(t, ., .) : V \times V \to \mathbb{C}$ satisfy (3.1) and (3.14) for $\lambda = 0$.

Corollary 3.5.19. Assume that for each $u \in V \overset{d}{\hookrightarrow} H = L^2(\Omega, \mathbb{R})$, one has $u^+ \in V$ and

$$a(t; u^+, u^-) \leq 0,$$

for almost all $t \in [0, T]$, then for all positive initial values the solution of the homogeneous non-autonomous Cauchy-Problem is positive.

Proof. First observe that for $u \in \mathcal{V}$, one has for almost all $t \in [0, T]$ that $u(t) \in V$, hence $u^+(t) = u(t)^+ \in V$. Moreover, one obtains $u^+ \in \mathcal{V}$ from the inequality $\|u^+\|_{\mathcal{V}} = \int_0^T \|u(t)^+\|_V^2 \, dt = \int_{\{t:0 \leq u(t)\}} \|u(t)^+\|_V^2 \leq \|u\|_{\mathcal{V}}$. Now, by the positivity of the form $a(t, ., .)$, we get

$$\begin{aligned}
\mathfrak{a}(u, u^+) &= \int_0^T a(t, u(t), u(t)^+) \, dt \\
&= \int_0^T a(t, u(t)^+, u(t)^+) \, dt - \int_0^T a(t, u(t)^-, u(t)^+) \, dt \\
&\geq -\int_0^T a(t, (-u(t))^+, (-u(t))^-) \, dt \\
&\geq 0,
\end{aligned}$$

since $a(t, (-u(t))^+, (-u(t))^-) \leq 0$ almost everywhere on $[0, T]$.
Therefore, the assumptions of Lemma 3.5.17 are satisfied and thus condition (iii) of Proposition 3.5.12, which says that the corresponding C_0-semigroup $(\mathcal{T}_{|\mathcal{H}}(t))_{t \geq 0}$ is positivity preserving. This means that $\mathcal{T}_{|\mathcal{H}}(t) L^2(0, T; H_+) \subset L^2(0, T; H_+)$, from which we obtain with Theorem 3.4.19 that $U(t, s)_{|H} H_+ \subset H_+$ for all $s \in [0, T]$ and almost all $t \in [s, T]$. In particular for all positive $x \in H_+$, the solution given by $u(t) := U(t, 0)x$ is positive. $\square$

Corollary 3.5.20. Assume that for all $u \in V \overset{d}{\hookrightarrow} L^2(\Omega, \mathbb{R})$ one has $u^+ \in V$, $u \wedge \mathbf{1} \in V$ and

$$a(t; u, u \wedge \mathbf{1}) \leq a(t; u, u)$$

for almost all $t \in [0, T]$. Then the solution of the non-autonomous Cauchy problem is sub-Markovian.

Proof. As before, we obtain that $u^+ \in \mathcal{V}$. Additionally, for $u \in \mathcal{V}$, one has for almost all $t \in [0, T]$ that $u(t) \in V$, hence $(u^+ \wedge \mathbf{1})(t) = u(t)^+ \wedge 1 \in V$. Moreover,

$$\begin{aligned} \|u^+ \wedge \mathbf{1}\|_{\mathcal{V}} &= \int_0^T \|u(t)^+ \wedge 1\|_V^2 \, dt \\ &= \int_{\{t: 0 \leq u(t) \leq 1\}} \|u(t)^+\|_V^2 \, dt + \int_{\{t: 0 \leq u(t) \leq 1\}} 1 \, dt \leq \|u\|_{\mathcal{V}} + T, \end{aligned}$$

which shows that $u^+ \wedge \mathbf{1} \in \mathcal{V}$. Now, observe that $a(t; u, u \wedge \mathbf{1}) \leq a(t; u, u)$ is equivalent to $a(t; u, u - u \wedge \mathbf{1}) \geq 0$. Since $u \wedge \mathbf{1}$ is the orthogonal projection of u onto the closed convex set $\{u \in V : u \leq 1\}$, we get by Remark 1.3.6, that also $a(t; u \wedge \mathbf{1}, u - u \wedge \mathbf{1}) \geq 0$. From that, and by the positivity of the form $a(t, ., .)$, we get

$$\begin{aligned} &\mathfrak{a}(u + u^+ \wedge 1, u - u^+ \wedge 1) \\ &= \int_0^T a(t, u(t) + u(t)^+ \wedge 1, u(t) - u(t)^+ \wedge 1) \, dt \\ &= \int_0^T a(t, u(t), u(t) - u(t)^+ \wedge 1) + a(t, u(t)^+ \wedge 1, u(t) - u(t)^+ \wedge 1) \, dt \\ &= \int_{\{t: u(t) \leq 0\}} a(t, u(t), u(t)) + a(t, 0, u(t)) \, dt \\ &\quad + \int_{\{t: 0 \leq u(t) \leq 1\}} a(t, u(t), 0) + a(t, u(t), 0) \, dt \\ &\quad + \int_{\{t: u(t) \geq 1\}} a(t, u(t), u(t) - u(t) \wedge 1) + a(t, u(t) \wedge 1, u(t) - u(t) \wedge 1) \, dt \\ &\geq 0, \end{aligned}$$

since each integrand is greater than or equal to 0 almost everywhere on $[0, T]$. Therefore, the assumptions of Lemma 3.5.18 are satisfied and thus condition (iii) of Proposition 3.5.13, which says that the corresponding C_0-semigroup $(\mathcal{T}_{|\mathcal{H}}(t))_{t \geq 0}$ is sub-Markovian. However, this means that $\mathcal{T}_{|\mathcal{H}}(t) L^2(0, T; K) \subset L^2(0, T; K)$, where $K = \{x \in H : 0 \leq x \leq 1\}$, from which we obtain with Theorem 3.4.19 that $U(t, s)_{|H} K \subset K$ for all $s \in [0, T]$ and almost all $t \in [s, T]$. In particular for all $0 \leq x \leq 1$, we have $x \in K$, and thus the solution given by $u(t) := U(t, 0)x$ belongs to K, which says that $0 \leq u(t) \leq 1$ for almost all $t \in [0, T]$. Hence, the solution is sub-Markovian. □

Chapter 4

Infinite Dimensional Evolution Equations

So far we only studied the theory of partial differential equations in finite dimensional spaces. However, stochastic differential equations are related to infinite dimensional evolution equations. For more details, we refer to [PZ].
Lately, there have been some analytic results on the heat equation in infinite dimensions, see [CP], [ArDE] and [AbEK]. Here we will consider the more general case of second order differential equations with mixed derivatives, to which we refer as the non-diagonal case.
Before we recall the results on the heat equation, we shall give the necessary background on the generators of the shift group and on Gaussian measures and semigroups. Then we give necessary and sufficient conditions on the coefficients a_{ij} for well-posedness of the problem

$$(P) \quad \begin{cases} \frac{\partial u(t,x)}{\partial t} = \sum_{i,j\in\mathbb{N}} a_{ij} \frac{\partial^2 u(t,x)}{\partial x_i \partial x_j}, & t > 0 \\ u(0,.) = f & \in BUC(l^p), \end{cases}$$

with $a_{ij} \in \mathbb{R}$.
Finally, we will study a generalization, where we replace the derivatives with respect to x_i by group generators.

4.1 The Heat Equation in Infinite Dimensions

Before studying the problem (P) we want to recall some results about the infinite dimensional heat equation

$$(HE_\infty) \quad \begin{cases} \frac{\partial u(t,x)}{\partial t} = \sum_{j\in\mathbb{N}} \lambda_j \frac{\partial^2 u(t,x)}{\partial x_j^2}, & t > 0 \\ u(0,.) = f & \in BUC(l^p) \end{cases}$$

established in the papers [ArDE] and [AbEK].
In the subsequent paragraph, we precise the notion of derivative. Then we recall the definition of Gaussian measures and semigroups and certain properties, which we shall use for our results.

4.1.1 The shift group and its generator

Let $1 \leq p < \infty$ and $l^p = \{(x_n)_{n\in\mathbb{N}} \subset \mathbb{R} : \|x\|_p := (\sum_{n=1}^{\infty} |x_n|^p)^{1/p} < \infty\}$, which is a Banach space for the norm $\|.\|_p$. Then its positive cone is given as the subspace $l^p_+ = \{(x_n)_{n\in\mathbb{N}} \in l^p : x_n > 0 \text{ for all } n \in \mathbb{N}\}$. We denote by e_k the k-th canonical basis vector of l^p. Let $BUC(l^p)$ be the space of functions $f : l^p \to \mathbb{C}$, which are bounded and uniformly continuous. Then $BUC(l^p)$ provided with the supremum norm $\|f\|_\infty = \sup_{x\in l^p} |f(x)|$ is a Banach space.

Definition 4.1.1. Let $f \in BUC(l^p)$ and $k \in \mathbb{N}$. We say that $f \in D(\frac{\partial}{\partial x_k})$, if for every $x \in l^p$,

$$\frac{\partial}{\partial x_k} f(x) := \lim_{t\to 0} \frac{f(x+te_k) - f(x)}{t}$$

exists and defines a function $\frac{\partial}{\partial x_k} f \in BUC(l^p)$.

Note that $\frac{\partial}{\partial x_k} : f \mapsto \frac{\partial}{\partial x_k} f$ defines a linear operator on $BUC(l^p)$ with domain $D(\frac{\partial}{\partial x_k})$.
For $k \in \mathbb{N}$ and $t \in \mathbb{R}$ consider the shift operator $T_k(t) : BUC(l^p) \to BUC(l^p)$ defined by $(T_k(t)f)(x) = f(x+te_k)$. Then $(T_k(t))_{t\in\mathbb{R}}$ defines a strongly continuous group of operators. Let D_k denote its generator, then

$$\begin{aligned} D(D_k) &= \left\{ f \in BUC(l^p) : \lim_{t\to 0} \frac{T_k(t)f - f}{t} \text{ exists in } BUC(l^p) \right\} \\ D_k(f) &= \lim_{t\to 0} \frac{T_k(t)f - f}{t} \end{aligned}$$

Lemma 4.1.2. *The two operators $\frac{\partial}{\partial x_k}$ and D_k on $BUC(l^p)$ coincide.*

Proof. If $f \in D(D_k)$, then for every $x \in l^p$,

$$D_k f(x) = \lim_{t\to 0} \frac{T_k(t)f(x) - f(x)}{t} = \lim_{t\to 0} \frac{f(x+te_k) - f(x)}{t}$$

exists and $D_k f \in BUC(l^p)$. Hence $f \in D(\frac{\partial}{\partial x_k})$ and $\frac{\partial}{\partial x_k} f = D_k f$.
Conversely, if $f \in D(\frac{\partial}{\partial x_k})$, we have to show that $f \in D(D_k)$ and $D_k f = \frac{\partial}{\partial x_k} f =: g$, which is equivalent to

$$T_k(t)f - f = \int_0^t T_k(s) g \, ds.$$

Since both terms belong to $BUC(l^p)$, it suffices to show equality pointwise. But, since $g(x+te_k) = \frac{\partial}{\partial x_k} f(x+te_k) = \frac{d}{dt} f(x+te_k)$,

$$T_k(t)f(x) - f(x) = f(x+te_k) - f(x) = \int_0^t g(x+se_k)\,ds = \int_0^t T_k(s)g(x)\,ds$$

follows immediately. □

Hence, in the sequel, we will no longer distinguish between $\frac{\partial}{\partial x_k}$ and D_k. For more convenience, we shall use the following definition for the functions belonging to $D(D_k)$ for all $k \in \mathbb{N}$.

Definition 4.1.3. We call a function f belonging to the space

$$BUC^1(l^p) := \{f \in BUC(l^p) : f \in D(D_k) \text{ for all } k \in \mathbb{N}\} = \bigcap_k D(D_k),$$

once partially differentiable in $BUC(l^p)$, and analogously we call a function $f \in BUC^2(l^p) := \{f \in BUC^1(l^p) : D_k f \in BUC^1(l^p) \text{ for all } k \in \mathbb{N}\}$ **twice partially differentiable** in $BUC(l^p)$.

Observe that the shift operators commute, i.e. for all $s, t \in \mathbb{R}$ and $k, l \in \mathbb{N}$, one has $T_k(t)T_l(s)f = T_l(s)T_k(t)f$ for all $f \in BUC(l^p)$. Hence the resolvents $R(\lambda, D_k)$ of the group generators D_k commute. This implies that $D_k D_l f = D_l D_k f$, whenever $f \in D(D_k) \cap D(D_l)$ such that $D_l f \in D(D_k)$ and $D_k f \in D(D_l)$.

Lemma 4.1.4. *Let $f \in BUC^2(l^p)$. Then $D_k D_l f = D_l D_k f$ for all $k, l \in \mathbb{N}$.*

Proof. Obviously, for all $f \in BUC^2(l^p)$ one has $f \in D(D_k) \cap D(D_l)$ such that $D_l f \in D(D_k)$ and $D_k f \in D(D_l)$ for all $k, l \in \mathbb{N}$. □

From [ArDE], Proposition 2.4, one can conclude, that $BUC^1(l^p)$ is dense in $BUC(l^p)$. We shall use the same argument in order to show, that $BUC^2(l^p)$ is dense. The proof is based on the abstract version of the Mittag-Leffler theorem, see [Es] for a proof and further applications, see also [Am], V.1.1 and [ArEK]. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Theorem 4.1.5. *(Mittag-Leffler) Let (M_n, d_n) be complete metric spaces and $\Theta_n : M_{n+1} \to M_n$ continuous mappings with dense image ($n \in \mathbb{N}_0$). Let $x_0 \in M_0$, $\varepsilon > 0$. Then there exist $y_n \in M_n$ ($n \in \mathbb{N}_0$) such that*

(a) $d_0(x_0, y_0) < \varepsilon$

(b) $\Theta_n y_{n+1} = y_n$.

Calling a sequence $(y_n)_{n\in\mathbb{N}_0}$ with $y_n \in M_n$ **projective** if $\Theta_n y_{n+1} = y_n$, and calling y_0 the **final point** of such a sequence, the theorem says that the set of all final points of projective sequences is dense in M_0.

Proposition 4.1.6. *The space $BUC^2(l^p)$ is dense in $BUC(l^p)$.*

Proof. Let $M_0 = BUC(l^p)$ with the supremum norm $\|f\| = \sup_{x\in l^p} |f(x)|$, and for $n \in \mathbb{N}$, let $M_n = \{f \in \bigcap_{k=1}^n D(D_k) : D_j f \in \bigcap_{k=1}^n D(D_k), 1 \le j \le n\}$ with the norm $\|f\|_n = \sum_{l,k=1}^n \|D_l D_k f\| + \sum_{k=1}^n \|D_k f\| + \|f\|$. Then $(M_n, \|.\|_n)$ is a Banach space, because the D_k are closed operators. Moreover, the injection $\Theta_n : M_{n+1} \to M_n$ is continuous. Let $x \in M_n$. Since the operators commute, one has $R(\lambda, D_{n+1})^2 x \in M_{n+1}$ and $\lim_{\lambda\to\infty} \lambda^2 R(\lambda, D_{n+1})^2 x \to x$ in M_n. Thus, Θ_n has dense image for all $n \in \mathbb{N}_0$. Here, every projective sequence is constant and and final points are the same as the elements of $BUC^2(l^p)$. Thus, the Mittag-Leffler theorem says that $BUC^2(l^p)$ is dense in $BUC(l^p)$. □

Remark 4.1.7. The approach to infinite dimensional evolution equations in [CP] requires Fréchet differentiability. Then for a Hilbert space H the space of all functions with continuous second derivative $C^2(H)$ is dense in $BUC(H)$, see [Ku]. However, the subspace of $C^2(H)$ consisting of all functions possessing bounded uniformly continuous second derivatives fails to be dense in $BUC(H)$, see [NS].

Let $f \in BUC(l^p)$ and fix $x, y \in l^p$, then $t \mapsto f(y+tx)$ defines a bounded uniformly continuous function from $\mathbb{R}$ to $\mathbb{C}$. This type of function plays an important role in the sequel. In particular, we will use the following differentiability result.

Lemma 4.1.8. *Let $1 \le p < \infty$ and $b \in l^p$. Assume, that $f \in BUC^1(l^p)$ and $\sum_j |b_j| \, \|D_j f\| < \infty$. Then for all $y \in l^p$*

$$\lim_{t\to 0} \frac{f(y+tb) - f(y)}{t} = \sum_j b_j D_j f(y).$$

Proof. Since $\sum_j |b_j| \, \|D_j f\| < \infty$, the map $s \mapsto \sum_j b_j D_j f(y+sb)$, is continuous. Thus, it suffices to show, that $f(y+tb) - f(y) = \int_0^t \sum_j b_j D_j f(y+sb)\, ds$ for all $t \in [0, \tau]$ and some $\tau > 0$.
First observe, that for $f \in BUC^1(l^p)$ and all $n \in \mathbb{N}$ and $y \in l^p$ fix, the map

$$\mathbb{R}^n \ni x \mapsto f(x_1, \dots, x_n, y_{n+1}, y_{n+2}, \dots) \in \mathbb{R}$$

is totally differentiable, because all partial derivatives exist and are continuous. Hence, if we set $b^n := (b_1, \dots, b_n, 0, \dots) \in l^p$, we obtain for $h \in \mathbb{R}$,

$$f(y + (t+h)b^n) - f(y + tb^n) = \sum_{j=1}^n D_j f(y + tb^n)\, h b_j + o(h).$$

Thus, $t \mapsto f(y + tb^n)$ is differentiable and $\frac{d}{dt} f(y+tb^n) = \sum_{j=1}^n b_j D_j f(y+tb^n)$, which is continuous, and we obtain $f(y+tb^n) - f(y) = \int_0^t \sum_{j=1}^n b_j D_j f(y+sb^n)\, ds$ for all $t \in \mathbb{R}$. From now on, let us fix $\tau \in \mathbb{R}$ and take $t \in [0, \tau]$.

Moreover, $f(y+tb^n) \to f(y+tb)$ as $n \to \infty$ uniformly in $t \in [0,\tau]$. Indeed, as $f \in BUC(l^p)$, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|x-y\|_{l^p} < \delta$ implies $|f(x) - f(y)| < \varepsilon$. Thus, it is sufficient to show, that for every $\delta > 0$ there exists an $n \in \mathbb{N}$ independent of $t \in [0,\tau]$, such that $\|tb^n - tb\| = t\left(\sum_{j=n+1}^{\infty} |b_j|^p\right)^{1/p} < \delta$, which is possible, since t is bounded and $b \in l^p$.
Finally, $\sum_{j=1}^{n} b_j D_j f(y+sb^n) \to \sum_{j=1}^{\infty} b_j D_j f(y+sb)$ as $n \to \infty$ uniformly in $s \in [0,t]$, because

$$
\begin{aligned}
&\Big|\sum_{j=1}^{\infty} b_j D_j f(y+sb) - \sum_{j=1}^{n} b_j D_j f(y+sb^n)\Big| \\
&\quad \le \Big|\sum_{j=1}^{\infty} b_j D_j f(y+sb) - \sum_{j=1}^{\infty} b_j D_j f(y+sb^n)\Big| \\
&\qquad + \Big|\sum_{j=1}^{\infty} b_j D_j f(y+sb^n) - \sum_{j=1}^{n} b_j D_j f(y+sb^n)\Big| \\
&\quad \le \Big|\sum_{j=1}^{\infty} b_j D_j f(y+sb) - \sum_{j=1}^{\infty} b_j D_j f(y+sb^n)\Big| + \Big|\sum_{j=n+1}^{\infty} b_j D_j f(y+sb^n)\Big|.
\end{aligned}
$$

The first term converges to 0 uniformly in s with the same argument as before for f, because $\sum_{j=1}^{\infty} b_j D_j f \in BUC(l^p)$. For the second, $\sum_{j=1}^{\infty} |b_j| \, \|D_j f\| < \infty$ implies convergence to 0 uniformly in s.
Now the uniform convergence allows to interchange integration and limit. Thus

$$
\begin{aligned}
f(y+tb) - f(y) &= \lim_{n\to\infty} f(y+tb^n) - f(y) \\
&= \lim_{n\to\infty} \int_0^t \sum_{j=1}^{n} b_j D_j f(y+sb^n) = \int_0^t \sum_{j=1}^{\infty} b_j D_j f(y+sb),
\end{aligned}
$$

which concludes the proof. □

Corollary 4.1.9. If $\varphi : \mathbb{R} \to \mathbb{R}$ is differentiable, then $t \mapsto f(y+\varphi(t)b)$ is differentiable with derivative $\sum_{j=1}^{\infty} b_j D_j f(y+\varphi(t)b)\varphi'(t)$. In particular one has $f(y+\sqrt{t}\,b) - f(y) = \int_0^t \sum_{j=1}^{\infty} \frac{b_j}{2\sqrt{s}} D_j f(y+\sqrt{s}\,b)$.

Proof. By Lemma 4.1.8, $t \mapsto f(y+tb)$ is differentiable with derivative given by $\sum_{j=1}^{\infty} b_j D_j f(y+tb)$ and we apply the chain rule to $t \mapsto f(y+\varphi(t)b)$. In particular $\varphi(t) = \sqrt{t}$ is differentiable for $t > 0$ and hence for all $t \ge \tau > 0$

$$
f(y+\sqrt{t}\,b) - f(y+\sqrt{\tau}\,b) = \int_\tau^t \sum_{j=1}^{\infty} \frac{b_j}{2\sqrt{s}} D_j f(y+\sqrt{s}\,b),
$$

and by letting $\tau \to 0$, the claim follows. □

We recall a simple result of semigroup theory. We refer to [Na1], A-II 1.13 or [Da1], Theorem 2.31 for the easy proof.

Proposition 4.1.10. *Let B be the generator of an isometric C_0-group $(T(t))_{t\in\mathbb{R}}$ on a Banach space E. Then B^2 generates a contractive C_0-semigroup S given by*

$$S(t)f = \frac{1}{(4\pi t)^{1/2}} \int_{\mathbb{R}} \mathrm{e}^{-s^2/4t}\, T(s)f\, ds \quad (f \in E).$$

Remark 4.1.11. Since $B^2 = (-B)^2$ and $-B$ generates the isometric C_0-group $(T(-t))_{t\in\mathbb{R}}$, we could write equivalently $S(t)f = \frac{1}{(4\pi t)^{1/2}} \int_{\mathbb{R}} \mathrm{e}^{-s^2/4t}\, T(-s)f\, ds$ for $f \in E$.

Recall that D_k is the generator of the isometric shift group T_k on the Banach space $BUC(l^p)$. Hence D_k^2 generates a C_0-semigroup of contractions given by

$$(S_k(t)f)(x) = \frac{1}{(4\pi t)^{1/2}} \int_{\mathbb{R}} \mathrm{e}^{-s^2/4t}\, f(x - se_k)\, ds$$

for $f \in BUC(l^p)$.
Moreover, the semigroups S_k commute, because the T_k commute. Thus, for all $n \in \mathbb{N}$, $(\prod_{k=1}^n S_k(t))_{t\geq 0}$ is a C_0-semigroup with generator $\overline{D_1^2 + \cdots + D_n^2}$, the closure of the operator $D_1^2 + \cdots + D_n^2$ defined on its domain $\bigcap_{k=1}^n D(D_k^2)$, see [Na1], A-I 3.8. Note that for $f \in BUC(l^p)$,

$$\Big(\prod_{k=1}^n S_k(t)f\Big)(x) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} \mathrm{e}^{-|s|^2/4t}\, f(x - \sum_{k=1}^n s_k e_k)\, ds.$$

The fact that $\frac{\mathrm{e}^{-|s|^2/4t}}{(4\pi t)^{1/2}}$ is the density of the Gaussian (or normal) distribution with mean zero (and variance $\sqrt{2t}$ in the one dimensional case) motivates the following definition.

Definition 4.1.12. On the Banach space $BUC(\mathbb{R})$ the semigroup S given by $(S(t)f)(x) = \frac{1}{(4\pi t)^{1/2}} \int_{\mathbb{R}} \mathrm{e}^{-s^2/4t}\, f(x-s)\, ds$ for $f \in BUC(\mathbb{R})$ is called the **Gaussian semigroup**.
On $BUC(l^p)$, we call $(S_k(t)f)(x) = \frac{1}{(4\pi t)^{1/2}} \int_{\mathbb{R}} \mathrm{e}^{-s^2/4t}\, f(x - se_k)\, ds$ the **one dimensional Gaussian semigroup in direction** k and $(\prod_{k=1}^n S_k(t))_{t\geq 0}$ the **n-dimensional Gaussian semigroup**.

The subsequent paragraph is devoted to a generalization of Gaussian semigroups in infinite dimensions.

4.1.2 Gaussian Measures and Semigroups in Infinite Dimensions

The approach to the infinite dimensional heat equation (HE_∞) in [AbEK] uses Gaussian semigroups. We will summarize the definitions and properties used in the sequel. Throughout this paragraph, let E be a separable Banach space.
The following definition is taken from the monograph [Bo]. There the measure is defined on the smallest σ-algebra, for which all continuous linear functionals on E are measurable. This, however, coincides for a separable Banach space E with the Borel σ-algebra $\mathcal{B}(E)$, see [Bo], Theorem A.3.7 or [VTC], Chapter I.

Definition 4.1.13. (i) A Borel probability measure μ on $\mathbb{R}$ is called **Gaussian measure** if it is either the Dirac measure δ_a at a point a or has density

$$p(., a, \sigma^2) : t \mapsto \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(t-a)^2}{2\sigma^2}\right)$$

with respect to Lebesgue measure. In the latter case μ is called **non-degenerate**. If the mean $a = 0$, then μ is called **centered** or **symmetric**, if additionally the variance $\sigma^2 = 1$, then μ is called **standard**.

(ii) Let E be a separable Banach space, and $\mathcal{B}$ its Borel σ-algebra. A probability measure μ defined on $(E, \mathcal{B})$ is called **Gaussian** if for any real valued $f \in E'$, the induced measure $\mu \circ f^{-1}$ on $\mathbb{R}$ is Gaussian. The measure μ is called **centered** or **symmetric**, if all measures $\mu \circ f^{-1}$, $f \in E'$, are centered.

A centered Gaussian measure μ is characterized by the following invariance property. We refer to [Bo], Proposition 2.2.10 for the proof. Here $\mu \otimes \mu$ denotes the product measure on $E \times E$, which is again Gaussian.

Proposition 4.1.14. *A probability measure μ on $(E, \mathcal{B})$ is centered Gaussian if and only if for every $\varphi \in \mathbb{R}$, the image of the measure $\mu \otimes \mu$ under the mapping*

$$E \times E \to E, (x, y) \mapsto \sin\varphi x + \cos\varphi y$$

coincides with μ.

We will also need the following theorem, which is due to Fernique, see [Fe] or [AbEK] with an alternative proof.

Theorem 4.1.15. *(Fernique) Let μ be a Gaussian measure on a Banach space E. Then there exist $\tau > 0$, $M \geq 1$ and $\omega > 0$ such that*

$$\mu(\|x\| > t) \leq M\, \mathrm{e}^{-\omega t^2},\ t > \tau.$$

Corollary 4.1.16. Let μ be a Gaussian measure on a Banach space E. Then $\int_E \|x\|^r d\mu(x) < \infty$ for each $r > 0$.

Proof. Let $R(k) = \{x \in E : k < \|x\| \leq k+1\}$, $k \in \mathbb{N}$. Then $E = \{0\} \cup \bigcup_k R(k)$ and

$$\begin{aligned}\int_E \|x\|^r d\mu(x) &\leq 1 + \sum_{k=1}^{\infty} \int_{R(k)} \|x\|^r d\mu(x) \leq 1 + \sum_{k=1}^{\infty} (k+1)^r \mu(\|x\| > k) \\ &\leq 1 + \sum_{k=1}^{\infty} (k+1)^r M \, \mathrm{e}^{-\omega k^2} < \infty.\end{aligned}$$

□

Observe that on the space $BUC(l^p)$ one has for the one dimensional Gaussian semigroup in direction k,

$$\begin{aligned}(S_k(t)f)(x) &= \frac{1}{(4\pi t)^{1/2}} \int_{\mathbb{R}} \mathrm{e}^{-s^2/4t} f(x - se_k) \, ds \\ &= \frac{1}{(4\pi)^{1/2}} \int_{\mathbb{R}} \mathrm{e}^{-s^2/4} f(x - \sqrt{t} s e_k) \, ds \\ &= \int_{\mathbb{R}} f(x - \sqrt{t} s e_k) \frac{\mathrm{e}^{-s^2/4}}{(4\pi)^{1/2}} \, ds \\ &= \int_{\mathbb{R}} f(x - \sqrt{t} s e_k) \, d\mu(s),\end{aligned}$$

where μ is a centered Gaussian measure on $\mathbb{R}$ given by the density $s \mapsto \frac{\mathrm{e}^{-s^2/4}}{(4\pi)^{1/2}}$. The great advantage of this writing is, that the Gaussian measure μ is independent of the parameter t. Analogously, one obtains

$$\Big(\prod_{k=1}^{n} S_k(t) f\Big)(x) = \int_{\mathbb{R}^n} f(x - \sqrt{t} \sum_{k=1}^{n} s_k e_k) \, d\mu_n(s),$$

where μ_n is a centered Gaussian measure on $\mathbb{R}^n$ given by the density $s \mapsto \frac{\mathrm{e}^{-|s|^2/4}}{(4\pi)^{n/2}}$, again independent of t.

Remark 4.1.17. We can extend μ_n to a centered Gaussian measure $\tilde{\mu}_n$ on l^p, if we define $\tilde{\mu}_n := \mu_n \otimes \delta_0^{n+1}$, where δ_0^{n+1} denotes the infinite product of Dirac measures at the point 0 for the coordinates x_k for $k \geq n+1$. Then for $A \subset l^p$ we have $\tilde{\mu}_n(A) := \mu_n(\{(x_1, \dots, x_n) \in \mathbb{R}^n : x = (x_1, \dots, x_n, 0, \dots) \in A\})$. Thus, $(\prod_{k=1}^{n} S_k(t)f)(x) = \int_{\mathbb{R}^n} f(x - \sqrt{t} \sum_{k=1}^{n} s_k e_k) \, d\mu_n(s) = \int_{l^p} f(x - \sqrt{t} s) \, d\tilde{\mu}_n(s)$. In the following, we shall only use the notation μ_n for either the measure on $\mathbb{R}^n$ or the one on l^p.

Therefore we wish to know, if we can extend this notion to general centered Gaussian measures and still obtain a C_0-semigroup.

Proposition 4.1.18. *Let μ be a centered Gaussian measure on E. For $t \geq 0$ and $f \in BUC(E)$ define*

$$(S(t)f)(x) := \int_E f(x + \sqrt{t}\, y)\, d\mu(y).$$

Then $(S(t))_{t\geq 0}$ is a strongly continuous semigroup of operators on $BUC(E)$. Moreover, $\|S(t)\| \leq 1$ holds for all $t \geq 0$.

Proof. For every $t \geq 0$, $S(t)$ defines obviously a linear operator from $BUC(E)$ to $BUC(E)$ with $\|S(t)\| \leq 1$, because μ is a probability measure. We also have $S(0) = Id$.
Note, that for all $s, t \in \mathbb{R}$ there exists a $\varphi \in \mathbb{R}$ such that $\sin\varphi = \frac{\sqrt{t}}{\sqrt{t+s}}$ and $\cos\varphi = \frac{\sqrt{s}}{\sqrt{t+s}}$. Then

$$\begin{aligned}
(S(t)S(s)f)(x) &= S(t)\int_E f(x + \sqrt{s}\, y)\, d\mu(y) \\
&= \int_E \int_E f(x + \sqrt{t}\, z + \sqrt{s}\, y)\, d\mu(y)\, d\mu(z) \\
&= \int_{E\times E} f\left(x + \sqrt{t+s}\left(\tfrac{\sqrt{t}}{\sqrt{t+s}}\, z + \tfrac{\sqrt{s}}{\sqrt{t+s}}\, y\right)\right)\, d(\mu\otimes\mu)(y,z) \\
&= \int_E f(x + \sqrt{t+s}\,\xi)\, d\mu(\xi) \\
&= (S(t+s)f)(x),
\end{aligned}$$

where $\xi = \frac{\sqrt{t}}{\sqrt{t+s}}\, z + \frac{\sqrt{s}}{\sqrt{t+s}}\, y = \sin\varphi z + \cos\varphi y$ for an appropriate φ, which enables us to use Proposition 4.1.14. Hence $S(t)S(s) = S(t+s)$.
Finally, we have to show, that for all $f \in BUC(E)$, the map $t \mapsto S(t)f$ is continuous in 0. Let $\varepsilon > 0$. Since f is uniformly continuous, there exists a $\delta > 0$ such that $\|x_1 - x_2\|_E < \delta$ implies $|f(x_1) - f(x_2)| < \frac{\varepsilon}{2}$.
Let τ, M and ω be the constants due to the Theorem of Fernique 4.1.15. Choose $R > \tau$, such that $2\|f\|_\infty M\, \mathrm{e}^{-\omega R^2} < \frac{\varepsilon}{2}$.
Then for all $0 \leq t < (\frac{\delta}{R})^2$

$$\begin{aligned}
\|S(t)f - f\|_\infty &= \sup_{x\in E}\left|\int_E [f(x+\sqrt{t}\, y) - f(x)]\, d\mu(y)\right| \\
&\leq \sup_{x\in E}\Big[\int_{\|y\|\leq R} |f(x+\sqrt{t}\, y) - f(x)|\, d\mu(y) \\
&\qquad + \int_{\|y\|>R} |f(x+\sqrt{t}\, y) - f(x)|\, d\mu(y)\Big] \\
&< \frac{\varepsilon}{2} + 2\|f\|_\infty M\, \mathrm{e}^{-\omega R^2} < \varepsilon.
\end{aligned}$$

For the first estimate we use that μ is a probability measure and the fact that $\|x + \sqrt{t}\,y - x\| \le \sqrt{t}\,\|y\| < \delta$ for $\|y\| \le R$ and hence $|f(x + \sqrt{t}\,y) - f(x)| < \frac{\varepsilon}{2}$ uniformly in $x \in E$. For the second, we have $|f(x + \sqrt{t}\,y) - f(x)| \le 2\|f\|$ for all $x \in E$, and $\mu(\|y\| > R) \le M\,\mathrm{e}^{-\omega R^2}$ by Fernique's Theorem 4.1.15.
Hence $t \mapsto S(t)f$ is continuous and $(S(t))_{t\ge 0}$ a strongly continuous semigroup. □

Since the Gaussian semigroups which occurred so far admit a Gaussian measure μ_n on $\mathbb{R}^n$ and a representation $(S(t)f)(x) = \int_{l^p} f(x+\sqrt{t}\,y)\,d\mu_n(y)$ for all functions $f \in BUC(l^p)$, we get in no conflict with the following definition.

Definition 4.1.19. We call a family $S(t)$ of operators on $BUC(E)$ a **Gaussian semigroup**, if there exists a centered Gaussian measure μ and $S(t)$ admits a representation

$$(S(t)f)(x) = \int_E f(x + \sqrt{t}\,y)\,d\mu(y).$$

Note that by the above proposition, a Gaussian semigroup is a strongly continuous semigroup of contractions.

Remark 4.1.20. Recall, that on the metric space E with two measures μ and ν such that the equality $\int_E f\,d\mu = \int_E f\,d\nu$ holds for all $f \in BUC(E)$, then $\mu = \nu$, see e.g. [Par], Theorem 5.9. Hence the measure μ of a Gaussian semigroup is unique.

In the sequel, we shall be concerned with Gaussian measures on the separable Banach space $E = l^p$. We have the following convenient relation between a Gaussian semigroup and the generators of the shift groups on the space $BUC(l^p)$.

Lemma 4.1.21. *Let $1 \le p < \infty$ and let $S = (S(t))_{t\ge 0}$ be a Gaussian semigroup on $BUC(l^p)$. Then for all $f \in D(D_k)$ and all $\tau \ge 0$ one has $S(\tau)f \in D(D_k)$ and $D_k S(\tau)f = S(\tau)D_k f$.*

Proof. Since D_k is the generator of the shift group T_k, we have to show for all $f \in D(D_k)$ and $\tau \ge 0$ that $T_k(t)S(\tau)f - S(\tau)f = \int_0^t T_k(s)S(\tau)D_k f\,ds$ in $BUC(l^p)$. Therefore, it is sufficient to show equality pointwise, i.e. for all $x \in l^p$, we have to show that $S(\tau)f(x + te_k) - S(\tau)f(x) = \int_0^t S(\tau)D_k f(x + se_k)\,ds$. But

$$S(\tau)f(x + te_k) - S(\tau)f(x) = \int_{l^p} f(x + te_k + \sqrt{\tau}y) - f(x + \sqrt{\tau}y)\,d\mu(y)$$

and by Tonelli's Theorem

$$\begin{aligned}\int_0^t S(\tau)D_k f(x + se_k)\,ds &= \int_0^t \int_{l^p} D_k f(x + se_k + \sqrt{\tau}y)\,d\mu(y)\,ds \\ &= \int_{l^p} \int_0^t D_k f(x + se_k + \sqrt{\tau}y)\,ds\,d\mu(y).\end{aligned}$$

Hence it is sufficient to show for each $y \in l^p$ that

$$f(y + te_k) - f(y) = \int_0^t D_k f(y + se_k)\, ds,$$

which holds by Lemma 4.1.8. □

4.1.3 Semigroup Theory for the Heat Equation

The finite dimensional heat equation in $\mathbb{R}^n$ is given as

$$(HE_n) \quad \begin{cases} \frac{\partial u(t,x)}{\partial t} = \sum_{j=1}^n \frac{\partial^2 u(t,x)}{\partial x_j^2}, & t > 0 \\ u(0,.) = f & \in BUC(\mathbb{R}^n), \end{cases}$$

which we regard as the abstract Cauchy problem for $u(t) := u(t,.) \in BUC(l^p)$ associated with the operator $D_1^2 + \cdots + D_n^2$. Recall that the closure $\overline{D_1^2 + \cdots + D_n^2}$ is the generator of the n-dimensional Gaussian semigroup on $BUC(\mathbb{R}^n)$,

$$\Big(\prod_{j=1}^n S_j(t) f\Big)(x) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} \mathrm{e}^{-|s|^2/4t} f(x - \sum_{j=1}^n s_j e_j)\, ds.$$

Thus, the orbits $t \mapsto (\prod_{j=1}^n S_j(t))f$ are the unique mild solution of the abstract Cauchy problem associated with $\overline{D_1^2 + \cdots + D_n^2}$. Moreover $\bigcap_{j=1}^n D(D_j^2)$ is invariant under $\prod_{j=1}^n S_j(t)$, such that for $f \in \bigcap_{j=1}^n D(D_j^2)$, $t \mapsto (\prod_{j=1}^n S_j(t))f$ solves (HE_n) in the classical sense.
Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ and consider the generalized heat equation

$$(HE_{\lambda,n}) \quad \begin{cases} \frac{\partial u(t,x)}{\partial t} = \sum_{j=1}^n \lambda_j \frac{\partial^2 u(t,x)}{\partial x_j^2}, & t > 0 \\ u(0,.) = f & \in BUC(\mathbb{R}^n). \end{cases}$$

Then the solution is given by the orbits of the n-dimensional Gaussian semigroup with **change of speed** λ on $BUC(\mathbb{R}^n)$,

$$(S^n(t)f)(x) = \prod_{j=1}^n S_j(\lambda_j t) f(x) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} \mathrm{e}^{-|s|^2/4t} f(x - \sum_{j=1}^n \sqrt{\lambda_j} s_j e_j)\, ds.$$

The natural extension from the n-dimensional setting to an infinite dimensional situation is achieved by replacing $\mathbb{R}^n$ by one of the real sequence spaces l^p with $1 \le p < \infty$. Then we obtain the infinite dimensional heat equation

$$(HE_\infty) \quad \begin{cases} \frac{\partial u(t,x)}{\partial t} = \sum_{j\in\mathbb{N}} \lambda_j \frac{\partial^2 u(t,x)}{\partial x_j^2}, & t > 0 \\ u(0,.) = f & \in BUC(l^p). \end{cases}$$

Since the finite product provides the solution for the finite dimensional problem, the infinite product of Gaussian semigroups seems to be a good candidate to solve the infinite dimensional heat equation. Therefore the convergence of the infinite product of Gaussian semigroups was thoroughly studied for example in [CP] and [ArDE]. From the latter we took the following result. In this context the existence of a change of speed $\lambda = (\lambda_j)_{j\in\mathbb{N}}$ is essential.

Proposition 4.1.22. *Let $1 \le p < \infty$ and $\lambda = (\lambda_n)_{n\in\mathbb{N}} \in l_+^{p/2}$. Define on the Banach space $BUC(l^p)$ the n-dimensional Gaussian semigroup S^n with change of speed λ by*

$$(S^n(t)f)(x) = \prod_{j=1}^{n} S_j(\lambda_j t) f(x) = (4\pi t)^{-n/2} \int_{\mathbb{R}^n} \mathrm{e}^{-|s|^2/4t} f(x - \sum_{j=1}^{n} \sqrt{\lambda_j} s_j e_j)\, ds$$

for $f \in BUC(l^p)$. Then

$$S(t)f = \lim_{n\to\infty} S^n(t)f$$

converges uniformly on $[0,\tau]$ in $BUC(l^p)$ for every $\tau > 0$, $f \in BUC(l^p)$ and defines a strongly continuous semigroup S on $BUC(l^p)$. We call it the **heat semigroup** *of speed λ.*

Remark 4.1.23. The above result was shown in [ArDE] even for $0 < p < 1$ and weighted l^p-spaces. But in the following, we need $1 \le p < \infty$.

Lately in [AbEK], there has been developed a different approach to the semigroup solving the infinite dimensional heat equation (HE_∞).
Recall that the finite dimensional Gaussian semigroup $S^n(t) = \prod_{j=1}^n S_j(\lambda_j t)$ with change of speed λ, has a representation

$$S^n(t)f(x) = \int_{l^p} f(x + \sqrt{t}\, y)\, d\mu_n(y),$$

where the measure μ_n is given by the density $s \mapsto \prod_{j=1}^n \frac{\mathrm{e}^{-s^2/4\lambda_j}}{(4\pi\lambda_j)^{1/2}}$, hence is a centered Gaussian measure.
The question, weather the heat semigroup $S(t) = \prod_{k=1}^\infty S_j(\lambda_j t)$ is also Gaussian, was answered positively in [AbEK] as a consequence of a characterization of Gaussian semigroups.

Proposition 4.1.24. *Let $1 \le p < \infty$ and $\lambda = (\lambda_n)_{n\in\mathbb{N}} \in l^{p/2}$. The semigroup $S = (S(t))_{t\ge 0} = (\prod_{j=1}^\infty S_j(\lambda_j t))_{t\ge 0}$ on $BUC(l^p)$ is Gaussian, i.e. there exists a unique Gaussian measure μ on l^p, such that S admits a representation*

$$(S(t)f)(x) = \int_{l^p} f(x + \sqrt{t}\, y)\, d\mu(y),$$

for all $f \in BUC(l^p)$.

Definition 4.1.25. For $1 \leq p < \infty$ we call this Gaussian measure μ on l^p **associated** to the sequence $\lambda = (\lambda_n)_{n \in \mathbb{N}} \in l^{p/2}$.

Remark 4.1.26. The crucial point is indeed to show the existence of a measure. Additional properties guarantee that it is Gaussian and represents the semigroup. Note that $\Lambda_{t,x} : f \mapsto (S(t)f)(x)$ defines a positive functional on $BUC(l^p)$. If it is order continuous, then by the Daniell-Stone theorem, there exists a measure $\mu_{t,x}$ such that $\Lambda_{t,x} f = \int f \, d\mu_{t,x}$. At the end of this chapter in Section 4.4, we show this result as a consequence of the better known Riesz representation theorem.

We shall soon see that indeed, the heat semigroup provides a solution of the infinite dimensional heat equation. For that we need the following lemma, which gives some analog to partial integration with respect to a Gaussian measure.

Lemma 4.1.27. *Let μ be the Gaussian measure on l^p associated with the sequence $\lambda_i \in l^{p/2}$. Then for a function $f \in BUC^1(l^p)$, one has*

$$\int_{l^p} f(y) y_i \, d\mu(y) = 2\lambda_i \int_{l^p} \frac{\partial}{\partial y_i} f(y) \, d\mu(y).$$

Remark 4.1.28. Note that because of Corollary 4.1.16, the left hand side has a sense.

Proof. The heat semigroup of speed λ is given as the Gaussian semigroup associated with the Gaussian measure μ on l^p, as well as the infinite product of one dimensional Gaussian semigroups. Therefore

$$\begin{aligned} (S(t)f)(x) &= \int_{l^p} f(x + \sqrt{t}\, y) \, d\mu(y) \\ &= \lim_{n \to \infty} \int_{\mathbb{R}^n} f(x + \sqrt{t}\, y) \, d\mu_n(y), \end{aligned}$$

where $\mu_n(y)$ is given by the density $y \mapsto \prod_{j=1}^n \frac{e^{-y_j^2/4\lambda_j}}{(4\pi\lambda_j)^{1/2}}$. Hence, if a function g depends only on a finite number of variables, then $\int_{l^p} g(y) \, d\mu(y) = \int_{\mathbb{R}^n} g(y) \, d\mu_n(y)$. Assume first, that f depends only on a finite number of variables. Then also

$\frac{\partial}{\partial y_i} f(y)$ depends only on a finite number of variables and

$$\begin{aligned}
\int_{l^p} f(y) y_i \, d\mu(y) &= \int_{\mathbb{R}^n} f(y) y_i \, d\mu_n(y) \\
&= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} f(y) y_i \, \frac{\mathrm{e}^{-y_i^2/4\lambda_i}}{(4\pi\lambda_i)^{1/2}} dy_i \prod_{j \neq i} \left(\frac{\mathrm{e}^{-y_j^2/4\lambda_j}}{(4\pi\lambda_j)^{1/2}} dy_j \right) \\
&= \int_{\mathbb{R}^{n-1}} 2\lambda_i \int_{\mathbb{R}} \frac{\partial}{\partial y_i} f(y) \frac{\mathrm{e}^{-y_i^2/4\lambda_i}}{(4\pi\lambda_i)^{1/2}} dy_i \prod_{j \neq i} \frac{\mathrm{e}^{-y_j^2/4\lambda_j}}{(4\pi\lambda_j)^{1/2}} \\
&= 2\lambda_i \int_{\mathbb{R}^n} \frac{\partial}{\partial y_i} f(y) \, d\mu_n(y) \\
&= 2\lambda_i \int_{l^p} \frac{\partial}{\partial y_i} f(y) \, d\mu(y),
\end{aligned}$$

since

$$\begin{aligned}
&\int_{\mathbb{R}} f(y) y_i \, \frac{\mathrm{e}^{-y_i^2/4\lambda_i}}{\sqrt{4\pi\lambda_i}} \, dy_i \\
&= \left[-f(y) 2\lambda_i \frac{\mathrm{e}^{-y_i^2/4\lambda_i}}{\sqrt{4\pi\lambda_i}} \right]_{y_i=-\infty}^{\infty} + 2\lambda_i \int_{\mathbb{R}} \frac{\partial}{\partial y_i} f(y) \frac{\mathrm{e}^{-y_i^2/4\lambda_i}}{\sqrt{4\pi\lambda_i}} \, dy_i \\
&= 2\lambda_i \int_{\mathbb{R}} \frac{\partial}{\partial y_i} f(y) \frac{\mathrm{e}^{-y_i^2/4\lambda_i}}{\sqrt{4\pi\lambda_i}} \, dy_i,
\end{aligned}$$

by partial integration, and because $\mathrm{e}^{-y_i^2/4\lambda_i}$ vanishes as $|y_i| \to \infty$.
Now if f depends on infinitely many variables, define the map $\Pi_n : l^p \to l^p$ by $(x_1, \ldots, x_n, x_{n+1}, \ldots) \mapsto (x_1, \ldots, x_n, 0, \ldots)$. Then $f_n := f \circ \Pi_n \in BUC(l^p)$ and depends only on a finite number of variables. Moreover, for $i \leq n$, one has $\frac{\partial}{\partial y_i} f_n = \left(\frac{\partial}{\partial y_i} f \right) \circ \Pi_n$, and for $i > n$, $\frac{\partial}{\partial y_i} f_n = 0$. Thus $f_n \in BUC^1(l^p)$ and also $\frac{\partial}{\partial y_i} f_n$ depends only on a finite number of variables. Hence by the above considerations,

$$\int_{l^p} f_n(y) y_i \, d\mu(y) = 2\lambda_i \int_{l^p} \frac{\partial}{\partial y_i} f_n(y) \, d\mu(y).$$

Now $f_n(y) \to f(y)$ as $n \to \infty$ and $\|f_n(y) y_i\| \leq \|f\| \, |y_i| \leq M \|y\|$, which is integrable by Fernique's theorem. Hence by Lebesgue's dominated convergence theorem

$$\int_{l^p} f_n(y) y_i \, d\mu(y) \to \int_{l^p} f(y) y_i \, d\mu(y).$$

On the other hand $\frac{\partial}{\partial y_i} f_n(y) = \left(\frac{\partial}{\partial y_i} f \right) \circ \Pi_n(y) \to \frac{\partial}{\partial y_i} f(y)$ and also one has $\left\| \frac{\partial}{\partial y_i} f_n \right\| \leq \left\| \left(\frac{\partial}{\partial y_i} f \right) \circ \Pi_n \right\| \leq \tilde{M}$, hence again by Lebesgue's dominated convergence theorem

$$\int_{l^p} \frac{\partial}{\partial y_i} f_n(y) \, d\mu(y) \to \int_{l^p} \frac{\partial}{\partial y_i} f(y) y_i \, d\mu(y),$$

and the claim follows. □

Corollary 4.1.29. Under the assumptions of Lemma 4.1.27, for $\alpha \in \mathbb{R}$,

$$\int_{l^p} \frac{1}{\alpha} f(\alpha y) y_i \, d\mu(y) = 2\lambda_i \int_{l^p} D_i f(\alpha y) \, d\mu(y).$$

Now we are given all the necessary results to establish a relation between the heat semigroup $S(t) = \prod_{j=1}^{\infty} S_j(\lambda_j t)$ on $BUC(l^p)$ and the infinite dimensional heat equation (HE_∞).

Theorem 4.1.30. *The generator* $\mathfrak{B}$ *of the heat semigroup* $S(t) = \prod_{j=1}^{\infty} S_j(\lambda_j t)$ *is an extension of the operator*

$$\begin{aligned} \mathcal{B} &= \sum_{j\in\mathbb{N}} \lambda_j D_j^2, \\ D(\mathcal{B}) &= \Big\{ u \in BUC^1(l^p) : D_j u \in D(D_j) \text{ and } \sum_{j\in\mathbb{N}} |\lambda_j| \, \|D_j^2 u\| < \infty \Big\}. \end{aligned}$$

Proof. We have to show for every $f \in D(\mathcal{B})$ that $f \in D(\mathfrak{B})$ and $\mathcal{B}f = \mathfrak{B}f$, which is equivalent to $S(t)f - f = \int_0^t S(s)\mathcal{B}f \, ds$. Since both terms are elements of $BUC(l^p)$, it suffices to show $S(t)f(x) - f(x) = \int_0^t S(s)\mathcal{B}f(x) \, ds$ for all $x \in l^p$. Since S is a Gaussian semigroup, one has $S(t)f(x) = \int_{l^p} f(x + \sqrt{t}y) \, d\mu(y)$ for all $f \in BUC(l^p)$, where μ is the Gaussian measure associated with $\lambda = (\lambda_j)_{j\in\mathbb{N}}$.
Let as before $\Pi_n : l^p \to l^p$, $(x_1, \ldots, x_n, x_{n+1}, \ldots) \mapsto (x_1, \ldots, x_n, 0, \ldots)$ and let $f_n = f \circ \Pi_n$. Then for $j \le n$, $D_j f_n = (D_j f) \circ \Pi_n$, and for $j > n$, $D_j f_n = 0$. Thus, $f_n \in D(\mathcal{B})$ and by Corollary 4.1.29,

$$\begin{aligned} S(s)\mathcal{B}f_n(x) &= \int_{l^p} \mathcal{B}f_n(x + \sqrt{s}\, y) \, d\mu(y) = \int_{l^p} \sum_{j\in\mathbb{N}} \lambda_j D_j^2 f_n(x + \sqrt{s}\, y) \, d\mu(y) \\ &= \int_{l^p} \sum_{j=1}^{n} \lambda_j D_j^2 f_n(x + \sqrt{s}\, y) \, d\mu(y) = \sum_{j=1}^{n} \int_{l^p} \lambda_j D_j (D_j f_n)(x + \sqrt{s}\, y) \, d\mu(y) \\ &= \sum_{j=1}^{n} \int_{l^p} \frac{y_j}{2\sqrt{s}} (D_j f_n)(x + \sqrt{s}\, y) \, d\mu(y) = \int_{l^p} \sum_{j=1}^{n} \frac{y_j}{2\sqrt{s}} (D_j f_n)(x + \sqrt{s}\, y) \, d\mu(y). \end{aligned}$$

Hence, by Tonelli's Theorem and Corollary 4.1.9,

$$\begin{aligned} \int_0^t S(s)\mathcal{B}f_n(x) \, ds &= \int_0^t \int_{l^p} \sum_{j=1}^{n} \frac{y_j}{2\sqrt{s}} (D_j f_n)(x + \sqrt{s}\, y) \, d\mu(y) \, ds \\ &= \int_{l^p} \int_0^t \sum_{j=1}^{n} \frac{y_j}{2\sqrt{s}} (D_j f_n)(x + \sqrt{s}\, y) \, ds \, d\mu(y) = \int_{l^p} f_n(x + \sqrt{t}\, y) - f_n(x) \, d\mu(y) \\ &= (S(t)f_n)(x) - f_n(x). \end{aligned} \tag{4.1}$$

Now $f_n(x) \to f(x)$ as $n \to \infty$ and $\|f_n\| \le \|f\|$. Hence by Lebesgue's dominated convergence theorem

$$(S(t)f_n)(x) = \int_{l^p} f_n(x + \sqrt{t}\,y)\,d\mu(y) \to \int_{l^p} f(x + \sqrt{t}\,y)\,d\mu(y) = (S(t)f)(x).$$

Moreover,

$$\begin{aligned}
|\mathcal{B}f(x) - \mathcal{B}f_n(x)| &= |\sum_{j\in\mathbb{N}} \lambda_j D_j^2 f(x) - \sum_{j=1}^{n} \lambda_j D_j^2 f_n(x)| \\
&= |\sum_{j\in\mathbb{N}} \lambda_j D_j^2 f(x) - \sum_{j=1}^{n} \lambda_j (D_j^2 f) \circ \Pi_n(x)| \\
&\le |\sum_{j\in\mathbb{N}} \lambda_j D_j^2 f(x) - \sum_{j\in\mathbb{N}} \lambda_j (D_j^2 f) \circ \Pi_n(x)| \\
&\quad + |\sum_{j=1}^{\infty} \lambda_j (D_j^2 f) \circ \Pi_n(x) - \sum_{j=1}^{n} \lambda_j (D_j^2 f) \circ \Pi_n(x)|,
\end{aligned}$$

where the first term converges to 0 as $n \to \infty$, because for $f \in D(\mathcal{B})$ one has $\sum_{j\in\mathbb{N}} \lambda_j D_j^2 f \in BUC(l^p)$, and additionally for the second term one has the estimate $|\sum_{j=n+1}^{\infty} \lambda_j (D_j^2 f) \circ \Pi_n(x)| \le \sum_{j=n+1}^{\infty} |\lambda_j| \|D_j^2 f\|$, which also vanishes as $n \to \infty$. Therefore $\mathcal{B}f_n(x) \to \mathcal{B}f(x)$ for all $x \in l^p$ and consequently as before $S(s)\mathcal{B}f_n(x) \to S(s)\mathcal{B}f(x)$ for all $s \ge 0$. Once more we apply Lebesgue's dominated convergence theorem to obtain $\int_0^t S(s)\mathcal{B}f_n(x)\,ds \to \int_0^t S(s)\mathcal{B}f(x)\,ds$. Uniqueness of the limit gives $S(t)f(x) - f(x) = \int_0^t S(s)\mathcal{B}f(x)\,ds$, hence $f \in D(\mathfrak{B})$ and $\mathfrak{B}f = \mathcal{B}f$. □

Remark 4.1.31. Note that one could prove (4.1) also using the definition of the heat semigroup given by $S(t)f = \lim_{n\to\infty} S^n(t)f$ for all $f \in BUC(l^p)$, where $S^n(t) = \prod_{j=1}^{n} S_j(\lambda_j t)$ is the n-dimensional Gaussian semigroup generated by the closure of the operator $\mathcal{B}^n := \sum_{j=1}^{n} \lambda_j D_j^2$. Observe that for all $f \in BUC(l^p)$ one has $S(t)f_n = (S^n(t)f)_n$ and, for $f \in D(\mathcal{B})$ one has $f_n \in D(\mathcal{B})$, $f \in D(\mathcal{B}^n)$ and $\mathcal{B}f_n = (\mathcal{B}^n f)_n$. Thus,

$$\begin{aligned}
S(t)f_n - f_n &= (S^n(t)f)_n - f_n = (S^n(t)f - f)_n = \Big(\int_0^t S^n(s)\mathcal{B}^n f\,ds\Big)_n \\
&= \int_0^t (S^n(s)\mathcal{B}^n f)_n\,ds = \int_0^t S(s)(\mathcal{B}^n f)_n\,ds = \int_0^t S(s)\mathcal{B}f_n\,ds.
\end{aligned}$$

As a consequence we obtain solutions of the heat equation

$$(HE_\infty)\quad \begin{cases} \frac{\partial u(t,x)}{\partial t} = \sum_{j\in\mathbb{N}} \lambda_j \frac{\partial^2 u(t,x)}{\partial x_j^2}, & t > 0 \\ u(0,.) = f \quad \in BUC(l^p) & (1 \le p < \infty). \end{cases}$$

Definition 4.1.32. A function $u : [0,\infty) \times l^p \to \mathbb{C}, (t,x) \mapsto u(t,x)$ is called a **solution** of (HE_∞), if for $t > 0$ the function $u(t) := u(t,.)$ has its values in $D(\mathcal{B})$ and is differentiable as a function with values in $BUC(l^p)$, such that (HE_∞) is satisfied.

Proposition 4.1.33. *Let $1 \le p < \infty$ and let $S(t) = \prod_{j=1}^{\infty} S_j(\lambda_j t)$ be the heat semigroup on $BUC(l^p)$. If $f \in D(\mathcal{B})$ then $(t,x) \mapsto u(t,x) := S(t)f(x)$ is the unique solution of (HE_∞).*

Proof. For every $f \in BUC(l^p)$ the orbit $u(t) := S(t)f$ is the unique mild solution of the abstract Cauchy problem associated with the operator $\mathfrak{B}$, which is an extension of $\mathcal{B}$. Hence it suffices to show, that the orbit $u(t) = S(t)f \in D(\mathcal{B})$ for every $f \in D(\mathcal{B})$. If $f \in D(\mathcal{B})$, then $f \in D(D_k)$ and also $D_k f \in D(D_k)$ for all $k \in \mathbb{N}$. Therefore applying Lemma 4.1.21 twice yields $S(t)f \in D(D_k)$ for all $k \in \mathbb{N}$ and $D_k S(t)f \in D(D_k)$. Moreover $D_k^2 S(t)f = S(t)D_k^2 f$, and one gets $\sum_{j\in\mathbb{N}} |\lambda_j| \, \|D_j^2 S(t)f\| = \sum_{j\in\mathbb{N}} |\lambda_j| \, \|S(t)D_j^2 f\| \le \sum_{j\in\mathbb{N}} |\lambda_j| \, \|D_j^2 f\| < \infty$. Thus $S(t)f \in D(\mathcal{B})$. □

4.2 Second Order Differential Equations

The previous section was devoted to the infinite dimensional heat equation. Now we want to allow also mixed second order derivatives, but no lower order terms. We consider the problem

$$(P) \quad \begin{cases} \frac{\partial u(t,x)}{\partial t} = \sum_{i,j\in\mathbb{N}} a_{ij} \frac{\partial^2 u(t,x)}{\partial x_i \partial x_j}, & t > 0 \\ u(0,.) = f \quad \in BUC(l^p), \end{cases}$$

with $a_{ij} \in \mathbb{R}$.

Definition 4.2.1. A function $u : [0,\infty) \times l^p \to \mathbb{C}, (t,x) \mapsto u(t,x)$ is called a **solution** of problem (P) if for all $t > 0$, the function $u(t) := u(t,.)$ has its values in $BUC^2(l^p)$ and is differentiable as a function with values in $BUC(l^p)$, such that (P) is satisfied with absolute convergence of the sum in $BUC(l^p)$.

In order to examine this problem and its solutions, we want to introduce some notations. Let the operator $\boldsymbol{A}$ be defined by

$$\begin{aligned} D(\boldsymbol{A}) &= \Big\{ u \in BUC^2(l^p) : \sum_{i,j\in\mathbb{N}} |a_{ij}| \Big\| \frac{\partial^2 u(t,x)}{\partial x_i \partial x_j} \Big\| < \infty \Big\} \\ \boldsymbol{A}u &= \sum_{i,j\in\mathbb{N}} a_{ij} \frac{\partial^2 u(t,x)}{\partial x_i \partial x_j}. \end{aligned}$$

Then $\boldsymbol{A}$ is a linear operator on $BUC(l^p)$. Hence the Problem (P) is the abstract Cauchy problem associated with $(\boldsymbol{A}, D(\boldsymbol{A}))$.

Definition 4.2.2. We say, that the Problem (P) is **well-posed**, if the operator $(\boldsymbol{A}, D(\boldsymbol{A}))$ is closable and its closure $(\bar{\boldsymbol{A}}, D(\bar{\boldsymbol{A}}))$ is the generator of a strongly continuous semigroup $(G(t))_{t\geq 0}$.

If the Problem (P) is well-posed, then the the orbits $u(t) = G(t)f$, $f \in BUC(l^p)$ (resp. $f \in D(\bar{\boldsymbol{A}})$) are the unique mild (resp. classical) solutions of the abstract Cauchy problem associated with $(\bar{\boldsymbol{A}}, D(\bar{\boldsymbol{A}}))$. See also [EN], Section II.6, for further details on the concept of well-posedness.
However, under certain assumptions we get solutions of the Problem (P) in the sense of Definition 4.2.1 whenever $f \in D(A)$.

Lemma 4.2.3. *Let the Problem (P) be well-posed and assume that the semigroup $(G(t))_{t\geq 0}$ generated by $(\bar{\boldsymbol{A}}, D(\bar{\boldsymbol{A}}))$ is Gaussian. Then for every $f \in D(A)$ the function $(t, x) \mapsto u(t, x) := G(t)f(x)$ is the unique solution of Problem (P) in the sense of Definition 4.2.1.*

Proof. For every $f \in BUC(l^p)$ the orbit $u(t) := G(t)f$ is the unique mild solution of the abstract Cauchy problem associated with $(\bar{\boldsymbol{A}}, D(\bar{\boldsymbol{A}}))$. Therefore, we only have to show, that $G(t)f \in D(\boldsymbol{A})$, whenever $f \in D(\boldsymbol{A}) \subset BUC^2(l^p)$. Applying Lemma 4.1.21, we get $G(t)f \in BUC^2(l^p)$ and $D_iD_jG(t)f = G(t)D_iD_jf$. Moreover,

$$\sum_{i,j} |a_{ij}|\,\|D_iD_jG(t)f\| = \sum_{i,j} |a_{ij}|\,\|G(t)D_iD_jf\| \leq \sum_{i,j} |a_{ij}|\,\|D_iD_jf\| < \infty,$$

since $f \in D(\boldsymbol{A})$, and the claim follows. □

Our aim is to characterize well-posedness with respect to the coefficients a_{ij}. We will first find a necessary condition and then sufficient conditions for well-posedness.

4.2.1 Necessary Condition for Well-Posedness

Assume, that the Problem (P) is well-posed. Further suppose, that the semigroup generated by $(\bar{\boldsymbol{A}}, D(\bar{\boldsymbol{A}}))$ is Gaussian. Although this is a restriction, under this additional assumption we know that the orbits of the semigroup define solutions of the Problem (P). Moreover, it enables us to deduce a property of the coefficients a_{ij}.

Proposition 4.2.4. *Let $1 \leq p < \infty$. Assume that the Problem (P) is well-posed and that the semigroup $G(t)$ generated by $(\bar{\boldsymbol{A}}, D(\bar{\boldsymbol{A}}))$ is a Gaussian semigroup on $BUC(l^p)$. Then the a_{ij} satisfy $a_{kk} \geq 0$ for all $k \in \mathbb{N}$ and $\sum_{k\in\mathbb{N}} a_{kk}^{(p/2)} < \infty$, for $p \geq 2$ and $\sum_{k\in\mathbb{N}} a_{kk}^{p} < \infty$, for $p < 2$.*

Proof. Since $G(t)$ is a Gaussian semigroup on $BUC(l^p)$, there exists a Gaussian measure μ on l^p such that for all $f \in BUC(l^p)$ one has the representation $G(t)f(x) = \int_{l^p} f(x+\sqrt{t}y)d\mu(y)$.
For the given initial function $f(x) = \mathrm{e}^{ix_k}$, which obviously belongs to $D(\boldsymbol{A})$, we can calculate the solution in two different ways.
On the one hand the solution is given as the orbit of the Gaussian semigroup and we obtain

$$\begin{aligned} u(t,x) &= G(t)f(x) = \int_{l^p} f(x+\sqrt{t}\,y)d\mu(y) = \int_{l^p} \mathrm{e}^{i(x_k+\sqrt{t}\,y_k)}\,d\mu(y) \\ &= \mathrm{e}^{ix_k}\int_{l^p} \mathrm{e}^{i\sqrt{t}\,y_k}\,d\mu(y). \end{aligned}$$

On the other hand for the initial function $f(x) = \mathrm{e}^{ix_k}$ the Problem (P) reduces to a one dimensional problem

$$(1-\mathrm{dim}) \qquad \begin{cases} \frac{\partial u(t,x_k)}{\partial t} = a_{kk}\frac{\partial^2 u(t,x_k)}{\partial x_k^2}, & t>0 \\ u(0,x_k) = \mathrm{e}^{ix_k}, \end{cases}$$

in the sense that if $u(t,x_k)$ solves (1-dim), then $\tilde{u}(t,x) := u(t,x_k)$ solves (P).
The solution of the one dimensional heat equation is well known to be given by the one dimensional Gaussian semigroup

$$u(t,x) = \frac{1}{\sqrt{4\pi t a_{kk}}}\int_{-\infty}^{\infty} \mathrm{e}^{-s^2/4ta_{kk}}\,\mathrm{e}^{i(x_k-s)}\,ds$$

which yields after substituting $r := \frac{s}{\sqrt{2ta_{kk}}}$

$$\begin{aligned} u(t,x) &= \frac{\mathrm{e}^{ix_k}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{-r^2/2}\,\mathrm{e}^{-ir\sqrt{2ta_{kk}}}\,dr \\ &= \mathrm{e}^{ix_k}\,\mathrm{e}^{-ta_{kk}} \end{aligned}$$

since $e^{x^2/2}$ is invariant under Fourier transformation.
Comparing the two results we get by the uniqueness of the solution

$$\int_{l^p} \mathrm{e}^{i\sqrt{t}y_k}\,d\mu(y) = \mathrm{e}^{-ta_{kk}}\,.$$

As $\int_{l^p} \mathrm{e}^{i\sqrt{t}y_k}\,d\mu(y) = \int_{l^p} \mathrm{e}^{-i\sqrt{t}y_k}\,d\mu(y)$ by the symmetry of the Gaussian measure, we obtain

$$\begin{aligned} \mathrm{e}^{-ta_{kk}} &= \frac{1}{2}\left(\int_{l^p} \mathrm{e}^{i\sqrt{t}\,y_k}\,d\mu(y) + \int_{l^p} \mathrm{e}^{-i\sqrt{t}\,y_k}\,d\mu(y)\right) \\ &= \int_{l^p} \cos(\sqrt{t}\,y_k)d\mu(y). \end{aligned}$$

Hence, we get

$$\begin{aligned} a_{kk} &= \frac{d}{dt}(\mathrm{e}^{-ta_{kk}})_{|t=0} = \frac{d}{dt}\left(\int_{l^p} \cos(\sqrt{t}\, y_k) d\mu(y)\right)_{|t=0} \\ &= \int_{l^p} \frac{d}{dt}(\cos(\sqrt{t}\, y_k))_{|t=0} d\mu(y) = \int_{l^p} \lim_{t\to 0}\left[\frac{\sin((\sqrt{t}\, y_k)}{\sqrt{t}\, y_k}\right] \frac{y_k^2}{2} d\mu(y) \\ &= \frac{1}{2}\int_{l^p} y_k^2 d\mu(y) \geq 0. \end{aligned}$$

Therefore, for $p \geq 2$,

$$(a_{kk})^{(p/2)} = \left(\frac{1}{2}\int_{l^p} y_k^2 d\mu(y)\right)^{(p/2)} \leq \left(\frac{1}{2}\right)^{(p/2)} \int_{l^p} |y_k|^p d\mu(y),$$

by Hölder's inequality and the fact that μ is a probability measure.
For $1 \leq p < 2$, however, we obtain by the same argument

$$(a_{kk})^p \leq \left(\frac{1}{2}\right)^p \int_{l^p} |y_k|^{2p} d\mu(y).$$

Consequently for $p \geq 2$

$$\begin{aligned} \sum_{k\in\mathbb{N}} a_{kk}^{(p/2)} &\leq \sum_{k\in\mathbb{N}} \left(\frac{1}{2}\right)^{(p/2)} \int_{l^p} |y_k|^p d\mu(y) = \left(\frac{1}{2}\right)^{(p/2)} \int_{l^p} \sum_{k\in\mathbb{N}} |y_k|^p d\mu(y) \\ &= \left(\frac{1}{2}\right)^{(p/2)} \int_{l^p} \|y\|_{l^p}^p d\mu(y) < \infty, \end{aligned}$$

and respectively for $1 \leq p < 2$, since $\sum_k |y_k|^{2p} \leq \left(\sum_k |y_k|^p\right)^2$,

$$\sum_{k\in\mathbb{N}} a_{kk}^p \leq \left(\frac{1}{2}\right)^p \int_{l^p} \|y\|_{l^p}^{2p} d\mu(y) < \infty.$$

In both cases the boundedness follows by the corollary to Fernique's theorem. □

Remark 4.2.5. Let $a_{ij} = \lambda_i \delta_{ij} = \begin{cases} \lambda_i & \text{if} \quad j = i \\ 0 & \text{for} \quad j \neq i \end{cases}$. Then the Problem (P) is the infinite dimensional heat equation. Thus the condition $\sum a_{kk}^{p/2} = \sum \lambda_k^{p/2} < \infty$ is sufficient for well-posedness, which is also necessary for $p \geq 2$. However, for $1 \leq p < 2$, the necessary condition $\sum a_{kk}^p = \sum \lambda_k^p < \infty$ is weaker.

4.2.2 Sufficient Condition for Well-Posedness

Let $1 \leq p < \infty$. Recall that the Problem

$$(P) \quad \begin{cases} \frac{\partial u(t,x)}{\partial t} = \sum_{k,l\in\mathbb{N}} a_{kl} \frac{\partial^2 u(t,x)}{\partial x_k \partial x_l}, & t > 0 \\ u(0,.) = f \quad \in BUC(l^p) \end{cases}$$

is well-posed, if the operator $\boldsymbol{A} = \sum_{k,l} a_{kl} \frac{\partial^2}{\partial x_k \partial x_l}$ is closable and its closure is the generator of a strongly continuous semigroup $G(t)$ on $BUC(l^p)$.
We will proceed as follows. We transform the semigroup S solving the infinite dimensional heat equation (HE_∞) into a new semigroup G and determine its generator $\mathcal{A}$. Then we will give sufficient conditions on the coefficients a_{ij}, with $i, j \in \mathbb{N}$, such that the closure of the operator $\boldsymbol{A}$ coincides with $\mathcal{A}$.
For that, consider

$$(HE_\infty) \quad \begin{cases} \frac{\partial u(t,x)}{\partial t} = \sum_{n\in\mathbb{N}} \lambda_n \frac{\partial^2 u(t,x)}{\partial x_n^2}, & t > 0, \\ u(0,.) = f \quad \in BUC(l^r) \quad (1 \leq r < \infty). \end{cases}$$

We use the results established in the papers [ArDE] and [AbEK] and already recalled in Section 4.1.
For $1 \leq r < \infty$ and $\lambda = (\lambda_n)_{n\in\mathbb{N}} \in l_+^{r/2}$,

$$S(t)f = \lim_{n\to\infty} S^n(t)f = \lim_{n\to\infty} \prod_{j=1}^{n} S_j(\lambda_j t)f$$

converges uniformly on $[0, \tau]$ in $BUC(l^p)$ for every $\tau > 0$, $f \in BUC(l^p)$ and defines the strongly continuous heat semigroup S on $BUC(l^p)$.
This semigroup $S = (S(t))_{t\geq 0}$ is Gaussian, hence it admits a representation

$$(S(t)f)(x) = \int_{l^r} f(x + \sqrt{t}y)d\mu(y),$$

where $f \in BUC(l^r)$ and μ is the Gaussian measure on l^r associated with $\lambda \in l^{r/2}$.
Moreover, the generator of S is an extension of the operator $\mathcal{B} = \sum_{n\in\mathbb{N}} \lambda_n \frac{\partial^2}{\partial x_n^2}$, with its appropriate domain.
We start with some heuristic considerations which will lead us from the heat equation to the non-diagonal problem (P).
Let Λ denote the infinite diagonal matrix with entries $(\lambda_n)_{n\in\mathbb{N}} \in l_+^{r/2}$. Further, let M be a bounded linear operator from l^r to l^p with the matrix representation $M = (m_{ij})_{i,j\in\mathbb{N}}$ with respect to the canonic bases of l^r and l^p. Note that for the matrix representation of its adjoint $M^\tau : l^{p'} \to l^{r'}$, where $l^{p'} = (l^p)'$ with $\frac{1}{p} + \frac{1}{p'} = 1$, one has $m_{ij}^\tau = m_{ji}$. Then $M\Lambda M^\tau$ defines a bounded linear operator

from $l^{p'}$ to l^p, because $M : l^r \to l^p$ is bounded, hence its adjoint $M^\tau \in \mathcal{L}(l^{p'}, l^{r'})$ and as $(\lambda_n)_n \in l_+^{r/2} \subset l^r$ the operator Λ maps $l^{r'}$ to $l^1 \subset l^r$. In particular for all $k, l \in \mathbb{N}$, one has $\sum_{i\in\mathbb{N}} \lambda_i m_{ki} m_{li} = (M\Lambda M^\tau)_{kl} = (M\Lambda M^\tau e_l)_k \in \mathbb{R}$.
Since for all $\varphi \in (l^p)'$ one has $\varphi \circ M \in (l^r)'$ and the image measure defined by $\mu_M(y) := \mu(M^{-1}y)$ on l^p is also a Gaussian measure.
Then we can define the Gaussian semigroup with respect to this image measure on $BUC(l^p)$,

$$G(t)f(x) := \int_{l^p} f(x + \sqrt{t}y) d\mu_M(y) = \int_{l^r} f(x + \sqrt{t}My) d\mu(y).$$

In particular $(G(t)f)\circ M = S(t)(f\circ M)$. We deduce a very useful relation between the generator $\mathcal{A} := \frac{\partial}{\partial t} G(t)_{|t=0}$ and the generator $\mathfrak{B}$ of the heat semigroup S.

Lemma 4.2.6. *(i) If $f \in D(\mathcal{A})$, then $f\circ M \in D(\mathfrak{B})$ and $\mathfrak{B}(f\circ M) = (\mathcal{A}f)\circ M$.*

(ii) Suppose additionally that M has dense image in l^p. If $f \circ M \in D(\mathfrak{B})$ and $\mathfrak{B}(f \circ M) = g \circ M$ for one $g \in BUC(l^p)$, then $f \in D(\mathcal{A})$ and $\mathcal{A}f = g$.

Proof. (i) We have to show that $S(t)(f \circ M) - f \circ M = \int_0^t S(s)((\mathcal{A}f) \circ M)\, ds$. Since $f \in D(\mathcal{A})$, we have

$$\begin{aligned} S(t)(f \circ M) - f \circ M &= (G(t)f) \circ M - f \circ M = (G(t)f - f) \circ M \\ &= \Big(\int_0^t G(s)(\mathcal{A}f)\, ds\Big) \circ M = \int_0^t (G(s)(\mathcal{A}f)) \circ M\, ds \\ &= \int_0^t S(s)((\mathcal{A}f) \circ M)\, ds. \end{aligned}$$

(ii) We have to show that $G(t)f - f = \int_0^t G(s)g\, ds$. But both terms are elements of $BUC(l^p)$, hence it suffices to show equality pointwise on the dense subset $\operatorname{Im} M \subset l^p$ or equivalently to show $(G(t)f) \circ M - f \circ M = (\int_0^t G(s)g\, ds) \circ M$ in $BUC(l^r)$. Since $f \circ M \in D(\mathfrak{B})$, we have

$$\begin{aligned} (G(t)f) \circ M - f \circ M &= S(t)(f \circ M) - f \circ M = \int_0^t S(s)(\mathfrak{B}(f \circ M))\, ds \\ &= \int_0^t S(s)(g \circ M)\, ds = \int_0^t (G(s)g) \circ M\, ds \\ &= \Big(\int_0^t G(s)g\, ds\Big) \circ M. \end{aligned}$$

□

Since we do not know the operator $\mathfrak{B}$, we shall use in the following the above result for $\mathfrak{B}$ replaced by $\mathcal{B}$.

Corollary 4.2.7. Suppose that M has dense image in l^p. If $f \circ M \in D(\mathcal{B})$ and $\mathcal{B}(f \circ M) = g \circ M$ for one $g \in BUC(l^p)$, then $f \in D(\mathcal{A})$ and $\mathcal{A}f = g$.

Our aim is to determine the generator $\mathcal{A} := \frac{\partial}{\partial t} G(t)_{|t=0}$, and in particular to find a relation with the operator $\boldsymbol{A}$.
For the following formal calculation, we assume, that $x \in \operatorname{Im} M \subset l^p$, then there exists a w in l^r such that $x = Mw$.

$$\begin{aligned}
\frac{\partial}{\partial t} G(t) f(x) &= \frac{\partial}{\partial t} G(t) f(Mw) = \frac{\partial}{\partial t} \int_{l^r} f(Mw + \sqrt{t} My) d\mu(y) \\
&= \frac{\partial}{\partial t} \int_{l^r} (f \circ M)(w + \sqrt{t} y) d\mu(y) = \frac{\partial}{\partial t} S(t)(f \circ M)(w) \\
&= S(t)\Big[\sum_{i \in \mathbb{N}} \lambda_i \frac{\partial^2}{\partial x_i^2} (f \circ M)\Big](w) \\
&= S(t)\Big[\sum_{i \in \mathbb{N}} \lambda_i \frac{\partial}{\partial x_i} \sum_l m_{li} (\frac{\partial}{\partial x_l} f) \circ M\Big](w) \\
&= S(t)\Big[\sum_{i \in \mathbb{N}} \lambda_i \sum_{k,l} m_{ki} m_{li} (\frac{\partial^2}{\partial x_k \partial x_l} f) \circ M\Big](w) \\
&= S(t)\Big[\sum_{k,l} \sum_{i \in \mathbb{N}} \lambda_i m_{ki} m_{li} (\frac{\partial^2}{\partial x_k \partial x_l} f) \circ M\Big](w) \\
&= G(t) \sum_{k,l} a_{kl} (\frac{\partial^2}{\partial x_k \partial x_l} f)(x),
\end{aligned} \tag{4.2}$$

provided that $a_{kl} := \sum_{i \in \mathbb{N}} \lambda_i m_{ki} m_{li}$. Then $A = (a_{kl})_{k,l \in \mathbb{N}} = M \Lambda M^\tau$ defines a bounded linear operator from $l^{p'}$ to l^p.
Therefore, to continue these investigations, we will restrict ourselves to the case, where the matrix of the coefficients has such a representation.
This formal calculation gives the idea that the generator $\mathcal{A}$ of the Gaussian semigroup G could be the closure of the operator $\boldsymbol{A}$. However, we need to precise the assumptions for the equalities in the formal calculation (4.2).
As we wish to use Corollary 4.2.7, we first establish conditions for $f \circ M$ to belong to $D(\mathcal{B})$. We shall automatically obtain that $\mathcal{B}(f \circ M) = g \circ M$ for some $g \in BUC(l^p)$.

Lemma 4.2.8. *Let $1 \leq p, r < \infty$ and $M : l^r \to l^p$ a bounded linear operator with matrix representation $(m_{ij})_{i,j \in \mathbb{N}}$ with respect to the canonic bases. Assume, that $f \in BUC^1(l^p)$ and $\sum_l |m_{lj}| \, \|D_l f\| < \infty$ for all $j \in \mathbb{N}$. Then $f \circ M \in BUC^1(l^r)$ and $D_j(f \circ M) = \sum_l m_{lj} D_l f \circ M$.*

Proof. First observe, that obviously $f \circ M \in BUC(l^r)$, whenever $f \in BUC(l^p)$ and $M : l^r \to l^p$ is bounded. We have to show for all $j \in \mathbb{N}$,

$$\lim_{t \to 0} \frac{f \circ M(x + te_j) - f \circ M(x)}{t} = \sum_l m_{lj} D_l f \circ M(x) \in BUC(l^r).$$

But if we fix $j \in \mathbb{N}$ and set $Mx = y$ and $Me_j = (m_{lj})_{l=1}^{\infty} = b \in l^p$, we obtain with Lemma 4.1.8,

$$\begin{aligned} \lim_{t\to 0} \frac{f \circ M(x + te_j) - f \circ M(x)}{t} &= \lim_{t\to 0} \frac{f(Mx + tMe_j) - f(Mx)}{t} \\ &= \lim_{t\to 0} \frac{f(y + tb) - f(y)}{t} \\ &= \sum_l b_l D_l f(y) = \sum_l m_{lj} D_l f(Mx) \\ &= \sum_l m_{lj} D_l f \circ M(x). \end{aligned}$$

As $\sum_l |m_{lj}| \, \|D_l f\| < \infty$ implies $\sum_l m_{lj} D_l f \in BUC(l^p)$, the right hand side is obviously in $BUC(l^r)$ and the claim follows. □

Corollary 4.2.9. Let $1 \leq p, r < \infty$ and $M : l^r \to l^p$ a bounded linear operator with matrix representation $(m_{ij})_{i,j\in\mathbb{N}}$ with respect to the canonic bases. Assume, that $f \in BUC^1(l^p)$ and $\sum_l |m_{lj}| \, \|D_l f\| < \infty$ for one $j \in \mathbb{N}$. Then $f \circ M \in D(D_j)$ and $D_j(f \circ M) = \sum_l m_{lj} D_l f \circ M$.

Lemma 4.2.10. *Let $1 \leq p < \infty$ and $g_l \in BUC^1(l^p)$, for all $l \in \mathbb{N}$. Assume that $\sum_l \|g_l\| < \infty$ and $\sum_l \|D_k g_l\| < \infty$ for all $k \in \mathbb{N}$. Then $\sum_l g_l \in BUC^1(l^p)$ and $D_k \sum_l g_l = \sum_l D_k g_l$.*

Proof. We immediately get $\sum_l g_l \in BUC(l^p)$, because $\sum_l \|g_l\| < \infty$, and further $\sum_l \|D_k g_l\| < \infty$ implies that $\sum_l D_k g_l(x)$ converges uniformly in x. This allows to interchange summation and differentiation, which gives $\sum_l g_l \in BUC^1(l^p)$ and $D_k \sum_l g_l = \sum_l D_k g_l$. □

Proposition 4.2.11. *Let $1 \leq p, r < \infty$ and $M : l^r \to l^p$ a bounded linear operator with matrix representation $(m_{ij})_{i,j\in\mathbb{N}}$ with respect to the canonic bases. Assume, that $f \in BUC^2(l^p)$ and for all $j \in \mathbb{N}$*

(i) $\sum_l |m_{lj}| \, \|D_l f\| < \infty$,

(ii) $\sum_l |m_{lj}| \, \|D_k D_l f\| < \infty$ *for all* $k \in \mathbb{N}$, *and*

(iii) $\sum_{k,l} |m_{kj}| \, |m_{lj}| \, \|D_k D_l f\| < \infty$.

Then $f \circ M \in BUC^1(l^r)$ and $D_j(f \circ M) \in D(D_j)$ and one has the formula $D_j^2(f \circ M) = \sum_{k,l} m_{kj} m_{lj} D_k D_l f \circ M$ for all $j \in \mathbb{N}$.

Proof. As $f \in BUC^2(l^p) \subset BUC^1(l^p)$ and $\sum_l |m_{lj}| \, \|D_l f\| < \infty$, we get from Lemma 4.2.8, that $f \circ M \in BUC^1(l^r)$ and $D_j(f \circ M) = \sum_l m_{lj} D_l f \circ M$ for all $j \in \mathbb{N}$. Let us fix $j \in \mathbb{N}$.

Let $g_l = m_{lj} D_l f$, then $g_l \in BUC^1(l^p)$, because $f \in BUC^2(l^p)$. Further we have $\sum_l \|g_l\| = \sum_l |m_{lj}| \, \|D_l f\| < \infty$ and $\sum_l \|D_k g_l\| = \sum_l |m_{lj}| \, \|D_k D_l f\| < \infty$ for all $k \in \mathbb{N}$. Hence, by Lemma 4.2.10 we get $\sum_l g_l \in BUC^1(l^p)$ and the equality $D_k \sum_l g_l = \sum_l D_k g_l$ holds.
Additionally, we have

$$\begin{aligned} \sum_k |m_{kj}| \left\| D_k \sum_l g_l \right\| &= \sum_k |m_{kj}| \left\| \sum_l D_k g_l \right\| = \sum_k |m_{kj}| \left\| \sum_l D_k m_{lj} D_l f \right\| \\ &\leq \sum_k \sum_l |m_{kj}| \, |m_{lj}| \, \|D_k D_l f\| < \infty \end{aligned}$$

Hence we can apply on the function $\sum_l g_l \in BUC^1(l^p)$ the corollary to Lemma 4.2.8 and obtain $\sum_l g_l \circ M \in D(D_j)$ and

$$\begin{aligned} D_j(g_l \circ M) &= \sum_k m_{kj} D_k \sum_l g_l \circ M = \sum_k m_{kj} \sum_l D_k g_l \circ M \\ &= \sum_k \sum_l m_{kj} D_k g_l \circ M. \end{aligned}$$

Substituting g_l by $m_{lj} D_l f$, we get $\sum_l m_{lj} D_l f \circ M = D_j(f \circ M) \in D(D_j)$ and

$$\begin{aligned} D_j(m_{lj} D_l f \circ M) = D_j^2(f \circ M) &= \sum_k \sum_l m_{kj} D_k m_{lj} D_l f \circ M \\ &= \sum_k \sum_l m_{kj} m_{lj} D_k D_l f \circ M, \end{aligned}$$

which concludes the proof. □

Lemma 4.2.12. *Let $1 \leq p, r < \infty$ and $M : l^r \to l^p$ a bounded linear operator with matrix representation $(m_{ij})_{i,j \in \mathbb{N}}$ with respect to the canonic bases. Assume, that $f \in BUC^2(l^p)$ and for all $j \in \mathbb{N}$*

(i) $\sum_l |m_{lj}| \, \|D_l f\| < \infty$, *and*

(ii) $\sum_l |m_{lj}| \, \|D_k D_l f\| < \infty$ *for all* $k \in \mathbb{N}$.

Assume further, that for the sequence $\lambda = (\lambda_n)_{n \in \mathbb{N}} \in l_+^{r/2}$, one has absolute convergence of the series $\sum_{k,l} \sum_{i \in \mathbb{N}} \lambda_i m_{ki} m_{li} (\frac{\partial^2}{\partial x_k \partial x_l} f)$ in $BUC(l^p)$, then $f \circ M \in D(\mathcal{B})$. Moreover $\mathcal{B}(f \circ M) = \sum_j \sum_{k,l} \lambda_j m_{kj} m_{lj} D_k D_l f \circ M$.

Proof. If $\sum_{k,l} \sum_{i \in \mathbb{N}} \lambda_i m_{ki} m_{li} (\frac{\partial^2}{\partial x_k \partial x_l} f)$ converges absolutely, then we can interchange the order of the sums and get absolute convergence in $BUC(l^p)$ of the series $\sum_{i \in \mathbb{N}} \lambda_i \sum_{k,l} m_{ki} m_{li} (\frac{\partial^2}{\partial x_k \partial x_l} f)$.
In particular $(|\lambda_i| \sum_{k,l} |m_{ki}| \, |m_{li}| \, \|\frac{\partial^2}{\partial x_k \partial x_l} f\|)_i \in l^1 \subset l^\infty$. As $\lambda = (\lambda_n)_{n \in \mathbb{N}} \in l_+^{r/2}$, we have that $\lambda_i > 0$ for each $i \in \mathbb{N}$ and hence $\sum_{k,l} |m_{ki}| \, |m_{li}| \, \|\frac{\partial^2}{\partial x_k \partial x_l} f\| < \infty$

for each $i \in \mathbb{N}$. Thus, all assumptions of Proposition 4.2.11 are satisfied and we can conclude that $f \circ M \in BUC^1(l^r)$ and $D_j(f \circ M) \in D(D_j)$ and one has the formula $D_j^2(f \circ M) = \sum_{k,l} m_{kj} m_{lj} D_k D_l f \circ M$ for all $j \in \mathbb{N}$. Therefore

$$\begin{aligned}
\sum_{i \in \mathbb{N}} |\lambda_i| \left\| \frac{\partial^2}{\partial x_i^2}(f \circ M) \right\| &\leq \sum_{i \in \mathbb{N}} |\lambda_i| \sum_{k,l} |m_{kj}|\,|m_{lj}|\,\|D_k D_l f \circ M\| \\
&= \sum_{k,l} \sum_{i \in \mathbb{N}} |\lambda_i|\,|m_{kj}|\,|m_{lj}|\,\|D_k D_l f\| < \infty,
\end{aligned}$$

Hence $f \circ M \in D(\mathcal{B})$ and $\mathcal{B}(f \circ M) = \sum_j \sum_{k,l} \lambda_j m_{kj} m_{lj} D_k D_l f \circ M$. □

Proposition 4.2.13. *Let $1 \leq p, r < \infty$ and let $(a_{ij})_{i,j \in \mathbb{N}} = M \Lambda M^\tau$, where $M : l^r \to l^p$ is a bounded linear operator with matrix representation $(m_{ij})_{i,j \in \mathbb{N}}$ with respect to the canonic bases and with dense image in l^p and $\Lambda = \operatorname{diag}(\lambda_n)$ with $(\lambda_n)_{n \in \mathbb{N}} \in l_+^{p/2}$. Assume, that $f \in BUC^2(l^p)$ and for all $j \in \mathbb{N}$*

(i) $\sum_l |m_{lj}|\,\|D_l f\| < \infty$, and

(ii) $\sum_l |m_{lj}|\,\|D_k D_l f\| < \infty$ for all $k \in \mathbb{N}$.

Assume further, that the series $\sum_{k,l} \sum_{i \in \mathbb{N}} \lambda_i m_{ki} m_{li} (\frac{\partial^2}{\partial x_k \partial x_l} f)$ converges absolutely in $BUC(l^p)$, then $f \in D(\mathcal{A}) \cap D(\boldsymbol{A})$ and $\mathcal{A}f = \boldsymbol{A}f$

Proof. $(a_{ij})_{i,j \in \mathbb{N}} = M \Lambda M^\tau$ implies $a_{kl} = \sum_i \lambda_i m_{ki} m_{li}$, hence the absolute convergence of $\sum_{k,l} \sum_{i \in \mathbb{N}} \lambda_i m_{ki} m_{li} (\frac{\partial^2}{\partial x_k \partial x_l} f)$ implies absolute convergence of the series $\sum_{k,l} a_{kl} (\frac{\partial^2}{\partial x_k \partial x_l} f)$ in $BUC(l^p)$. As $f \in BUC^2(l^p)$, we get $f \in D(\boldsymbol{A})$. In particular $\boldsymbol{A}f \in BUC(l^p)$.
Observe further that the absolute convergence of $\sum_{k,l} \sum_{i \in \mathbb{N}} \lambda_i m_{ki} m_{li} (\frac{\partial^2}{\partial x_k \partial x_l} f)$ implies the absolute convergence of $\sum_{i \in \mathbb{N}} \sum_{k,l} \lambda_i m_{ki} m_{li} (\frac{\partial^2}{\partial x_k \partial x_l} f)$. Hence all assumptions of Lemma 4.2.12 are satisfied, which gives $f \circ M \in D(\mathcal{B})$ and additionally $\mathcal{B}(f \circ M) = g \circ M$, where $g = \sum_j \sum_{k,l} \lambda_j m_{kj} m_{lj} D_k D_l f \in BUC(l^p)$. Hence by Corollary 4.2.7, $f \in D(\mathcal{A})$ and

$$\begin{aligned}
\mathcal{A}f &= \sum_j \sum_{k,l} \lambda_j m_{kj} m_{lj} D_k D_l f = \sum_{k,l} \sum_j \lambda_j m_{kj} m_{lj} D_k D_l f \\
&= \sum_{k,l} a_{kl} D_k D_l f = \boldsymbol{A}f,
\end{aligned}$$

because of the absolute convergence, which enables to interchange the order of summation. □

Lemma 4.2.14. *Let $D \subset D(\mathcal{A}) \cap D(\boldsymbol{A})$, such that D is dense in $BUC(l^p)$ and $\mathcal{A}f = \boldsymbol{A}f$ for all $f \in D$. If $G(t)D \subset D$ for all $t \geq 0$, then $\mathcal{A} = \bar{\boldsymbol{A}}$.*

Proof. The closed operator $\mathcal{A}$ is uniquely determined on a core. Since the dense subset D is invariant under the semigroup, it is by [EN], Proposition II.1.7, a core for the generator $\mathcal{A}$. Thus $\mathcal{A}f = \boldsymbol{A}f$ for all $f \in D$ implies that $\mathcal{A} = \bar{\boldsymbol{A}}$. □

We will make use of [AbEK], Corollary 4.3. For $\nu \in l^p_+$, the subspace

$$D^\nu := \left\{ f \in BUC^2(l^p) : \sup_k \nu_k \|D_k f\| < \infty \text{ and } \sup_{k,l} \nu_k \nu_l \|D_k D_l f\| < \infty \right\}$$

is dense in $BUC(l^p)$.
In the next section, we also give a proof to this result in a more general context, where the operators D_k are replaced by general group generators, see Lemma 4.3.7.

Lemma 4.2.15. *D^ν is invariant under the Gaussian semigroup $G(t)$, i.e. for all $f \in D^\nu$ one has $G(t)f \in D^\nu$ for all $t \geq 0$.*

Proof. Let $f \in D^\nu$, then in particular $f \in BUC^2(l^p)$. Since G is Gaussian, we can apply Lemma 4.1.21 twice and obtain $G(t)f \in BUC^2(l^p)$ and the equalities $D_k D_l G(t)f = D_k G(t) D_l f = G(t) D_k D_l f$. Moreover, G is contractive. Then one has $\sup_k \nu_k \|D_k G(t)f\| = \sup_k \nu_k \|G(t) D_k f\| \leq \sup_k \nu_k \|D_k f\| < \infty$ and also $\sup_{k,l} \nu_k \nu_l \|D_k D_l G(t) f\| = \sup_{k,l} \nu_k \nu_l \|G(t) D_k D_l f\| \leq \sup_{k,l} \nu_k \nu_l \|D_k D_l f\| < \infty$. Thus $G(t)f \in D^\nu$, whenever $f \in D^\nu$. □

We summarize the above results.

Corollary 4.2.16. Assume there exists a decomposition of the matrix of coefficients $A = (a_{ij})_{i,j \in \mathbb{N}} = M \Lambda M^\tau$, where $\Lambda = \operatorname{diag}(\lambda_n)$ with $(\lambda_n)_{n \in \mathbb{N}} \in l^{p/2}$ and $M : l^r \to l^p$ is a bounded linear operator with matrix representation $(m_{ij})_{i,j \in \mathbb{N}}$ with respect to the canonic bases and with dense image. Let $\nu \in l^p$. If for all $f \in D^\nu$ and for all $j \in \mathbb{N}$

(i) $\sum_l |m_{lj}| \, \|D_l f\| < \infty$,

(ii) $\sum_l |m_{lj}| \, \|D_k D_l f\| < \infty$ for all $k \in \mathbb{N}$,

and the series $\sum_{k,l} \sum_{i \in \mathbb{N}} \lambda_i m_{ki} m_{li} (\frac{\partial^2}{\partial x_k \partial x_l} f)$ converges absolutely in $BUC(l^p)$, then $\mathcal{A} = \bar{\boldsymbol{A}}$, i.e. the Problem (P) is well-posed.

Proof. The matrix A and the functions $f \in D^\nu$ satisfy the assumptions of Proposition 4.2.13. Hence $D^\nu \subset D(\mathcal{A}) \cap D(\boldsymbol{A})$. Since by Lemma 4.2.15, $G(t)D^\nu \subset D^\nu$, we conclude with Lemma 4.2.14, that $\mathcal{A} = \bar{\boldsymbol{A}}$ and hence the Problem (P) is well-posed. □

Hence, given the coefficients $a_{ij},\ i,j \in \mathbb{N}$, we first have to verify, that the matrix $A = (a_{ij})_{i,j\in\mathbb{N}}$ is a bounded operator from $l^{p'}$ to l^p and admits a decomposition $A = M\Lambda M^\tau$, where $M : l^r \to l^p$ is a bounded operator with matrix representation $M = (m_{ij})_{i,j\in\mathbb{N}}$ and dense image in l^p, M^τ its adjoint and Λ a diagonal operator given by a series $\lambda = (\lambda_n)_{n\in\mathbb{N}} \in l_+^{r/2}$. Finally one has to find a $\nu \in l^p$, such that for all $f \in D^\nu$, and all $j \in \mathbb{N}$, $\sum_l |m_{lj}|\,\|D_l f\| < \infty$, $\sum_l |m_{lj}|\,\|D_k D_l f\| < \infty$ for all $k \in \mathbb{N}$, and the series $\sum_{k,l}\sum_{i\in\mathbb{N}} \lambda_i m_{ki} m_{li} (\frac{\partial^2}{\partial x_k \partial x_l} f)$ converges absolutely in $BUC(l^p)$, in order to conclude well-posedness of the Problem (P).

Absolute convergence of the series

From the Problem (P), we are given $1 \le p < \infty$ and $a_{kl} \in \mathbb{R}$ for $k,l \in \mathbb{N}$. In order to concentrate on the absolute convergence of the series,we will suppose in this paragraph the following assumptions.

(A1) The matrix $(a_{kl})_{k,l\in\mathbb{N}}$ admits a decomposition $M\Lambda M^\tau$.

(A2) $M : l^r \to l^p$, $1 \le r < \infty$, is a bounded linear operator with dense image.

(A3) Λ is a diagonal matrix with entries $(\lambda_n)_{n\in\mathbb{N}} \in l_+^{r/2}$.

Moreover, let μ denote the Gaussian measure on l^r associated with the sequence $(\lambda_n)_{n\in\mathbb{N}} \in l_+^{r/2}$.
We shall examine the series one after the other. For that we fix $\nu \in l_+^p$ and take $f \in D^\nu$. Then for all $i \in \mathbb{N}$,

$$\sum_l \Big\| m_{li} \frac{\partial}{\partial x_l} f \Big\| = \sum_l \frac{|m_{li}|}{\nu_l} \nu_l \Big\| \frac{\partial}{\partial x_l} f \Big\| \le C_1 \sum_l \frac{|m_{li}|}{\nu_l} < \infty,$$

provided that $\left(\frac{m_{li}}{\nu_l}\right)_{l\in\mathbb{N}} \in l^1$. And analogously under this assumption, we obtain for all $i,k \in \mathbb{N}$,

$$\begin{aligned} \sum_l \Big\| m_{li} \frac{\partial^2}{\partial x_k \partial x_l} f \Big\| &= \sum_k \frac{|m_{li}|}{\nu_k \nu_l} \nu_k \nu_l \Big\| \frac{\partial^2}{\partial x_k \partial x_l} f \Big\| \\ &\le C_2 \sum_l \frac{|m_{li}|}{\nu_k \nu_l} = C_2 \frac{1}{\nu_k} \sum_l \frac{|m_{li}|}{\nu_l} < \infty. \end{aligned}$$

For simplicity, we denote $\gamma_i := \sum_k \frac{|m_{ki}|}{\nu_k}$. Further we get

$$\begin{aligned} &\sum_i \lambda_i \sum_{k,l} \Big\| m_{ki} m_{li} \frac{\partial^2}{\partial x_k \partial x_l} f \Big\| = \sum_i \lambda_i \sum_{k,l} \frac{|m_{ki} m_{li}|}{\nu_k \nu_l} \nu_k \nu_l \Big\| \frac{\partial^2}{\partial x_k \partial x_l} f \Big\| \\ &\le \ C_2 \sum_i \lambda_i \sum_{k,l} \frac{|m_{ki} m_{li}|}{\nu_k \nu_l} = C_2 \sum_i \lambda_i \sum_k \frac{|m_{ki}|}{\nu_k} \sum_l \frac{|m_{li}|}{\nu_l} = C_2 \sum_i \lambda_i \gamma_i^2 < \infty, \end{aligned}$$

if for $r > 2$, $\gamma_i^2 \in l^q$, with $q = \frac{r}{r-2}$ and for $r \leq 2$, $\gamma_i \in l^\infty$.
In particular, we get absolute convergence of $\sum_{i\in\mathbb{N}} \lambda_i \sum_{k,l} m_{ki} m_{li} (\frac{\partial^2}{\partial x_k \partial x_l} f)$ and by interchanging the order of summation, we obtain absolute convergence of $\sum_{k,l} \sum_{i\in\mathbb{N}} \lambda_i m_{ki} m_{li} (\frac{\partial^2}{\partial x_k \partial x_l} f)$. Hence all assumptions of Corollary 4.2.16 are satisfied and the Problem (P) is well-posed.
We summarize the result so far.

Proposition 4.2.17. *Let $1 \leq p < \infty$ and $a_{kl} \in \mathbb{R}$ for $k, l \in \mathbb{N}$. Suppose (A1), (A2), and (A3). Assume further, that there exists a sequence $\nu \in l^p_+$, such that $\left(\frac{m_{li}}{\nu_l}\right)_{l\in\mathbb{N}} \in l^1$ and $\left(\sum_l \frac{|m_{li}|}{\nu_l}\right)^2_{i\in\mathbb{N}} \in l^q$ with $q = \frac{r}{r-2}$ for $r > 2$ and $q = \infty$ otherwise. Then the generator $\mathcal{A}$ of the Gaussian semigroup $G(t)$ associated with the image measure μ_M is given by*

$$\mathcal{A} f(x) = \sum_{k,l} a_{kl} \left(\frac{\partial^2}{\partial x_k \partial x_l} f \right)(x),$$

for $f \in D^\nu$. Moreover the operator $\boldsymbol{A}$ is closable and its closure coincides with $\mathcal{A}$, i.e. the Problem (P) is well-posed.

The conditions so far require a complete knowledge of the matrix M. We shall give conditions only on the coefficients a_{ij}. Note that by assumption (A1), the coefficients are given as $a_{kl} = \sum_{i\in\mathbb{N}} \lambda_i m_{ki} m_{li}$. In particular, one has $a_{kk} = \sum_{i\in\mathbb{N}} \lambda_i m_{ki}^2 \geq 0$, thus $|m_{ki}| \leq (\frac{a_{kk}}{\lambda_i})^{1/2}$.

Proposition 4.2.18. *Let $1 \leq p < \infty$ and $a_{kl} \in \mathbb{R}$ for $k, l \in \mathbb{N}$. Suppose (A1), (A2), and (A3). Assume further, that there exists a sequence $\nu \in l^p_+$, such that $\sum_k \frac{\sqrt{a_{kk}}}{\nu_k} < \infty$. Then the Problem (P) is well-posed, which is equivalent to saying that the operator $\boldsymbol{A} = \sum_{k,l} a_{kl} \frac{\partial^2}{\partial x_k \partial x_l}$ is closable and its closure is the generator of a strongly continuous semigroup on $BUC(l^p)$.*

Proof. If $\sum_k \frac{\sqrt{a_{kk}}}{\nu_k} < \infty$, then for $f \in D^\nu$ and all $i \in \mathbb{N}$,

$$\begin{aligned} \sum_l |m_{li}| \left\| \frac{\partial f}{\partial x_l} \right\| &= \sum_l \frac{|m_{li}|}{\nu_l} \nu_l \left\| \frac{\partial f}{\partial x_l} \right\| \leq C_1 \sum_l \frac{1}{\nu_l} |m_{li}| \\ &\leq C_1 \sum_l \frac{1}{\nu_l} \frac{a_{ll}^{1/2}}{\sqrt{\lambda_i}} = \frac{C_1}{\sqrt{\lambda_i}} \sum_l \frac{a_{ll}^{1/2}}{\nu_l} < \infty. \end{aligned}$$

Analogously, we obtain for all $i, k \in \mathbb{N}$,

$$\begin{aligned} \sum_l |m_{li}| \left\| \frac{\partial^2 f}{\partial x_k \partial x_l} \right\| &= \sum_l \frac{|m_{li}|}{\nu_k \nu_l} \nu_k \nu_l \left\| \frac{\partial^2 f}{\partial x_k \partial x_l} \right\| \leq C_2 \sum_l \frac{1}{\nu_k \nu_l} |m_{li}| \\ &\leq C_2 \sum_l \frac{1}{\nu_k \nu_l} \frac{a_{ll}^{1/2}}{\sqrt{\lambda_i}} = \frac{C_2}{\sqrt{\lambda_i} \nu_k} \sum_l \frac{a_{ll}^{1/2}}{\nu_l} < \infty. \end{aligned}$$

Finally, one has

$$\sum_{k,l}\sum_i \lambda_i |m_{ki}|\,|m_{li}| \left\| \frac{\partial^2 f}{\partial x_k \partial x_l} \right\| = \sum_{k,l} \frac{\sum_i \lambda_i |m_{ki}|\,|m_{li}|}{\nu_k \nu_l} \nu_k \nu_l \left\| \frac{\partial^2 f}{\partial x_k \partial x_l} \right\|$$

$$\leq C \sum_{k,l} \frac{1}{\nu_k \nu_l} \sum_i \lambda_i |m_{ki}|\,|m_{li}| \leq C \sum_{k,l} \frac{1}{\nu_k \nu_l} \left(\sum_i \lambda_i m_{ki}^2 \right)^{1/2} \left(\sum_i \lambda_i m_{li}^2 \right)^{1/2}$$

$$= C \sum_{k,l} \frac{1}{\nu_k \nu_l} a_{kk}^{1/2} a_{ll}^{1/2} = C \left(\sum_k \frac{a_{kk}^{1/2}}{\nu_k} \right)^2 < \infty.$$

Hence, $\sum_{k,l}\sum_i \lambda_i m_{ki}\, m_{li} \frac{\partial^2 f}{\partial x_k \partial x_l}$ is absolutely convergent and by Corollary 4.2.16 the claim follows. □

However, so far we needed the assumptions (A1), (A2), and (A3). Our aim is certainly to deduce these properties directly from the assumptions on the coefficients $a_{ij}, i, j \in \mathbb{N}$.

Observe that under the assumptions (A1), (A2), and (A3) in the case, where $p = 2$, the matrix $A = (a_{kl})_{k,l \in \mathbb{N}}$ defines a bounded linear operator on l^2, which is positive and self-adjoint. This explains our assumption for the next paragraph.

The matrix A defining a bounded operator on l^2

Throughout this paragraph, we shall suppose the following assumption.

(A4) The matrix $A = (a_{ij})_{ij \in \mathbb{N}}$ defines a bounded strictly positive self-adjoint operator from l^2 to l^2.

Remark 4.2.19. Recall, that an operator A on a Hilbert space is called strictly positive, if $(Ax, x) > 0$ for all $x \neq 0$. This assumption corresponds to ellipticity in finite dimensions. In particular we get $a_{kk} > 0$, by taking for x the elements of the canonical basis.

Since A is a positive self-adjoint operator on the Hilbert space l^2, it admits a square root, i.e. a unique positive self-adjoint operator $M : l^2 \to l^2$ such that $A = M \cdot M$, see [RS], Theorem VI.9. Let $M = (m_{ij})_{i,j \in \mathbb{N}}$ be the matrix representation, then $a_{kl} = \sum_{j \in \mathbb{N}} m_{kj} m_{jl} = \sum_{j \in \mathbb{N}} m_{kj} m_{lj} = \sum_{j \in \mathbb{N}} m_{jk} m_{jl}$, and hence all these series are finite. In particular $a_{kk} = \sum_{j \in \mathbb{N}} m_{kj}^2$, which implies $|m_{kj}| \leq \sqrt{a_{kk}}$, and by symmetry $|m_{kj}| \leq \sqrt{a_{jj}}$, and one gets $|m_{kj}| \leq a_{kk}^{1/4} a_{jj}^{1/4}$.
Let $\omega = (\omega_k)_{k \in \mathbb{N}}$ be a bounded strictly positive sequence and define $m'_{ij} = \frac{m_{ij}}{\sqrt{\omega_j}}$ and let $M' = (m'_{ij})_{i,j \in \mathbb{N}}$. Then the matrix of coefficients admits a decomposition $A = (a_{ij})_{ij \in \mathbb{N}} = M \cdot M = M' \Lambda (M')^\tau$, where Λ is a diagonal matrix with entries ω_k, $k \in \mathbb{N}$.

Lemma 4.2.20. *Let* $\sum_k \frac{a_{kk}^{1/2}}{\omega_k} < \infty$. *Then for* $2 \leq p \leq \infty$ *the matrix* M' *defines a bounded linear operator from* l^p *to* l^p. *If in addition* $\sum_k a_{kk}^{1/4} < \infty$, *then* M' *defines a bounded linear operator from* l^p *to* l^p *for* $1 \leq p < 2$.

Proof. Observe first, that the operator defined by $M' = (m'_{ij})_{i,j\in\mathbb{N}}$ is bounded from l^2 to l^2. Indeed, let $x \in l^2$, then

$$
\begin{aligned}
\|M'x\|_2 &= \Big(\sum_i (M'x)_i^2\Big)^{1/2} = \Big(\sum_i \Big(\sum_k m'_{ik}x_k\Big)^2\Big)^{1/2} \\
&\leq \Big(\sum_i \Big(\sum_k m'^2_{ik}\Big)\Big(\sum_k x_k^2\Big)\Big)^{1/2} = \|x\|_2 \Big(\sum_i \sum_k m'^2_{ik}\Big)^{1/2} \\
&= \|x\|_2 \Big(\sum_k \sum_i \Big(\frac{m_{ik}}{\sqrt{\omega_k}}\Big)^2\Big)^{1/2} = \|x\|_2 \Big(\sum_k \frac{1}{\omega_k} \sum_i m_{ik}^2\Big)^{1/2} \\
&= \|x\|_2 \Big(\sum_k \frac{1}{\omega_k} a_{kk}\Big)^{1/2} = \|x\|_2 \Big(\sum_k \frac{\sqrt{a_{kk}}}{\omega_k}\sqrt{a_{kk}}\Big)^{1/2} \\
&\leq C\|x\|_2, \qquad (4.3)
\end{aligned}
$$

because $\sum_k \frac{a_{kk}^{1/2}}{\omega_k} < \infty$, which also implies, that $(\sqrt{a_{kk}})_{k\in\mathbb{N}}$ is bounded.
A similar calculation gives boundedness from l^∞ to l^∞. Indeed,

$$
\begin{aligned}
\|M'x\|_\infty &= \sup_i |(M'x)_i| = \sup_i \Big|\sum_k m'_{ik}x_k\Big| \\
&\leq \|x\|_\infty \sup_i \sum_k |m'_{ik}| = \|x\|_\infty \sup_i \Big|\sum_k \frac{|m_{ik}|}{\sqrt{\omega_k}}\Big| \\
&\leq \|x\|_\infty \sup_i \sum_k \frac{\sqrt{a_{kk}}}{\sqrt{\omega_k}},
\end{aligned}
$$

and $\sup_i \sum_k \frac{\sqrt{a_{kk}}}{\sqrt{\omega_k}} = \sum_k \frac{\sqrt{a_k}}{\omega_k}\sqrt{\omega_k}$ is finite, because $\sum_k \frac{\sqrt{a_{kk}}}{\omega_k}$ is finite and further $\omega \in l^{p/2} \subset l^\infty$.
Hence by the Riesz-Thorin interpolation theorem, $M' : l^p \to l^p$ is bounded for all $2 \leq p \leq \infty$.
Let $1 \leq p < 2$, and assume that $\sum_i a_{ii}^{1/4}$ is finite.

Then

$$\begin{aligned}
\|M'x\|_1 &= \sum_i |(M'x)_i| = \sum_i \Big|\sum_k m'_{ik}x_k\Big| \\
&\leq \sum_i\sum_k |m'_{ik}x_k| = \sum_k\sum_i |m'_{ik}x_k| = \sum_k |x_k| \sum_i |m'_{ik}| \\
&\leq \|x_k\|_1 \sup_k \sum_i \frac{|m_{ik}|}{\sqrt{\omega_k}} \\
&\leq \|x_k\|_1 \sup_k \sum_i \frac{a_{kk}^{1/4} a_{ii}^{1/4}}{\sqrt{\omega_k}} = \|x_k\|_1 \sup_k \frac{a_{kk}^{1/4}}{\sqrt{\omega_k}} \sum_i a_{ii}^{1/4},
\end{aligned}$$

and $\sup_k \frac{a_{kk}^{1/4}}{\sqrt{\omega_k}}$ is bounded, because $\frac{a_{kk}^{1/4}}{\sqrt{\omega_k}} = \left(\frac{a_{kk}^{1/2}}{\omega_k}\right)^{1/2}$, and $\frac{a_{kk}^{1/2}}{\omega_k} \in l^1 \subset l^\infty$. The boundedness of $\sum_i a_{ii}^{1/4}$, follows from the additional assumption.
Hence again by the interpolation theorem, if $\sum_i a_{ii}^{1/4}$ is finite, then $M' : l^p \to l^p$ is bounded for $1 \leq p \leq \infty$. □

Lemma 4.2.21. *Under the assumptions of Lemma 4.2.20, and the additional assumption, that $\omega = (\omega_k)_{k\in\mathbb{N}} \in l^p$ for $1 \leq p < 2$, the operator $A : l^{p'} \to l^p$ is bounded.*

Proof. By Lemma 4.2.20, $M' : l^p \to l^p$ is bounded. Since the transposed matrix defines the adjoint operator, $(M')^\tau : l^{p'} \to l^{p'}$ is bounded. For $p \geq 2$, the condition that $\omega = (\omega_k)_{k\in\mathbb{N}}$ is bounded, implies $\Lambda = \mathrm{diag}(\omega_k) : l^{p'} \to l^{p'} \subset l^p$ is bounded, and for $1 \leq p < 2$ the assumption $\omega = (\omega_k)_{k\in\mathbb{N}} \in l^p$ guarantees, that the application $\Lambda = \mathrm{diag}(\omega_k) : l^{p'} \to l^1 \subset l^p$ is bounded. Hence the operator $A = M'\Lambda(M')^\tau : l^{p'} \to l^p$ is bounded. □

Lemma 4.2.22. *Let $2 \leq p < \infty$ and $\omega = (\omega_k)_{k\in\mathbb{N}} \in l^\infty_+$. Assume as before that $\sum_k \frac{a_{kk}^{1/2}}{\omega_k} < \infty$. Then the operator $M' : l^p \to l^p$ has dense image.*

Proof. First observe, that the strict positivity of A implies, that M and hence A is an injective operator from l^2 to l^2. Indeed, for all $0 \neq x \in l^2$,

$$0 < (Ax, x) = (M \cdot Mx, x) = (Mx, Mx) = \|Mx\|.$$

Hence $Mx = 0$ implies $x = 0$, and $Ax = 0$ implies $M(Mx) = 0$, hence $Mx = 0$ and therefore $x = 0$.
Assume, that $\overline{\mathrm{Im}\, M'} \neq l^p$. Then there exists $0 \neq y^* \in l^{p'}$ such that $y^*_{|\mathrm{Im}\, M'} = 0$, i.e. $0 = y^*(M'x) = (M')^\tau y^*(x)$ for all $x \in l^p$. Hence $(M')^\tau$ is not injective as an operator from $l^{p'}$ to $l^{p'}$.
Therefore, if $p \geq 2$, there exists an $0 \neq x \in l^{p'} \subset l^2$, such that $(M')^\tau x = 0$. Hence $Ax = M'\Lambda(M')^\tau x = 0$, which contradicts the injectivity of A as an operator from l^2 to l^2. □

Lemma 4.2.23. *Let $1 \leq p < 2$ and $\omega = (\omega_k)_{k\in\mathbb{N}} \in l^p_+$. Assume as before that $\sum_k \frac{a_{kk}^{1/2}}{\omega_k} < \infty$ and $\sum_k a_{kk}^{1/4} < \infty$ and additionally that $A : l^{p'} \to l^p$ is injective. Then the operator $M' : l^p \to l^p$ has dense image.*

Proof. Assume, that $\overline{\operatorname{Im} M'} \neq l^p$. Then there exists $0 \neq y^* \in l^{p'}$ such that $y^*_{|\operatorname{Im} M'} = 0$, i.e. $0 = y^*(M'x) = (M')^\tau y^*(x)$ for all $x \in l^p$. Hence $(M')^\tau$ is not injective as an operator from $l^{p'}$ to $l^{p'}$.
Therefore, there exists an $0 \neq x \in l^{p'}$, such that $(M')^\tau x = 0$. Hence, one has $Ax = M'\Lambda(M')^\tau x = 0$, which contradicts the injectivity of A as an operator from $l^{p'}$ to l^p. □

Now we are ready to formulate the main result of this section.

Proposition 4.2.24. *Let $1 \leq p < \infty$. Let the matrix $A = (a_{ij})_{ij\in\mathbb{N}}$ define a bounded strictly positive self-adjoint operator from l^2 to l^2. Assume, that there exists a sequence $\omega = (\omega_k)_{k\in\mathbb{N}} \in l^{p/2}_+$, such that*

$$\sum_k \frac{a_{kk}^{1/2}}{\omega_k} < \infty.$$

Assume additionally for $p < 2$, that $\sum_k a_{kk}^{1/4} < \infty$ and that $A : l^{p'} \to l^p$ is injective. Then the Problem (P) is well-posed in $BUC(l^p)$.

Proof. Since the matrix $A = (a_{ij})_{ij\in\mathbb{N}}$ defines a bounded strictly positive self-adjoint operator from l^2 to l^2, it admits a square root, i.e. there exists a decomposition $A = M \cdot M = M'\Lambda(M')^\tau$, where M' is the matrix given by $m'_{ij} = \frac{m_{ij}}{\sqrt{\omega_j}}$ and Λ is the diagonal matrix with entries $\omega = (\omega_k)_{k\in\mathbb{N}} \in l^{p/2}_+$. By the lemmas above, $M' : l^p \to l^p$ has dense image and is bounded, hence (A1), (A2), and (A3) are satisfies and the claim follows by Proposition 4.2.18, because $\omega = (\omega_k)_{k\in\mathbb{N}} \in l^{p/2}_+ \subset l^p_+$. □

4.3 Evolution Equations with Group Generators

For a generalization of the previous result, we will replace the space of bounded uniformly continuous functions on a sequence space by a general Banach space E and the operators D_k by generators of C_0-groups of operators on E.
We proceed as before. First, we construct a strongly continuous semigroup. Then we give the relationship between its generator and the generalized problem.

4.3.1 Construction of a Semigroup

In order to obtain a similar result, the group generators should have properties analogous to those of the operators D_k. We need the concept of l^p-continuity, see [ArDE], Definition 3.3.

Definition 4.3.1. Let $0 < p < \infty$. A commuting family $(T_k)_{k\in\mathbb{N}}$ of contraction semigroups (resp. groups) on a Banach space E is called **l^p-continuous**, if for every $x \in E$ and every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sum_{k=1}^{n} |\lambda_k|^p \le \delta \text{ implies } \left\| \left(\prod_{k=1}^{n} T_k(\lambda_k) \right) x - x \right\| \le \varepsilon$$

for all finite sequences $(\lambda_1, \ldots, \lambda_n)$ in $\mathbb{R}_+$ (in $\mathbb{R}$, respectively in the group case).

Example 4.3.2. The shift groups T_k on the Banach space $BUC(l^p)$ are l^p-continuous. Indeed, let $f \in BUC(l^p)$ and $\varepsilon > 0$. Choose $\delta > 0$ such that $\|x-y\|_{l^p}^p \le \delta$ implies $|f(x) - f(y)| \le \varepsilon$. Let $(\lambda_1, \ldots, \lambda_n)$ be a finite sequences in $\mathbb{R}$, such that $\sum_{k=1}^n |\lambda_k|^p \le \delta$. Then $\|x + \sum_{k=1}^n \lambda_k e_k - x\|^p = \sum_{k=1}^n |\lambda_k|^p \le \delta$ for all $x \in l^p$ and $\|(\prod_{k=1}^n T_k(\lambda_k))\, f - f\| = \sup_{x\in l^p} |f(x + \sum_{k=1}^n \lambda_k e_k) - f(x)| \le \varepsilon$.

Throughout this section, we assume the following hypothesis.

(H1) Let $(A_j)_{j\in\mathbb{N}}$ be a family of group generators, such that the associated family of groups $(T_j)_{j\in\mathbb{N}}$ is l^p-continuous, commuting and contractive on the Banach space E.

Then we will consider the general problem

$$(P_g) \quad \begin{cases} \frac{\partial u(t)}{\partial t} = \sum_{i,j\in\mathbb{N}} a_{ij} A_i\, A_j u(t), \quad t > 0 \\ u(0) = f \quad \in E, \end{cases}$$

where $a_{ij} \in \mathbb{R}$ and we seek a solution $u : [0, \infty) \to E$.
In this situation, we define

$$\begin{aligned} D(\boldsymbol{A}) &= \Big\{ x \in E : x \in \bigcap_i D(A_i),\ A_j x \in \bigcap_i D(A_i), j \in \mathbb{N}, \\ &\qquad\qquad \text{and } \sum_{i,j\in\mathbb{N}} |a_{ij}|\, \|A_i\, A_j x\| < \infty \Big\} \\ \boldsymbol{A}x &= \sum_{i,j\in\mathbb{N}} a_{ij} A_i\, A_j x. \end{aligned}$$

According to Definition 4.2.2, the problem (P_g) is well-posed, if the operator $(\boldsymbol{A}, D(\boldsymbol{A}))$ is closable, and its closure is the generator of a strongly continuous semigroup.
We shall achieve a condition on the coefficients a_{ij}, which is similar to that in Proposition 4.2.24, in order to obtain well-posedness of the Problem (P_g). Hence, we suppose throughout this section the following assumption. Here $1 \le p < \infty$.

(G1) The matrix of coefficients $A = (a_{ij})_{ij\in\mathbb{N}}$ defines a bounded strictly positive selfadjoint operator from l^2 to l^2.

(G2) There exists a sequence $\omega = (\omega_k)_{k\in\mathbb{N}} \in l_+^{p/2}$, such that $\sum_k \frac{a_{kk}^{1/4}}{\omega_k} < \infty$.

Remark 4.3.3. Assumption (G2) implies $\sum_k a_{kk}^{1/4} < \infty$ and $\sum_k \frac{a_{kk}^{1/2}}{\omega_k} < \infty$. Therefore (G2) is more restrictive, than the assumptions in the previous section.

Under these assumptions, the matrix of coefficients admits a decomposition given as $A = (a_{ij})_{i,j\in\mathbb{N}} = M'\Lambda(M')^{\tau}$, where Λ is the diagonal matrix with entries $\omega = (\omega_k)_{k\in\mathbb{N}} \in l_+^{p/2}$, and $M' : l^p \to l^p$ is bounded with dense image. Recall that if $(m'_{ij})_{i,j\in\mathbb{N}}$ is the matrix representation of M' one has the estimate $|m'_{ij}| \leq \frac{a_{ii}^{1/4} a_{jj}^{1/4}}{\sqrt{\omega_j}}$.
Let μ be the Gaussian measure on l^p associated with $\omega = (\omega_k)_{k\in\mathbb{N}} \in l_+^{p/2}$ in the sense of Definition 4.1.25 and $\mu_{M'}$ the image measure, which is also Gaussian on l^p.
We will see, that under the above assumptions, the Problem (P_g) is well-posed. For that, we construct a semigroup G on E and show, that its generator coincides with the closure of the operator $\boldsymbol{A}$.
Assume (H1) and let $\lambda = (\lambda_k)_{k\in\mathbb{N}} \in l^p$. Then by [ArDE], Proposition 3.4, the group product

$$T_\lambda(t) := \prod_k T_k(\lambda_k t)$$

exists, i.e. for all $x \in E$, $T_\lambda(t)x := \lim_{n\to\infty} \prod_{k=1}^n T_k(\lambda_k t)x$ converges uniformly on compact subsets of $\mathbb{R}$. Moreover, $T_\lambda(t)$ defines a strongly continuous group on E.

Lemma 4.3.4. *For $x \in E$ and $t \in \mathbb{R}$, the map $l^p \ni \lambda \mapsto T_\lambda(t)x \in E$ is bounded and uniformly continuous.*

Proof. Since each of the T_k is a contraction, $T_\lambda(t)$ is also a contraction. Thus $\|T_\lambda(t)x\|_E \leq \|x\|$, which implies the boundedness. Let $\varepsilon > 0$. As the family of groups $(T_j)_{j\in\mathbb{N}}$ is l^p-continuous, there exists a $\delta > 0$, such that for every $n \in \mathbb{N}$, $\sum_{k=1}^n |\alpha_k|^p \leq \delta$ implies $\|(\prod_{k=1}^n T_k(\alpha_k))\, x - x\| \leq \varepsilon$. Then for all $\lambda^1, \lambda^2 \in l^p$ such that $\|\lambda^1 - \lambda^2\|_p \leq \delta/\sqrt{t}$, one has

$$\begin{aligned} \|T_{\lambda^1}(t)x - T_{\lambda^2}(t)x\| &\leq \left\| \prod_{k=1}^{\infty} T_k(\lambda_k^1 t)x - \prod_{k=1}^{n} T_k(\lambda_k^1 t)x \right\| \\ &\quad + \left\| \prod_{k=1}^{n} T_k(\lambda_k^1 t)x - \prod_{k=1}^{n} T_k(\lambda_k^2 t)x \right\| \\ &\quad + \left\| \prod_{k=1}^{n} T_k(\lambda_k^2 t)x \prod_{k=1}^{\infty} T_k(\lambda_k^2 t)x \right\| \\ &\leq 3\varepsilon, \end{aligned}$$

if we choose n large enough. □

Thus, the integral in the following definition has a sense.

Definition 4.3.5. For $t \geq 0$ and $x \in E$ define

$$G(t)x = \int_{l^p} T_\lambda(\sqrt{t})x \, d\mu_{M'}(\lambda).$$

Lemma 4.3.6. *The family of operators $(G(t))_{t\geq 0}$ is a C_0-semigroup on E.*

Proof. As the groups T_k are contractive, the product is also contractive. Hence $\|G(t)x\| \leq \int_{l^p} \|T_\lambda(\sqrt{t})\| \, \|x\| \, d\mu_{M'}(\lambda) \leq \|x\|$, because $\mu_{M'}$ is Gaussian. Hence, for all $t \geq 0$, $G(t)$ is a bounded linear operator on E.
Further $G(0)x = \int_{l^p} T_\lambda(0)x \, d\mu_{M'}(\lambda) = \int_{l^p} x \, d\mu_{M'}(\lambda) = x$, hence $G(0) = Id$.
For the semigroup property, we use Proposition 4.1.14, which says, that μ is centered Gaussian, if and only if $\mu \otimes \mu(x, y) = \mu(x \sin\varphi + y \cos\varphi)$ for all $\varphi \in \mathbb{R}$. Observe that for $t, s > 0$ there exists a $\varphi \in \mathbb{R}$ such that $\sin\varphi = \frac{\sqrt{t}}{\sqrt{t+s}}$ and $\cos\varphi = \frac{\sqrt{s}}{\sqrt{t+s}}$. Then

$$\begin{aligned}
G(t)G(s)x &= G(t) \int_{l^p} T_{\tilde{\lambda}}(\sqrt{s})x \, d\mu_{M'}(\tilde{\lambda}) \\
&= \int_{l^p} T_{\hat{\lambda}}(\sqrt{t}) \int_{l^p} T_{\tilde{\lambda}}(\sqrt{s})x \, d\mu_{M'}(\tilde{\lambda}) d\mu_{M'}(\hat{\lambda}) \\
&= \int_{l^p \times l^p} T_{\hat{\lambda}}(\sqrt{t}) T_{\tilde{\lambda}}(\sqrt{s})x \, d(\mu_{M'} \otimes \mu_{M'})(\hat{\lambda}, \tilde{\lambda}) \\
&= \int_{l^p \times l^p} \prod_k T_k(\hat{\lambda}_k \sqrt{t}) \prod_k T_k(\tilde{\lambda}_k \sqrt{s})x \, d(\mu_{M'} \otimes \mu_{M'})(\hat{\lambda}, \tilde{\lambda}) \\
&= \int_{l^p \times l^p} \prod_k T_k(\hat{\lambda}_k \sqrt{t} + \tilde{\lambda}_k \sqrt{s})x \, d(\mu_{M'} \otimes \mu_{M'})(\hat{\lambda}, \tilde{\lambda}) \\
&= \int_{l^p \times l^p} \prod_k T_k\Big(\sqrt{t+s}\Big(\hat{\lambda}_k \frac{\sqrt{t}}{\sqrt{t+s}} + \tilde{\lambda}_k \frac{\sqrt{s}}{\sqrt{t+s}}\Big)\Big)x \, d(\mu_{M'} \otimes \mu_{M'})(\hat{\lambda}, \tilde{\lambda}) \\
&= \int_{l^p \times l^p} T_{\hat{\lambda}\frac{\sqrt{t}}{\sqrt{t+s}} + \tilde{\lambda}\frac{\sqrt{s}}{\sqrt{t+s}}}(\sqrt{t+s})x \, d(\mu_{M'} \otimes \mu_{M'})(\hat{\lambda}, \tilde{\lambda}) \\
&= \int_{l^p} T_\lambda(\sqrt{t+s})x \, d\mu_{M'}(\lambda) \\
&= G(t+s)x.
\end{aligned}$$

We still have to show the strong continuity. But

$$\|G(t)x - x\|_E \leq \int_{l^p} \|T_\lambda(\sqrt{t})x - x\|_E \, d\mu_{M'}(\lambda) \to 0,$$

as $t \to 0$ by Lebesgue's dominated convergence theorem, because one has pointwise convergence $\|T_\lambda(\sqrt{t})x - x\|_E \to 0$ and $\|T_\lambda(\sqrt{t})x - x\|_E \leq 2\|x\|_E$. □

4.3.2 Properties of the Semigroup

Lemma 4.3.7. *Let $1 \le p < \infty$. Let $\{T_j : j \in \mathbb{N}\}$ be an l^p-continuous, commuting family of contraction semigroups with generators A_j, $j \in \mathbb{N}$. Let $\nu = (\nu_n)_{n\in\mathbb{N}} \in l^p$, such that $\nu_n > 0$ for all $n \in \mathbb{N}$. Then the set*

$$\begin{aligned} D^\nu \;=\; & \Big\{ x \in \bigcap_i D(A_i) : A_j x \in \bigcap_i D(A_i) \text{ for all } j \in \mathbb{N}, \\ & \sup_i \nu_i \|A_i x\| < \infty, \text{ and } \sup_{i,j} \nu_i \nu_j \|A_i A_j x\| < \infty \Big\} \end{aligned}$$

is dense in E.

Proof. Let $n \in \mathbb{N}$. Let

$$\begin{aligned} M_m^n\, x &= \int_{(0,1)^m} T_1\left(\frac{\nu_1 s_1}{n}\right) T_2\left(\frac{\nu_2 s_2}{n}\right) \cdots T_m\left(\frac{\nu_m s_m}{n}\right) x\, ds_1 \ldots ds_m \\ &= n^m \int_{(0,\frac{1}{n})^m} T_1(\nu_1 s_1) \cdots T_m(\nu_m s_m)\, x\, ds_1 \ldots ds_m. \end{aligned}$$

Then $M_m^n \in \mathcal{L}(E)$ and $\|M_m^n\| \le 1$. Moreover,

$$\begin{aligned} & \|M_{m+k}^n\, x - M_m^n\, x\| \\ \le\; & \int_{(0,1)^k} \left\| T_{m+1}\left(\frac{\nu_{m+1} s_{m+1}}{n}\right) \cdots T_{m+k}\left(\frac{\nu_{m+k} s_{m+k}}{n}\right) x - x \right\| ds_{m+1} \ldots ds_{m+k} \\ =\; & n^k \int_{(0,\frac{1}{n})^k} \|T_{m+1}(\nu_{m+1} s_{m+1}) \cdots T_{m+k}(\nu_{m+k} s_{m+k})\, x - x\|\, ds_{m+1} \ldots ds_{m+k}. \end{aligned}$$

The right hand side converges to 0 as $m \to \infty$ uniformly in n by the l^p-continuity of the family $\{T_k : k \in \mathbb{N}\}$. Indeed, for $\varepsilon > 0$, let δ be such that $\sum_{j=1}^l |\lambda_j| < \delta$ implies $\left\| \prod_{j=1}^l T_j(\lambda_j) x - x \right\| < \varepsilon$. Then for $\nu \in l^p$ choose m large enough such that $\sum_{j=m}^\infty |\nu_j| < \delta$, and one gets

$$\begin{aligned} & n^k \int_{(0,\frac{1}{n})^k} \|T_{m+1}(\nu_{m+1} s_{m+1}) \cdots T_{m+k}(\nu_{m+k} s_{m+k})\, x - x\|\, ds_{m+1} \ldots ds_{m+k} \\ & \le\; n^k \left(\frac{1}{n}\right)^k \varepsilon = \varepsilon. \end{aligned}$$

Hence, $M^n x := \lim_{m\to\infty} M_m^n\, x$ exists for all $x \in E$ uniformly in n. Therefore $M^n \in \mathcal{L}(E)$ and $\|M^n\| \le 1$. Since $\lim_{n\to\infty} M_m^n x = x$ for all $m \in \mathbb{N}$, by Lebesgue's dominated convergence theorem, it follows, that $\lim_{n\to\infty} M^n x = x$ for all $x \in E$.

Let $j \in \mathbb{N}$. Then for $x \in E$ and $m \geq j$,

$$\begin{aligned}
A_j M_m^n x &= A_j n^m \int_{(0,\frac{1}{n})^m} T_1(\nu_1 s_1) \cdots T_m(\nu_m s_m)\, x\, ds_1 \ldots ds_m \\
&= n A_j \int_0^{\frac{1}{n}} T_j(\nu_j s_j) \left[n^{m-1} \int_{(0,\frac{1}{n})^{m-1}} \prod_{k \neq j} T_k(\nu_k s_k)\, x \prod_{k \neq j} ds_k \right] ds_j \\
&= n A_j \int_0^{\frac{\nu_j}{n}} T_j(\sigma_j) \left[n^{m-1} \int_{(0,\frac{1}{n})^{m-1}} \prod_{k \neq j} T_k(\nu_k s_k)\, x \prod_{k \neq j} ds_k \right] \frac{d\sigma_j}{\nu_j} \\
&= \frac{n}{\nu_j} \left[T_j\left(\frac{\nu_j}{n}\right) - Id \right] \left[n^{m-1} \int_{(0,\frac{1}{n})^{m-1}} \prod_{k \neq j} T_k(\nu_k s_k)\, x \prod_{k \neq j} ds_k \right] \\
&= \frac{n}{\nu_j} n^{m-1} \int_{(0,\frac{1}{n})^{m-1}} \prod_{k \neq j} T_k(\nu_k s_k) \left[T_j\left(\frac{\nu_j}{n}\right) x - x \right] \prod_{k \neq j} ds_k \\
&= \frac{n}{\nu_j} n^{m-1} \int_{(0,\frac{1}{n})^{m-1}} \prod_{k \neq j} T_k(\nu_k s_k) \prod_{k \neq j} ds_k \left[T_j\left(\frac{\nu_j}{n}\right) x - x \right],
\end{aligned}$$

which converges, as $m \to \infty$, by the same argument as above. Since A_j is closed, we obtain $M^n x \in D(A_j)$ and further $\nu_j \|A_j M^n x\| \leq 2n\|x\|$, because T_j is a contraction. Moreover, for $x \in D(A_j)$ the identity $T_j(t)x - x = \int_0^t T_j(s) A_j x$ implies $A_j M^n x = M^n A_j x$.

Hence for all $x \in E$, and all $n \in \mathbb{N}$, one has $M^n M^n x \in \bigcap_k D(A_k)$ as well as $A_j M^n M^n x = M^n A_j M^n x \in \bigcap_k D(A_k)$. As $\nu_j \|A_j M^n M^n x\| \leq 2n\|M^n x\| \leq 2n\|x\|$ and $\nu_i \nu_j \|A_i A_j M^n M^n x\| = \nu_i \nu_j \|A_i M^n A_j M^n x\| \leq \lambda_j 2n \|A_j M^n x\| \leq 4n^2\|x\|$, we obtain $M^n M^n x \in D^\nu$. Since

$$\begin{aligned}
\|M^n M^n x - x\| &\leq \|M^n M^n x - M^n x\| + \|M^n x - x\| \\
&\leq \|M^n\| \, \|M^n x - x\| + \|M^n x - x\| \to 0
\end{aligned}$$

as $n \to \infty$, D^ν is dense in E. □

Lemma 4.3.8. *If $x \in D(A_j)$, then $G(t)x \in D(A_j)$ and $A_j G(t)x = G(t)A_j x$ for all $t \geq 0$.*

Proof. We have to show $T_j(\tau)G(t)x - G(t)x = \int_0^\tau T_j(\sigma)G(t)A_j x\, d\sigma$. Recall that $G(t)x = \int_{l^p} T_\lambda(\sqrt{t})x\, d\mu(\lambda)$, where T_λ is the group product of the commuting groups T_k. Hence, $T_j(\tau)$ and $T_\lambda(\sqrt{t})$ commute also. Thus, by Tonelli's theorem,

$$\begin{aligned} T_j(\tau)G(t)x - G(t)x &= T_j(\tau)\int_{l^p} T_\lambda(\sqrt{t})x\,d\mu(\lambda) - \int_{l^p} T_\lambda(\sqrt{t})x\,d\mu(\lambda) \\ &= \int_{l^p} (T_j(\tau)T_\lambda(\sqrt{t})x - T_\lambda(\sqrt{t})x)\,d\mu(\lambda) = \int_{l^p} T_\lambda(\sqrt{t})(T_j(\tau)x - x)\,d\mu(\lambda) \\ &= \int_{l^p} T_\lambda(\sqrt{t})\Big(\int_0^\tau T_j(\sigma)A_jx\,d\sigma\Big)\,d\mu(\lambda) = \int_{l^p}\int_0^\tau T_\lambda(\sqrt{t})T_j(\sigma)A_jx\,d\sigma\,d\mu(\lambda) \\ &= \int_0^\tau\int_{l^p} T_j(\sigma)T_\lambda(\sqrt{t})A_jx\,d\mu(\lambda)\,d\sigma = \int_0^\tau T_j(\sigma)\Big(\int_{l^p} T_\lambda(\sqrt{t})A_jx\,d\mu(\lambda)\Big)\,d\sigma \\ &= \int_0^\tau T_j(\sigma)G(t)A_jx\,d\sigma. \end{aligned}$$

□

Corollary 4.3.9. If $x \in D(\sum_j A_j)$, i.e. $x \in \bigcap_j D(A_j)$ and $\sum_j \|A_jx\| < \infty$, then $G(t)x \in D(\sum_j A_j)$ and $\sum_j A_jG(t)x = G(t)\sum_j A_jx$ for all $t \geq 0$. The same holds for $\sum_j A_j$ replaced by $\boldsymbol{A}$.

Remark 4.3.10. In the lemma and corollary above, one can replace the semigroup G by either the infinite product $T_\lambda(t) := \prod_k T_k(\lambda_k t)$ or the finite product $\prod_{k=1}^n T_k(\lambda_k t)$ for any $n \in \mathbb{N}$ and any $\lambda \in l^p$.

Lemma 4.3.11. *For all $\nu \in l^p$, D^ν is invariant under the semigroup G.*

Proof. Let $x \in D^\nu$, then in particular $x \in D(A_j)$ for all $j \in \mathbb{N}$. Hence for all $t \geq 0$, by Lemma 4.3.8, $G(t)x \in D(A_j)$ and $A_jG(t)x = G(t)A_jx$ for all $j \in \mathbb{N}$. Moreover $A_jx \in D(A_i)$ for all $i \in \mathbb{N}$, and we can apply Lemma 4.3.8 once more and obtain $G(t)A_jx = A_jG(t)x \in D(A_i)$ and $A_iA_jG(t)x = A_iG(t)A_jx = G(t)A_iA_jx$ for all $i,j \in \mathbb{N}$. Finally $\sup_i \nu_i\|A_iG(t)x\| = \sup_i \nu_i\|G(t)A_ix\| \leq \sup_i \nu_i\|A_ix\| < \infty$ and $\sup_{i,j}\nu_i\nu_j\|A_iA_jG(t)x\| = \sup_{i,j}\nu_i\nu_j\|G(t)A_iA_jx\| \leq \sup_{i,j}\nu_i\nu_j\|A_iA_jx\| < \infty$, which shows that $G(t)x \in D^\nu$. □

Hence D^ν is a core for the generator $\mathcal{A}$ of the semigroup $(G(t))_{t\geq 0}$. Thus, in order to show, that the generator $\mathcal{A}$ coincides with the closure of the operator $\boldsymbol{A}$, it is sufficient to show $\mathcal{A}x = \boldsymbol{A}x$ for all $x \in D^\nu$.
Recall that $\mathcal{A}x = \frac{d}{dt}G(t)x_{|t=0}$ for $x \in D(\mathcal{A})$ and

$$\begin{aligned} G(t)x &= \int_{l^p} T_\lambda(\sqrt{t})x\,d\mu_{M'}(\lambda) = \int_{l^p} T_{M'\lambda}(\sqrt{t})x\,d\mu(\lambda) \\ &= \int_{l^p}\prod_k T_k(\sqrt{t}(M'\lambda)_k)x\,d\mu(\lambda) = \int_{l^p}\prod_k T_k\Big(\sqrt{t}\Big(\sum_j m'_{kj}\lambda_j\Big)\Big)x\,d\mu(\lambda). \end{aligned}$$

Therefore we are interested, if the function $t \mapsto \prod_k T_k(\sqrt{t}(\sum_j m'_{kj}\lambda_j))x$ is differentiable, what we shall obtain as a corollary to the following lemma.

Lemma 4.3.12. *Let* $\beta = (\beta_k)_{k\in\mathbb{N}} \in l^p$ *and* $x \in \bigcap_k D(A_k)$. *Assume further that* $\sum_k |\beta_k| \, \|A_k x\| < \infty$. *Then for all* $\alpha = (\alpha_k)_{k\in\mathbb{N}} \in l^p$,

$$\lim_{h\to 0} \frac{\prod_k T_k(\alpha_k + h\beta_k)x - \prod_k T_k(\alpha_k)x}{h} = \sum_j \beta_j A_j \prod_k T_k(\alpha_k)x.$$

Proof. First, observe that $x \in \bigcap_k D(A_k)$ and $\sum_k |\beta_k| \, \|A_k x\| < \infty$ implies that $\theta \mapsto \sum_j \beta_j A_j \prod_k T_k(\alpha_k + \theta\beta_k)x = \prod_k T_k(\alpha_k + \theta\beta_k) \sum_j \beta_j A_j x$ is continuous, by the strong continuity of the group product. Thus, it suffices to show that

$$\prod_k T_k(\alpha_k + h\beta_k)x - \prod_k T_k(\alpha_k)x = \int_0^h \sum_j \beta_j A_j \prod_k T_k(\alpha_k + \theta\beta_k)x \, d\theta.$$

Recall that the group $\mathcal{T}_n(h) := \prod_{k=1}^n T_k(h\beta_k)$ is generated by the closure of the operator $\sum_{j=1}^n b_j A_j$. Hence one has $\mathcal{T}_n(h)y - y = \int_0^h \mathcal{T}_n(\theta) \sum_{j=1}^n b_j A_j y \, d\theta$ for all $y \in \bigcap_{k=1}^n D(A_k)$. Now let $\beta^n := (\beta_1, \ldots, \beta_n, 0, \ldots)$. Then

$$\prod_k T_k(\alpha_k + h\beta_k^n)x = \prod_k T_k(\alpha_k) \prod_{k=1}^n T_k(h\beta_k^n) = \prod_k T_k(\alpha_k)\mathcal{T}_n(h) = \mathcal{T}_n(h) \prod_k T_k(\alpha_k).$$

If we set $y = \prod_k T_k(\alpha_k)x$, which obviously belongs to $\bigcap_{k=1}^n D(A_k)$, then

$$\prod_k T_k(\alpha_k + h\beta_k^n)x - \prod_k T_k(\alpha_k)x = \mathcal{T}_n(h)y - y = \int_0^h \mathcal{T}_n(\theta) \sum_{j=1}^n \beta_j A_j y \, d\theta$$
$$= \int_0^h \sum_{j=1}^n \beta_j A_j \mathcal{T}_n(\theta) \prod_k T_k(\alpha_k)x \, d\theta = \int_0^h \sum_{j=1}^n \beta_j A_j \prod_k T_k(\alpha_k + \theta\beta_k^n)x \, d\theta.$$

Note that by [ArDE], Proposition 3.4, $\mathcal{T}_n(h)y = \prod_{k=1}^n T_k(h\beta_k)y \to \prod_k T_k(h\beta_k)y$ as $n \to \infty$ uniformly on compact subsets of $[0, \infty)$. Hence, the left hand side $\prod_k T_k(\alpha_k + h\beta_k^n)x - \prod_k T_k(\alpha_k)x \to \prod_k T_k(\alpha_k + h\beta_k)x - \prod_k T_k(\alpha_k)x$, as $n \to \infty$. For the right hand side, observe that for every $\theta \in [0, h]$ and with the notation

$\xi = \sum_{j=1}^n \beta_j A_j x$, one has

$$\begin{aligned}
&\Big\| \sum_{j=1}^n \beta_j A_j \prod_k T_k(\alpha_k + \theta\beta_k^n)x - \sum_{j=1}^\infty \beta_j A_j \prod_k T_k(\alpha_k+\theta\beta_k)x \Big\| \\
&\le \Big\| \sum_{j=1}^n \beta_j A_j \prod_k T_k(\alpha_k + \theta\beta_k^n)x - \sum_{j=1}^n \beta_j A_j \prod_k T_k(\alpha_k+\theta\beta_k)x \Big\| \\
&\quad + \Big\| \sum_{j=1}^n \beta_j A_j \prod_k T_k(\alpha_k + \theta\beta_k)x - \sum_{j=1}^\infty \beta_j A_j \prod_k T_k(\alpha_k+\theta\beta_k)x \Big\| \\
&\le \Big\| \prod_k T_k(\alpha_k + \theta\beta_k^n) \sum_{j=1}^n \beta_j A_j x - \prod_k T_k(\alpha_k+\theta\beta_k) \sum_{j=1}^n \beta_j A_j x \Big\| \\
&\quad + \Big\| \prod_k T_k(\alpha_k+\theta\beta_k)\big(\sum_{j=1}^n \beta_j A_j x - \sum_{j=1}^\infty \beta_j A_j x \big) \Big\| \\
&\le \Big\| \prod_k T_k(\alpha_k + \theta\beta_k^n)\xi - \prod_k T_k(\alpha_k+\theta\beta_k)\xi \Big\| + \sum_{j=n+1}^\infty |\beta_j| \|A_j x\| \to 0,
\end{aligned}$$

as $n \to \infty$, with the same argument as before for the first term and as a consequence of the assumption for the second. Hence by Lebesgue's theorem $\int_0^h \sum_{j=1}^n \beta_j A_j \prod_k T_k(\alpha_k + \theta\beta_k^n)x\, d\theta \to \int_0^h \sum_j \beta_j A_j \prod_k T_k(\alpha_k+\theta\beta_k)x\, d\theta$ and the claim follows by the uniqueness of limits. $\square$

Corollary 4.3.13. Let $x \in \bigcap_k D(A_k)$, $\beta \in l^p$ such that $\sum_j |\beta_j|\, \|A_j x\| < \infty$. Then the function defined by $t \mapsto \prod_k T_k(\sqrt{t}\beta_k)x$ is differentiable for $t > 0$ and the derivative is given by the formula $\frac{d}{dt}\prod_k T_k(\sqrt{t}\beta_k)x = \sum_j \frac{\beta_j}{2\sqrt{t}} A_j \prod_k T_k(\sqrt{t}\beta_k)x$.

Proof. By the above lemma, the application $t \mapsto \prod_k T_k(t\beta_k)x$ is differentiable and has derivative $\sum_j \beta_j A_j \prod_k T_k(t\beta_k)x$. Moreover $t \mapsto \sqrt{t}$ is differentiable for $t > 0$ and we apply the chain rule. $\square$

4.3.3 Well-Posedness

Lemma 4.3.14. *Assume (G1) and (G2) and let μ be the Gaussian measure on l^p associated with $\omega \in l_+^{p/2}$. Then for every $x \in D^\omega$ and $t > 0$ one has $\frac{d}{dt}G(t)x = \frac{1}{2\sqrt{t}} \int_{l^p} \sum_n \prod_k T_k(\sqrt{t}(\sum_l m'_{kl}\lambda_l)) \sum_i m'_{in} A_i x\, \lambda_n \, d\mu(\lambda)$.*

Proof. Fix $\lambda \in l^p$ and let $\beta_k = \sum_l m'_{kl}\lambda_l \in l^p$. Then

$$\begin{aligned}
\sum_j |\beta_j|\, \|A_j x\| &\le \sum_j \sum_l |m'_{jl}|\, |\lambda_l|\, \frac{1}{\omega_j}\omega_j \|A_j x\| \le C \sum_j \frac{1}{\omega_j} \sum_l |m'_{jl}| \|\lambda\|_\infty \\
&\le C \sum_j \frac{1}{\omega_j} \sum_l \frac{a_{jj}^{1/4} a_{ll}^{1/4}}{\sqrt{\omega_l}} \|\lambda\|_\infty \le C \sum_j \frac{a_{jj}^{1/4}}{\omega_j} \Big(\sum_l \frac{a_{ll}^{1/4}}{\omega_l} \sqrt{\omega_l} \Big) \|\lambda\|_\infty \le \tilde{C} \|\lambda\|_p,
\end{aligned}$$

and we can apply the corollary above and obtain

$$\begin{aligned}
\frac{d}{dt}\prod_k T_k(\sqrt{t}\sum_l m'_{kl}\lambda_l)x &= \frac{d}{dt}\prod_k T_k(\sqrt{t}\beta_k)x = \sum_j \frac{\beta_j}{2\sqrt{t}}A_j\prod_k T_k(\sqrt{t}\beta_k)x \\
&= \sum_j \frac{\sum_n m'_{jn}\lambda_n}{2\sqrt{t}}A_j\prod_k T_k(\sqrt{t}\sum_l m'_{kl}\lambda_l)x \\
&= \frac{1}{2\sqrt{t}}\sum_j\sum_n m'_{jn}\lambda_n\prod_k T_k(\sqrt{t}\sum_l m'_{kl}\lambda_l)A_jx \\
&= \frac{1}{2\sqrt{t}}\sum_n\prod_k T_k(\sqrt{t}\sum_l m'_{kl}\lambda_l)\sum_j m'_{jn}A_jx\lambda_n,
\end{aligned}$$

because

$$\begin{aligned}
&\sum_j\sum_n |m'_{jn}|\,|\lambda_n|\,\Big\|\prod_k T_k(\sqrt{t}\sum_l m'_{kl}\lambda_l)A_jx\Big\| \\
&\quad\leq \sum_j\sum_n |m'_{jn}|\,|\lambda_n|\,\|A_jx\| \leq \tilde{C}\|\lambda\|_p < \infty,
\end{aligned}$$

which allows to interchange the order of summation. Moreover, we used the fact that for $n \in \mathbb{N}$ fixed, $\sum_j m'_{jn}A_j\prod_k T_k(.)x = \prod_k T_k(.)\sum_j m'_{jn}A_jx$ for all $x \in \bigcap_j D(A_j)$ such that $\sum_j |m'_{jn}|\,\|A_jx\| < \infty$.
Thus $\prod_k T_k(\sqrt{t}\sum_l m'_{kl}\lambda_l)x$ is differentiable with respect to t for every $\lambda \in l^p$. Moreover, since $\lambda \mapsto M'\lambda$ is continuous, with Lemma 4.3.4 we conclude that the map $\lambda \mapsto \prod_k T_k(\sqrt{t}\sum_l m'_{kl}\lambda_l)x$ is continuous. Also, $\lambda \mapsto \sum_n\sum_j m'_{jn}\lambda_nA_jx$, and thus $\lambda \mapsto \frac{d}{dt}\prod_k T_k(\sqrt{t}\sum_l m'_{kl}\lambda_l)x = \prod_k T_k(\sqrt{t}\sum_l m'_{kl}\lambda_l)\sum_n\sum_j m'_{jn}\lambda_nA_jx$, is continuous. The latter is bounded by $\tilde{C}\|\lambda\|_p$, which is integrable with respect to the Gaussian measure μ by Fernique's theorem.
We conclude with Lebesgue's Theorem that $\int_{l^p}\prod_k T_k(\sqrt{t}\sum_l m'_{kl}\lambda_l)x\,d\mu(\lambda)$ is differentiable and

$$\begin{aligned}
\frac{d}{dt}G(t)x &= \frac{d}{dt}\int_{l^p}\prod_k T_k(\sqrt{t}\sum_l m'_{kl}\lambda_l)x\,d\mu(\lambda) \\
&= \int_{l^p}\frac{d}{dt}\prod_k T_k(\sqrt{t}\sum_l m'_{kl}\lambda_l)x\,d\mu(\lambda) \\
&= \frac{1}{2\sqrt{t}}\int_{l^p}\sum_n\prod_k T_k(\sqrt{t}(\sum_l m'_{kl}\lambda_l))\sum_i m'_{in}A_ix\,\lambda_n\,d\mu(\lambda).
\end{aligned}$$

□

We shall soon see, that

$$\begin{aligned}\int_{l^p} \sum_n \prod_k T_k(\sqrt{t}(\sum_l m'_{kl}\lambda_l)) \sum_i m'_{in} A_i x \, \lambda_n \, d\mu(\lambda) \\ = \ 2\int_{l^p} \sum_n \omega_n \frac{\partial}{\partial \lambda_n} \prod_k T_k(\sqrt{t}(\sum_l m'_{kl}\lambda_l)) \sum_i m'_{in} A_i x \, d\mu(\lambda).\end{aligned}$$

Lemma 4.3.15. *Assume (G1) and (G2) and let μ be the Gaussian measure on l^p associated with $\omega \in l_+^{p/2}$. Then for every $x \in D^\omega$ and $t > 0$, the function $f_n(\lambda) := \prod_k T_k(\sqrt{t}(\sum_l m'_{kl}\lambda_l)) \sum_i m'_{in} A_i x$ belongs to $BUC^1(l^p; E)$ and*

$$\frac{d}{d\lambda_n} f_n(\lambda) = \sqrt{t} \sum_j m'_{jn} A_j f_n(\lambda).$$

Proof. Since $x \in D^\omega$, we have $y := \sum_i m'_{in} A_i x \in \bigcap_k D(A_k)$. Indeed, take a $k \in \mathbb{N}$, then for every $N \in \mathbb{N}$, we have $D(A_k) \ni \sum_{i=1}^N m'_{in} A_i x \to \sum_i m'_{in} A_i x$ and $A_k \sum_{i=1}^N m'_{in} A_i x = \sum_i^N m'_{in} A_k A_i x \to \sum_i m'_{in} A_k A_i x$, because one has the estimate $\sum_i |m'_{in}| \, \|A_k A_i x\| \le \frac{C a_{nn}^{1/4}}{\omega_k \sqrt{\omega_n}} \sum_i \frac{a_{ii}^{1/4}}{\omega_i} < \infty$. Now, since the A_k are closed and k was arbitrary, $y := \sum_i m'_{in} A_i x \in \bigcap_k D(A_k)$ and $A_k \sum_i m'_{in} A_i x = \sum_i m'_{in} A_k A_i x$. Let $r \in \mathbb{N}$ and let $\beta = (\sqrt{t}\, m'_{kr})_k \in l^p$, then

$$\begin{aligned}\sum_k |\beta_k| \, \|A_k y\| &= \sum_k \sqrt{t}\, |m'_{kr}| \left\| A_k \sum_i m'_{in} A_i x \right\| \\ &\le \sqrt{t} \sum_k |m'_{kr}| \sum_i |m'_{in}| \, \|A_k A_i x\| \\ &\le \sqrt{t}\, C \sum_k \frac{|m'_{kr}|}{\omega_k} \sum_i \frac{|m'_{in}|}{\omega_i} < \infty.\end{aligned}$$

Hence for $\alpha = (\sqrt{t}(\sum_l m'_{kl}\lambda_l))_{k\in\mathbb{N}} \in l^p$, we obtain with Lemma 4.3.12,

$$\begin{aligned}\frac{d}{d\lambda_r} f_n(\lambda) &= \lim_{h\to 0} \frac{f_n(\lambda + h e_r) - f_n(\lambda)}{h} \\ &= \lim_{h\to 0} \frac{\prod_k T_k(\alpha_k + \sqrt{t}(m'_{kr} h e_r))y - \prod_k T_k(\alpha_k) y}{h} \\ &= \sum_j \beta_j A_j \prod_k T_k(\alpha_k) y \in BUC(l^p; E).\end{aligned}$$

Since r was arbitrary, $f_n \in BUC^1(l^p; E)$ and in particular we obtain the desired formula $\frac{d}{d\lambda_n} f_n(\lambda) = \sqrt{t} \sum_j m'_{jn} A_j f_n(\lambda)$, if we take $r = n$. □

Corollary 4.3.16. For every $N \in \mathbb{N}$

$$\begin{aligned}
&\int_{l^p} \sum_{n=1}^{N} \prod_k T_k\Big(\sqrt{t}\Big(\sum_l m'_{kl}\lambda_l\Big)\Big) \sum_i m'_{in} A_i x \, \lambda_n \, d\mu(\lambda) \\
&= 2\int_{l^p} \sum_{n=1}^{N} \omega_n \frac{\partial}{\partial \lambda_n} \prod_k T_k\Big(\sqrt{t}\Big(\sum_l m'_{kl}\lambda_l\Big)\Big) \sum_i m'_{in} A_i x \, d\mu(\lambda).
\end{aligned}$$

Proof. We use Lemma 4.1.27 on the functions $f_n \in BUC^1(l^p; E)$, $1 \le n \le N$, where again $f_n(\lambda) = \prod_k T_k(\sqrt{t}(\sum_l m'_{kl}\lambda_l)) \sum_i m'_{in} A_i x \in BUC^1(l^p; E)$. Then

$$\begin{aligned}
&\int_{l^p} \sum_{n=1}^{N} \prod_k T_k\Big(\sqrt{t}\Big(\sum_l m'_{kl}\lambda_l\Big)\Big) \sum_i m'_{in} A_i x \, \lambda_n \, d\mu(\lambda) \\
&= \sum_{n=1}^{N} \int_{l^p} \prod_k T_k\Big(\sqrt{t}\Big(\sum_l m'_{kl}\lambda_l\Big)\Big) \sum_i m'_{in} A_i x \, \lambda_n \, d\mu(\lambda) \\
&= 2\sum_{n=1}^{N} \int_{l^p} \omega_n \frac{\partial}{\partial \lambda_n} \prod_k T_k\Big(\sqrt{t}\Big(\sum_l m'_{kl}\lambda_l\Big)\Big) \sum_i m'_{in} A_i x \, d\mu(\lambda) \\
&= 2\int_{l^p} \sum_{n=1}^{N} \omega_n \frac{\partial}{\partial \lambda_n} \prod_k T_k\Big(\sqrt{t}\Big(\sum_l m'_{kl}\lambda_l\Big)\Big) \sum_i m'_{in} A_i x \, d\mu(\lambda).
\end{aligned}$$

□

We shall use Lebesgue's dominated convergence theorem twice, in order to obtain the desired result.

Lemma 4.3.17. *Assume (G1) and (G2) and let μ be the Gaussian measure on l^p associated with $\omega \in l_+^{p/2}$. Then for every $x \in D^\omega$ and $t > 0$,*

$$\begin{aligned}
&\lim_{N\to\infty} \int_{l^p} \sum_{n=1}^{N} \prod_k T_k\Big(\sqrt{t}\Big(\sum_l m'_{kl}\lambda_l\Big)\Big) \sum_i m'_{in} A_i x \, \lambda_n \, d\mu(\lambda) \\
&= \int_{l^p} \sum_{n=1}^{\infty} \prod_k T_k\Big(\sqrt{t}\Big(\sum_l m'_{kl}\lambda_l\Big)\Big) \sum_i m'_{in} A_i x \, \lambda_n \, d\mu(\lambda)
\end{aligned}$$

Proof. Since $\prod_k T_k(\sqrt{t}(\sum_l m'_{kl}\lambda_l)) \sum_i m'_{in} A_i x = \prod_k T_k(\sqrt{t}(M'\lambda)_k) \sum_i m'_{in} A_i x$, the application $\lambda \mapsto f_n(\lambda) = \prod_k T_k(\sqrt{t}(\sum_l m'_{kl}\lambda_l)) \sum_i m'_{in} A_i x$ is continuous, because $M' : \lambda \mapsto M'\lambda$ is continuous from l^p to l^p and $\lambda \mapsto T_\lambda(\tau)y$ is continuous for all $\tau \in \mathbb{R}$ and $y \in E$ by Lemma 4.3.4. Moreover f_n is bounded, thus for all $N \in \mathbb{N}$, $\sum_{n=1}^N f_n(\lambda)\lambda_n$ is integrable. Further, $\sum_n \sum_i m'_{in} \lambda_n A_i x$ is absolutely convergent, (see e.g. the proof of Lemma 4.3.14), and $\prod_k T_k(\sqrt{t}(\sum_l m'_{kl}\lambda_l))$ is bounded. Therefore

$$\begin{aligned}\lambda \ \mapsto\ & \sum_{n=1}^{\infty}\prod_k T_k\Big(\sqrt{t}\Big(\sum_l m'_{kl}\lambda_l\Big)\Big)\sum_i m'_{in}A_i x\,\lambda_n \\ &= \prod_k T_k\Big(\sqrt{t}\Big(\sum_l m'_{kl}\lambda_l\Big)\Big)\sum_n\sum_i m'_{in}\lambda_n A_i x,\end{aligned}$$

is continuous. Indeed, the map $\lambda \mapsto \sum_n \sum_i m'_{in}\lambda_n A_i x$ is linear and bounded, i.e. $\| \sum_n \sum_i m'_{in}\lambda_n A_i x\| \leq C\|\lambda\|_p$, hence continuous, and $\lambda \mapsto \prod_k T_k(\sqrt{t}(\sum_l m'_{kl}\lambda_l))$ is bounded and strongly continuous. Finally, for every $N \in \mathbb{N}$, one has the estimate $\| \sum_{n=1}^{N}\prod_k T_k(\sqrt{t}(\sum_l m'_{kl}\lambda_l))\sum_i m'_{in}A_i x\,\lambda_n\| \leq C\|\lambda\|_p$, and $\|\lambda\|_p$ is integrable by Fernique's theorem. Hence we can apply Lebesque's dominated convergence theorem, which concludes the proof. □

Lemma 4.3.18. *Assume (G1) and (G2) and let μ be the Gaussian measure on l^p associated with $\omega \in l_+^{p/2}$. Then for every $x \in D^\omega$ and $t > 0$,*

$$\begin{aligned}&\lim_{N\to\infty}\int_{l^p}\sum_{n=1}^{N}\omega_n\frac{\partial}{\partial\lambda_n}\prod_k T_k\Big(\sqrt{t}\Big(\sum_l m'_{kl}\lambda_l\Big)\Big)\sum_i m'_{in}A_i x\,d\mu(\lambda)\\ &= \int_{l^p}\sum_{n=1}^{\infty}\omega_n\frac{\partial}{\partial\lambda_n}\prod_k T_k\Big(\sqrt{t}\Big(\sum_l m'_{kl}\lambda_l\Big)\Big)\sum_i m'_{in}A_i x\,d\mu(\lambda)\\ &= \sqrt{t}\int_{l^p}\prod_k T_k\Big(\sqrt{t}\Big(\sum_l m'_{kl}\lambda_l\Big)\Big)\sum_n\sum_j\sum_i \omega_n\, m'_{jn}\, m'_{in}\, A_j A_i x\,d\mu(\lambda).\end{aligned}$$

Proof. Let $g_n(\lambda) := \frac{\partial}{\partial\lambda_n}\prod_k T_k(\sqrt{t}(\sum_l m'_{kl}\lambda_l))\sum_i m'_{in}A_i x\,d\mu(\lambda)$. Then by Lemma 4.3.15,

$$\begin{aligned}g_n(\lambda) &= \sqrt{t}\sum_j m'_{jn}A_j\prod_k T_k\Big(\sqrt{t}\Big(\sum_l m'_{kl}\lambda_l\Big)\Big)\sum_i m'_{in}A_i x\\ &= \sqrt{t}\prod_k T_k\Big(\sqrt{t}\Big(\sum_l m'_{kl}\lambda_l\Big)\Big)\sum_j\sum_i m'_{jn}m'_{in}A_jA_i x,\end{aligned}$$

as $\sum_j\sum_i |m'_{jn}|\,|m'_{in}|\,\|A_jA_ix\| \leq C\left(\sum_i \frac{|m'_{in}|}{\omega_i}\right)^2 < \infty$, and $\prod_k T_k(\sqrt{t}(\sum_l m'_{kl}\lambda_l))$ is bounded. With the same arguments as in the proof before, g_n is continuous and bounded, hence for every $N \in \mathbb{N}$, $\sum_{i=1}^{N}\omega_n g_n(\lambda)$ is integrable. Moreover,

$$\begin{aligned}\sum_n\sum_j\sum_i |\omega_n|\,|m'_{jn}|\,|m'_{in}|\,\|A_jA_ix\| &\leq C\sum_n\sum_j\sum_i \omega_n\frac{|m'_{jn}|}{\omega_j}\frac{|m'_{in}|}{\omega_i}\\ \leq\ C\sum_n\sum_j\sum_i\omega_n\frac{a_{jj}^{1/4}a_{nn}^{1/4}}{\omega_j\sqrt{\omega_n}}\frac{a_{ii}^{1/4}a_{nn}^{1/4}}{\omega_i\sqrt{\omega_n}} &= C\sum_n a_{nn}^{1/2}\sum_j\frac{a_{jj}^{1/4}}{\omega_j}\sum_i\frac{a_{ii}^{1/4}}{\omega_i} < \infty.\end{aligned}$$

Thus, $\lambda \mapsto \sum_n \omega_n g_n(\lambda) = \prod_k T_k(\sqrt{t}(\sum_l m'_{kl}\lambda_l)) \sum_n \sum_j \sum_i \omega_n\, m'_{jn}\, m'_{in} A_j A_i x$ is also continuous. Finally, for all $N \in \mathbb{N}$, $\sum_{n=1}^N \omega_n g_n(\lambda)$ is bounded, hence we obtain the result with Lebesgue's dominated convergence theorem. □

From uniqueness of limits, we immediately get the announced result.

Corollary 4.3.19. Assume (G1) and (G2) and let μ be the Gaussian measure on l^p associated with $\omega \in l_+^{p/2}$. Then for every $x \in D^\omega$ and $t > 0$

$$\begin{aligned}\int_{l^p} \sum_n \prod_k T_k\Big(\sqrt{t}\Big(\sum_l m'_{kl}\lambda_l\Big)\Big) \sum_i m'_{in} A_i x\, \lambda_n \, d\mu(\lambda) \\ = 2\int_{l^p} \sum_n \omega_n \frac{\partial}{\partial \lambda_n} \prod_k T_k\Big(\sqrt{t}\Big(\sum_l m'_{kl}\lambda_l\Big)\Big) \sum_i m'_{in} A_i x\, d\mu(\lambda).\end{aligned}$$

Now we are well prepared to proof the following well-posedness result.

Proposition 4.3.20. *Assume (G1) and (G2). Then Problem (P_g) is well-posed.*

Proof. We have to show that the operator $(\boldsymbol{A}, D(\boldsymbol{A}))$ is closable and its closure is the generator of a strongly continuous semigroup. For that it suffices to show, that for every $x \in D^\omega$, which is a core for the generator $\mathcal{A}$ of the semigroup G, one has $x \in D(\boldsymbol{A})$ and $\boldsymbol{A}x = \mathcal{A}x$.
If $x \in D^\omega$, then $x \in \bigcap_i D(A_i)$ and $A_j x \in \bigcap_i D(A_i)$ for all $j \in \mathbb{N}$. Moreover

$$\begin{aligned}\sum_{i,j\in\mathbb{N}} |a_{ij}|\, \|A_i\, A_j x\| &= \sum_{i,j\in\mathbb{N}} \Big|\sum_n \omega_n\, m'_{jn}\, m'_{in}\Big|\, \|A_i\, A_j x\| \\ &\le \sum_{i,j\in\mathbb{N}} \sum_n \omega_n\, |m'_{jn}|\, |m'_{in}|\, \|A_i\, A_j x\| < \infty,\end{aligned}$$

because $\sum_n \sum_j \sum_i \omega_n\, m'_{jn}\, m'_{in} A_j A_i x$ is absolutely convergent. Therefore, one has $\boldsymbol{A}x = \sum_n \sum_j \sum_i \omega_n\, m'_{jn}\, m'_{in} A_j A_i x$.
With the previous lemmas, we get for every $x \in D^\omega$ and every $t > 0$,

$$\begin{aligned}\frac{d}{dt} G(t)x &= \frac{1}{2\sqrt{t}} \int_{l^p} \sum_n \prod_k T_k\Big(\sqrt{t}\Big(\sum_l m'_{kl}\lambda_l\Big)\Big) \sum_i m'_{in} A_i x\, \lambda_n\, d\mu(\lambda) \\ &= \frac{1}{2\sqrt{t}} 2 \int_{l^p} \sum_n \omega_n \frac{\partial}{\partial\lambda_n} \prod_k T_k\Big(\sqrt{t}\Big(\sum_l m'_{kl}\lambda_l\Big)\Big) \sum_i m'_{in} A_i x\, d\mu(\lambda) \\ &= \frac{1}{\sqrt{t}} \sqrt{t} \int_{l^p} \prod_k T_k\Big(\sqrt{t}\Big(\sum_l m'_{kl}\lambda_l\Big)\Big) \sum_n \sum_j \sum_i \omega_n\, m'_{jn}\, m'_{in}\, A_j A_i x\, d\mu(\lambda) \\ &= G(t) \sum_n \sum_j \sum_i \omega_n\, m'_{jn}\, m'_{in}\, A_j A_i x = G(t)\boldsymbol{A}x.\end{aligned}$$

Since $t \mapsto G(t)Ax$ is continuous, we can consider the limit, as $t \to 0$. Thus, $x \in D(\mathcal{A})$ and $\mathcal{A}x = \frac{d}{dt} G(t) x_{|t=0} = G(0)\boldsymbol{A}x = \boldsymbol{A}x$. □

4.4 Order Continuous Linear Forms on $BUC(E)$

Order continuous positive linear forms on the space of bounded uniformly continuous functions on a complete metric space are characterized by a notion of tightness. This is equivalent to the functional being represented by a measure. These results are also presented in [ET].
This characterization is used to construct a Gaussian measure for the representation of the heat semigroup as a Gaussian semigroup, see [AbDE] and [AbEK], and therefore presented in this chapter.

4.4.1 Characterization of Order Continuous Linear Forms

Let E be a separable, complete metric space. We denote by $BUC(E)$ the space of functions $f : E \to \mathbb{R}$, which are bounded and uniformly continuous. Then $BUC(E)$ provided with the supremum norm $\|f\|_\infty := \sup_{x\in E} |f(x)|$ is a Banach space. We shall denote by $\mathbf{1}$ the constant-one-function.
We call a sequence $(f_n)_{n\in\mathbb{N}}$ in $BUC(E)$ **decreasing to** 0, if for all $m \geq n \in \mathbb{N}$, $f_m(x) \leq f_n(x)$ and $f_n(x) \to 0$ as $n \to \infty$ for all $x \in E$. We write $f_n \downarrow 0$.

Definition 4.4.1. A functional $\Lambda : BUC(E) \to \mathbb{R}$ is called **positive**, if $\Lambda f \geq 0$, whenever $f \geq 0$.
A positive functional Λ on $BUC(E)$ is called **order continuous**, if for each sequence $(f_n)_{n\in\mathbb{N}}$ in $BUC(E)$ decreasing to 0, the real sequence $\Lambda(f_n)$ is decreasing to 0.
We call a functional Λ on $BUC(E)$ **tight**, if for all $\varepsilon > 0$, there exists a compact set $K \subset E$, such that $|\Lambda(f)| \leq \sup_{x\in K} |f(x)| + \varepsilon\|f\|_\infty$ for all $f \in BUC(E)$.

The aim of this section is to give a direct proof of the following result.

Theorem 4.4.2. *A positive functional Λ on the space $BUC(E)$ mapping the constant-one-function $\mathbf{1}$ onto 1, i.e. $\Lambda\mathbf{1} = 1$, is order continuous if and only if it is tight.*

As a consequence, we will deduce from the usual Riesz representation theorem (for positive functionals on $C(K)$ with K compact) that each order continuous positive functional can be represented by a measure. This result is used in [AbDE]. Below we indicate an alternative way of proof based on the Daniell-Stone theorem.

Proof. ($\Rightarrow$) Let $\{y_i, i \in \mathbb{N}\}$ be a countable, dense subset in E. For $n, k \geq 1$, we set

$$C_{n,k} = \bigcup_{i=1}^{k} \overline{B}(y_i, \tfrac{1}{n}), \quad D_{n,k} = \bigcup_{i=1}^{k} B(y_i, \tfrac{1}{n}).$$

Where $B(x,r)$ is the open ball in E with center x and radius r and $\overline{B}(x,r)$ its closure. Remark that these sets are increasing in k with respect to inclusion.
Let $(f_{n,k})_{n,k}$ be the sequence of functions defined on E by

$$f_{n,k}(x) = \frac{d(x, {}^{C}D_{n,k})}{d(x, C_{2n,k}) + d(x, {}^{C}D_{n,k})}.$$

Then $0 \leq f_{n,k} \leq 1$, $f_{n,k}(x) = 1$ if $x \in C_{2n,k}$ and $f_{n,k}(x) = 0$ if $x \notin D_{n,k}$.
Further $d(C_{2n,k}, {}^{C}D_{n,k}) \geq \frac{1}{2n} > 0$. Consequently the functions $f_{n,k}$ are uniformly continuous on E. Additionally for fixed n, the sequence $(f_{n,k})_k$ is increasing. Indeed, $f_{n,k}(x) = \frac{1}{1+\alpha_{n,k}(x)}$ with $\alpha_{n,k}(x) = \frac{d(x,C_{2n,k})}{d(x,{}^{C}D_{n,k})}$, and since $C_{2n,k} \subset C_{2n,k+1}$ and ${}^{C}D_{n,k+1} \subset {}^{C}D_{n,k}$ we get $\alpha_{n,k+1}(x) \leq \alpha_{n,k}(x)$. As the function $x \longmapsto \frac{1}{1+x}$ is decreasing for $x \geq 0$, it follows that $f_{n,k}(x) \leq f_{n,k+1}(x)$. On the other hand $(f_{n,k})_k \nearrow \mathbf{1}$ as $k \to \infty$.
Now, let $\varepsilon > 0$ and $n \in \mathbb{N}$. Since Λ is order continuous and $\Lambda\mathbf{1} = 1$, there exists $l_n \in \mathbb{N}$ such that

$$\mid \Lambda f_{n,l_n} - 1 \mid \leq \frac{\varepsilon}{2^n}.$$

We can choose the l_n increasing and we set $K_n = \bigcap_{k=1}^{n} C_{k,l_k}$ and $K' = \bigcap_{k=1}^{\infty} C_{\frac{k}{2},l_k}$.
Then K' is compact, since it is closed in E and therefore complete and totally bounded (precompact), because for each arbitrary small radius it can be covered by a finite number of balls.
We set $g_n(x) = f_{1,l_1}(x)f_{2,l_2}(x)...f_{n,l_n}(x)$, then $\operatorname{supp}(g_n) \subset K_n$ and with $g_0 := \mathbf{1}$,

$$\mid \Lambda g_n - 1 \mid \leq \sum_{k=0}^{n-1} \mid \Lambda g_{k+1} - \Lambda g_k \mid \leq \sum_{k=0}^{n-1} \mid \Lambda f_{k+1,l_{k+1}} - 1 \mid \leq \varepsilon \sum_{k=0}^{n-1} \frac{1}{2^{k+1}} \leq \varepsilon.$$

Hence for every $f \in BUC(E)$,

$$\begin{aligned} \mid \Lambda f \mid &\leq \mid \Lambda(f(1-g_n)) \mid + \mid \Lambda f g_n \mid \leq \parallel f \parallel_\infty \mid 1 - \Lambda(g_n) \mid + \sup_{x \in K_n} \mid f(x) \mid \\ &\leq \varepsilon \parallel f \parallel_\infty + \sup_{x\in K_n} \mid f(x) \mid . \end{aligned}$$

We claim that $\liminf_{n\to\infty} \sup_{x \in K_n} \mid f(x) \mid \leq \sup_{z\in K'} |f(z)|$, for every function $f \in BUC(E)$.
It is sufficient to show, that for all $\varepsilon > 0$, there exists an $n \in \mathbb{N}$ such that

$$\sup_{x\in K_n} |f(x)| \leq \sup_{z \in K'} |f(z)| + \varepsilon,$$

which holds, if for all $x \in K_n$, there exists a $z \in K'$, such that $|f(x)| \leq |f(z)| + \varepsilon$.
Since f is uniformly continuous, for $\varepsilon > 0$, there exists a $\delta > 0$, such that $\|x - z\|_E < \delta$ implies $|f(x) - f(z)| < \varepsilon$, which again implies $\big||f(x)| - |f(z)|\big| < \varepsilon$.

Therefore the claim follows, if for all $n \in \mathbb{N}$ and all $x \in K_n$, there exists a $z \in K'$ such that $\|x - z\|_E < \frac{1}{n}$. Indeed, let $n \in \mathbb{N}$ and $x \in K_n = \bigcap_{k=1}^n \bigcup_{i=1}^{l_k} \overline{B}(y_i, \frac{1}{k})$, then for every $1 \leq k \leq n$ there exists a $1 \leq i_k \leq l_k$ such that $x \in \overline{B}(y_{i_k}, \frac{1}{k})$. Observe that for all $1 \leq k \leq n$, $\|y_{i_n} - y_{i_k}\| \leq \|y_{i_n} - x\| + \|x - y_{i_k}\| \leq \frac{1}{n} + \frac{1}{k} \leq \frac{2}{k}$. Hence, if we set for $k > n$, $i_k := i_n$, then

$$z := y_{i_n} \in \bigcap_{k=1}^{\infty} \overline{B}(y_{i_k}, \tfrac{2}{k}) \subset \bigcap_{k=1}^{\infty} \bigcup_{i=1}^{l_k} \overline{B}(y_i, \tfrac{2}{k}) = K'.$$

In particular one has $\|z - x\| \leq \frac{1}{n}$, and the claim follows.
($\Leftarrow$) Let $(f_n)_{n\geq 1}$ be a sequence not identically zero in $BUC(E)$ which decreases to 0. We will show that the sequence of real numbers $(\Lambda(f_n))_{n\geq 1}$ decreases also towards 0. Let $\varepsilon > 0$ and K a compact subset of E such that for all $f \in BUC(E)$

$$|\Lambda f| \leq \frac{\varepsilon}{2 \parallel f_1 \parallel_\infty} \parallel f \parallel_\infty + \sup_{x\in K} \mid f(x) \mid .$$

By the theorem of Dini, the sequence $(f_n)_{n\geq 1}$ converges uniformly to 0 on K. Hence there exists an integer n_ε such that $\sup_{x\in K} |f_n(x)| < \varepsilon/2$ for every $n \geq n_\varepsilon$. It follows that $\mid \Lambda f_n \mid \leq \varepsilon$ for every $n \geq n_\varepsilon$. □

4.4.2 The BUC - Compactification

In order to apply the Riesz representation theorem to $BUC(E)$, we need an identification with some $C(K)$, where K is a compact space, which we call the BUC **- compactification**. The construction is the same as the Stone-Čech compactification, which can be found in [Con], §V.6.
Let E be a metric space and $BUC(E)$ the space of bounded and uniformly continuous functions $f : E \to \mathbb{C}$. If $x \in E$, let $\delta_x : BUC(E) \to \mathbb{C}$ be defined by $\delta_x(f) = f(x)$ for every $f \in BUC(E)$. It is easy to see that $\delta_x \in BUC(E)^*$ and $\|\delta_x\| = 1$.

Lemma 4.4.3. *Let $\varphi : E \to BUC(E)^*$ be defined by $\varphi(x) = \delta_x$. Denote by w^* the weak*-topology in $BUC(E)^*$. Then $\varphi : E \to (\varphi(E), w^*)$ is a homeomorphism, i.e. a bijective continuous mapping with continuous inverse.*

Proof. Let $x_n \to x$ in E, then $f(x_n) \to f(x)$ for all f in $BUC(E)$. This says that $\delta_{x_n} \to \delta_x$ (w^*) in $BUC(E)^*$. Hence $\varphi : E \to (BUC(E)^*, w^*)$ is continuous.
If $x_1 \neq x_2$, then there is an f in $BUC(E)$ such that $f(x_1) = 1$ and $f(x_2) = 0$. Thus $\delta_{x_1} \neq \delta_{x_2}$, and $\varphi : E \to (\varphi(E), w^*)$ is injective. The surjectivity is obvious. For the continuity of the inverse, it suffices to show that $\varphi : E \to (\varphi(E), w^*)$ is an open map. Let U be an open subset of E and let $x_0 \in U$. Then there is an f in $BUC(E)$ such that $f(x_0) = 1$ and $f \equiv 0$ on $E \setminus U$. Let V_1 be the subset of $BUC(E)^*$ defined by $V_1 = \{\psi \in BUC(E)^* : \psi(f) > 0\}$. Then V_1 is w^* open in

$BUC(E)^*$ and $V_1 \cap \varphi(E) = \{\delta_x : f(x) > 0\}$. So if $V = V_1 \cap \varphi(E)$, V is w^* open in $\varphi(E)$ and $\delta_{x_0} \in V \subset \varphi(U)$. Since x_0 was arbitrary, $\varphi(U)$ is open in $\varphi(E)$. Therefore $\varphi : E \to (\varphi(E), w^*)$ is a homeomorphism. □

Theorem 4.4.4. *Let E be a metric space. Then there exists a compact space K such that:*

(i) there is a continuous map $\varphi : E \to K$ with the property that $\varphi : E \to \varphi(E)$ is a homeomorphism;

(ii) $\varphi(E)$ is dense in K;

(iii) the map $C(K) \to BUC(E), f \mapsto f \circ \varphi$ is an isomorphism.

Proof. Let $\varphi : E \to BUC(E)^*$ be the map defined by $\varphi(x) = \delta_x$ and let K be the w^* closure of $\varphi(E)$ in $BUC(E)^*$. By Alaoglu's Theorem and the fact that evidently $\|\delta_x\| = 1$ for all x, K is compact. By the preceding lemma, (i) holds. Assertion (ii) is true by definition. For (iii) it remains to show, that $C(K) \to BUC(E), f \mapsto f \circ \varphi$ is surjective.
Fix g in $BUC(E)$ and define $f : K \to \mathbb{C}$ by $f(\psi) = \langle g, \psi \rangle$ for every ψ in $K \subset BUC(E)^*$. Clearly f is continuous and $f \circ \varphi(x) = f(\delta_x) = \langle g, \delta_x \rangle = g(x)$ and (iii) holds. □

4.4.3 Existence of a probability measure

Now we return to the situation, where E is a separable, complete metric space. Let $\mathfrak{B}$ be the Borel σ-algebra on E. Recall, that a measure μ on $(E, \mathfrak{B})$ is called **regular**, if for every $B \in \mathfrak{B}$

$$\mu(B) = \inf\{\mu(O) : B \subset O \text{ open}\} \text{ and}$$

$$\mu(B) = \sup\{\mu(K) : B \supset K \text{ compact}\}, \text{ if } \mu(B) < \infty.$$

Since E is a separable, complete metric space, every finite measure μ on $(E, \mathfrak{B})$ is regular, see [Coh], Proposition 8.1.10. In particular, a finite measure μ on $(E, \mathfrak{B})$ is **tight**, i.e. for each $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subset E$, such that $\mu(E \setminus K_\varepsilon) < \varepsilon$.
If μ is a probability measure on $(E, \mathfrak{B})$ and if for every $f \in BUC(E)$ we define $\Lambda f := \int_E f \, d\mu$, then Λ is a positive functional on $BUC(E)$, satisfying $\Lambda(\mathbf{1}) = 1$. Moreover, Λ is tight. Indeed, since μ is a tight probability measure, for all $\varepsilon > 0$,

$$\begin{aligned} |\Lambda f| &\leq \int_E |f| \, d\mu = \int_{K_\varepsilon} |f| \, d\mu + \int_{\complement K_\varepsilon} |f| \, d\mu \\ &\leq \sup_{x \in K_\varepsilon} |f(x)| \mu(K_\varepsilon) + \sup_{x \notin K_\varepsilon} |f(x)| \mu(E \setminus K_\varepsilon) \\ &\leq \sup_{x \in K_\varepsilon} |f(x)| + \varepsilon \|f\|_\infty. \end{aligned}$$

Conversely, the following proposition shows, that all tight positive functionals Λ on $BUC(E)$, which satisfy $\Lambda\mathbf{1} = 1$, are integrals with respect to a probability measure.

Proposition 4.4.5. *Let Λ be a positive functional on $BUC(E)$ satisfying $\Lambda\mathbf{1} = 1$. If Λ is tight, then there exists a unique probability measure μ on E, such that $\Lambda f = \int_E f\, d\mu$ for all f in $BUC(E)$.*

Proof. Assume, that there exist two probability measures μ and ν on E such that $\int_E f\, d\mu = \int_E f\, d\nu$ for all f in $BUC(E)$. As E is a metric space, we obtain by [Par], Theorem 5.9, that $\mu = \nu$, hence the measure is unique.
By Theorem 4.4.4, there exists a compact K, such that $BUC(E)$ is isomorphic to $C(K)$. We can view E as a dense subset of K, and then the isomorphism is given as $j : C(K) \to BUC(E),\ f \mapsto f_{|E}$.
Hence Λ defines a positive functional on $C(K)$, mapping the constant-one-function onto 1. Therefore, by the Riesz representation theorem, see [Coh], Theorem 7.2.8, there exists a unique regular probability Borel measure $\tilde{\mu}$ on $(K, \mathfrak{B}(K))$, such that

$$\Lambda f = \int_K f\, d\tilde{\mu} \quad \text{for all } f \in C(K).$$

As Λ is a tight functional on $BUC(E)$, for every $n \in \mathbb{N}$, there exists a compact $K_n \subset E$, such that for all $f \in BUC(E)$,

$$|\Lambda f| \leq \sup_{x \in K_n} |f(x)| + \frac{1}{n}\|f\|_\infty. \tag{4.4}$$

We obtain a sequence $(K_n)_{n\in\mathbb{N}}$ of compact sets, satisfying $K_n \subset K_{n+1} \subset E$. Observe for $f \in C(K)$ that $\sup_{x\in K} |f(x)| = \sup_{x\in E} |f(x)|$, because E is dense in K. Hence (4.4) is also satisfied for all $f \in C(K)$.
As compact subsets of E are also compact in K, the K_n are in particular closed in K, and therefore $K_n \in \mathfrak{B}(K)$ for all $n \in \mathbb{N}$.
Since $\tilde{\mu}$ is regular, for every $\varepsilon > 0$, there exists a compact set $C_\varepsilon \subset K \setminus K_n$, such that $\tilde{\mu}(K \setminus K_n) \leq \tilde{\mu}(C_\varepsilon) + \varepsilon$. Since $C_\varepsilon \cap K_n = \emptyset$, there exists a function $f_\varepsilon \in C(K)$, such that $0 \leq f_\varepsilon \leq 1$, $f_\varepsilon \equiv 1$ on C_ε and $f_\varepsilon \equiv 0$ on K_n.
Therefore

$$\begin{aligned} \tilde{\mu}(K \setminus K_n) &\leq \tilde{\mu}(C_\varepsilon) + \varepsilon \leq \int_K f_\varepsilon\, d\tilde{\mu} + \varepsilon = \Lambda f_\varepsilon + \varepsilon \\ &\leq \sup_{x\in K_n} |f_\varepsilon(x)| + \frac{1}{n}\|f_\varepsilon\|_\infty + \varepsilon \leq \frac{1}{n} + \varepsilon. \end{aligned}$$

As $\varepsilon > 0$ was arbitrary, $\tilde{\mu}(K \setminus K_n) \leq \frac{1}{n}$ for all $n \in N$. Since $K_n \subset K_{n+1}$, $\tilde{\mu}(K \setminus (\bigcup_{n\in\mathbb{N}} K_n)) \leq \inf_{n\in\mathbb{N}} \tilde{\mu}(K \setminus K_n) = 0$.
We claim that $\mathfrak{B}(E) = \{A \in \mathfrak{B}(E) : A \cap \bigcup_{n\in\mathbb{N}} K_n \in \mathfrak{B}(K)\} =: \mathfrak{S}$. It is sufficient to show that $\mathfrak{S}$ is a σ-algebra in E, which contains the closed subsets of E. Indeed,

$E \in \mathfrak{B}(E)$ and $E \cap \bigcup_{n\in\mathbb{N}} K_n = \bigcup_{n\in\mathbb{N}} K_n \in \mathfrak{B}(K)$. Further, if $A \in \mathfrak{B}(E)$ such that $A \cap \bigcup_{n\in\mathbb{N}} K_n \in \mathfrak{B}(K)$, then ${}^{C}\!A = E \setminus A \in \mathfrak{B}(E)$ and $K \setminus (A \cap \bigcup_{n\in\mathbb{N}} K_n) \in \mathfrak{B}(K)$. Hence, because $\bigcup_{n\in\mathbb{N}} K_n \subset E \subset K$,

$$\begin{aligned} {}^{C}\!A \cap \bigcup_{n\in\mathbb{N}} K_n &= (E \setminus A) \cap \bigcup_{n\in\mathbb{N}} K_n = (K \setminus A) \cap \bigcup_{n\in\mathbb{N}} K_n \\ &= K \setminus (A \cap \bigcup_{n\in\mathbb{N}} K_n) \cap \bigcup_{n\in\mathbb{N}} K_n \in \mathfrak{B}(K), \end{aligned}$$

which shows ${}^{C}\!A \in \mathfrak{S}$. Finally, for $A_k \in \mathfrak{S}$, $\bigcup_{k\in\mathbb{N}} A_k \in \mathfrak{B}(E)$ and

$$(\bigcup_{k\in\mathbb{N}} A_k) \cap \bigcup_{n\in\mathbb{N}} K_n = \bigcup_{k\in\mathbb{N}} (A_k \cap \bigcup_{n\in\mathbb{N}} K_n) \in \mathfrak{B}(K).$$

Now, let $C \subset E$ be closed. Then for every $n \in \mathbb{N}$, $C \cap K_n$ is compact in E and K. In particular $C \cap K_n \in \mathfrak{B}(K)$, which implies $C \cap \bigcup_{n\in\mathbb{N}} K_n = \bigcup_{n\in\mathbb{N}} C \cap K_n \in \mathfrak{B}(K)$, i.e. each closed subset of E is an element of the σ-algebra $\mathfrak{S}$.
Now define on $(E, \mathfrak{B}(E))$ the measure μ by $\mu(A) = \tilde{\mu}(A \cap \bigcup_{n\in\mathbb{N}} K_n)$ for every $A \in \mathfrak{B}(E)$. Then for every $f \in BUC(E)$,

$$\Lambda f = \Lambda(j^{-1} f) = \int_K j^{-1} f \, d\tilde{\mu} = \int_{\bigcup_n K_n} f_{|\bigcup_n K_n} \, d\tilde{\mu} = \int_E f \, d\mu,$$

which concludes the proof. □

Obviously, by Lebesgue's dominated convergence theorem, a probability measure on E defines an order continuous linear form on $BUC(E)$ satisfying $\Lambda \mathbf{1} = 1$. Conversely, combining the above results, we obtain

Corollary 4.4.6. For each positive order continuous functional Λ on $BUC(E)$ satisfying $\Lambda \mathbf{1} = 1$, there exists a unique probability measure μ on E, such that $\Lambda f = \int_E f \, d\mu$.

This is also a consequence of the Daniell-Stone Theorem, which can be found in the monographs [Bau], Theorem 7.1.4, [HL], Section 2.2, and [Kö], Chapter V. See also [Ki] for a very short proof. Note that by the Daniell-Stone Theorem the measure μ is given on the smallest σ-algebra, with respect to which all functions in $BUC(E)$ are measurable. In this context, this σ-algebra coincides with the Borel σ-algebra $\mathfrak{B}(E)$ on E, see e.g. the proof of [AbDE], Proposition 2.
Hence for the proof of the one implication in Theorem 4.4.2 one could also argue as follows. The order continuity of Λ implies the existence of a unique measure μ on E, such that $\Lambda f = \int_E f \, d\mu$ for all f in $BUC(E)$. Moreover, μ is a probability measure, because $\Lambda \mathbf{1} = 1$. Since E is a separable, complete metric space, the finite measure μ is tight. Thus, by the considerations before Proposition 4.4.5, Λ is tight.

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Zusammenfassung in deutscher Sprache

Die vorliegende Dissertation stellt einige Resultate vor, die aus Untersuchungen von Formmethoden hervorgingen. Diese spielen eine wichtige Rolle für Evolutionsgleichungen, wenn sich diese als abstraktes Cauchy Problem schreiben lassen, bei dem der lineare Operator zu einer Sesquilinearform assoziiert ist. Dies ist der Fall für die meisten partiellen Differentialgleichungen. Falls diese Form dicht definiert, stetig und elliptisch ist, so ist das zugehörige abstrakte Cauchy Problem wohlgestellt, denn der Operator ist der Generator einer analytischen Kontraktionshalbgruppe. Dann ist man an den Eigenschaften der Lösung interessiert, die direkt von der Form abgelesen werden können, wie Positivität, Kontraktivität und Regularität. Die Situation ist heikler für nicht autonome Cauchy Probleme, wenn die Form auch vom Zeitparameter abhängt. Unter geeigneten Meßbarkeitsannahmen definiert die Familie der assoziierten Operatoren einen Multiplikationsoperator.

Im ersten Kapitel werden wichtige Definitionen und Resultate wiederholt, die im Laufe der Dissertation benutzt werden. Zunächst gehen wir dabei auf vektorwertige Funktionenräume und die Definition des Bochnerintegrals ein. Dann stellen wir die Grundlagen für abstrakte Cauchy Probleme und die Halbgruppentheorie zusammen. Zum Abschluß geben wir eine Einführung in Formmethoden.

Trotz des gemeinsamen Ausgangspunktes hat die Forschung in drei verschiedene Richtungen geführt, die hier in unabhängigen Kapiteln dargestellt werden.

Im zweiten Kapitel beschäftigen wir uns mit Multiplikationsoperatoren. In skalaren Funktionenräumen bieten sie einfache Beispiele, während operatorwertige Multiplikationsoperatoren auf vektorwertigen Räumen komplizierter sind. Ihre Bedeutung erlangen sie von nicht-autonomen Cauchy Problemen. Nach einer Einführung in Vektorverbände untersuchen wir Operatoren auf Vektor- und Banachverbänden, wobei wir uns dabei auf das Zentrum und seine Eigenschaften konzentrieren. Auf dem Banachverband der skalaren p-integrierbaren Funktionen stimmen die Zentrumsoperatoren mit den Multiplikationsoperatoren überein. Wir geben eine passende Definition von Zentrumsoperatoren auf verktorwertigen Funktionenräumen, um eine analoge Charakterisierung bezüglich beschränkter Multiplikationsoperatoren zu erhalten. Dann führt die Betrachtung unbeschränkter Multiplikationsoperatoren über die Spektraltheorie zu Multiplikationshalbgruppen. Auf der Grundlage dieser Resultate erhalten wir eine Charakterisierung von Multiplikationsoperatoren, die zu Sesquilinearformen assoziiert sind, sowohl im skalaren Fall, als auch für operatorwertige Multiplikationsoperatoren.

Das dritte Kapitel behandelt nicht-autonome variationelle Cauchy Probleme. Solche sind assoziiert zu einer Familie von zeitabhängigen linearen Operatoren, von denen jeder von einer stetigen elliptischen Sesquilinearform kommt. Nach einer Einführung in die zugrunde liegenden Räume und ihre Eigenschaften formulieren wir das betrachtete Problem auf verschiedene äquivalente Weisen. Wir erinnern an einen Darstellungssatz, der auf J. L. Lions zurückgeht, und aus dem wir Wohlgestelltheit für das Problem ableiten können. Nach der Existenz und Eindeutigkeit von Lösungen, die stetig von den gegebenen Werten abhängen, interessieren wir uns für ihre Eigenschaften. Zunächst studieren wir Verbandoperationen auf bestimmten zugrunde liegenden Räumen und erhalten dann hinreichende Bedingungen an die Formen, so dass die Lösungen positiv oder submarkovsch sind. Dabei erhalten wir dieselben Bedingungen an die Formen wie sie im autonomen Fall durch die Beurling-Deny Kriterien gegeben sind. Die Untersuchungen zur Regularität ermöglichen uns einen alternativen Beweis der maximalen Regularität im autonomen Fall. Allerdings können diese Methoden nur in einem ganz bestimmten nicht-autonomen Fall angewandt werden. Nicht-autonome Cauchy Probleme können auch mittels Halbgruppenmethoden untersucht werden, wobei hier die Definition der Wohlgestelltheit strenger ist. Wir geben eine Einführung in Evolutionshalbgruppen und -familien. Unsere Charakterisierung von Wohlgestelltheit verallgemeinert das bekannte Resultat für stetige Funktionen. Ein Generationssatz für surjektive und dissipative Operatoren führt uns zur Wohlgestelltheit des nicht-autonomen variationellen Cauchy Problems in einem größeren Raum, und wir können diese Lösung auf den ursprünglichen Raum einschränken. Schließlich untersuchen wir die Invarianz abgeschlossener konvexer Mengen. Um dafür Formmethoden direkt auf nicht-autonome Cauchy Probleme anzuwenden, braucht man verallgemeinerte Formen, zu denen wir eine kurze Einführung geben. Wir charakterisieren die Invarianz abgeschlossener konvexer Mengen unter der zugehörigen Halbgruppe bezüglich der Form. Daraus erhalten wir Beurling-Deny Kriterien, die wir dann auf nicht-autonome Evolutionsgleichungen anwenden können.

Im letzten Kapitel beschäftigen wir uns mit partiellen Differentialgleichungen in unendlichdimensionalen Räumen. Wir benutzen zwar keine Formmethoden, aber ihre Anwendung auf Differentialoperatoren zweiter Ordnung motivierte die Untersuchungen. Nach einer Einführung in Gaußmaße untersuchen wir Gaußhalbgruppen und deren Beziehung zur unendlichdimensionalen Wärmeleitungsgleichung. Auf dieser Basis studieren wir Wohlgestelltheit von partiellen Differentialgleichungen zweiter Ordnung in unendlichdimensionalen Räumen. Die Idee dabei ist, die Matrix der Koeffizienten zu diagonalisieren, wobei die Diagonalmatrix die Bedingungen der Wärmeleitungsgleichung erfüllt, und somit ein Gaußmaß definiert.

Aus dem Bildmaß unter der Transformationsmatix erhält man wiederum eine starkstetige Halbgruppe, die das Problem zweiter Ordnung löst. Dann verallgemeinern wir diese Ergebnisse, indem wir die Ableitungen durch allgemeine Gruppengeneratoren ersetzen. Wie für die Differentialoperatoren beginnen wir mit der Konstruktion einer Halbgruppe und geben Bedingungen an, unter denen der Generator mit dem gewünschten Operator übereinstimmt. Schließlich betrachten wir ordnungsstetige Linearformen auf dem Raum der beschränkten, gleichmäßig stetigen Funktionen auf einem Banachraum. Die dargestellte Charakterisierung wird benutzt, um ein Gaußmaß für die Darstellung der Wärmehalbgruppe zu konstruieren, was den Zusammenhang zum restlichen Kapitel erklärt.

Erklärung

Die vorliegende Arbeit habe ich selbständig angefertigt und keine anderen als die angegebenen Quellen und Hilfmittel benutzt sowie die wörtlich oder inhaltlich übenommene Stellen als solche erkenntlich gemacht.

Ulm, den 28. April 2003

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