Summability of formal solutions for abstract Cauchy problems and related convolution equations

Werner Balser,[*] Anne Duval,[†] and Stephane Malek[†]

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Abstract

In this article we shall study an abstract inhomogeneous Cauchy problem $u' = A\,u + f(t)$, $u(0) = u_0$, where $f(t)$ is a Banach-space-valued function that is holomorphic in a sectorial region G and has an asymptotic power series expansion $\hat{f}(t)$ as $t \to 0$ in G, while A is a possibly unbounded operator. In fact, it shall be natural to consider a slightly more general *integral equation* that, due to the presence of the operator A, is best considered as a singular one. However, note that here we restrict ourselves to the situation where A does not depend upon t, leaving the general case to be discussed later.

Replacing $f(t)$ by $\hat{f}(t)$, the corresponding (formal) Cauchy problem has a unique formal solution $\hat{u}(t)$ that is a power series in t, and we shall investigate its multisummability in the sense of *Jean Ecalle*. If $\hat{u}(t)$ is so summable, then the sum $u(t)$ of $\hat{u}(t)$ is a solution for the inhomogeneous Cauchy problem which is a (Banach-space-valued) holomorphic function in a sectorial region of large opening and asymptotic to $\hat{u}(t)$ as $t \to 0$. However, even if $\hat{u}(t)$ fails to be multisummable, there may still be a solution $u(t)$ that is asymptotic to $\hat{u}(t)$ in G or a subregion of G, and we shall also address the question of existence and/or uniqueness of $u(t)$. Moreover, we shall give a review of some existing results on summability of formal power series solutions of PDE in two and more variables, reformulated to fit into this abstract frame.

Introduction

Throughout this article, we shall consider a fixed, but arbitrary Banach space $\mathbb{X}$ over the field $\mathbb{C}$ of complex numbers, as well as a closed linear operator A, mapping some, typically dense, subspace $\mathbb{D} \subset \mathbb{X}$ into $\mathbb{X}$. In this setting, we shall

[*]Abteilung Angewandte Analysis, Universität Ulm, 89069 Ulm, Germany
[†]UFR de Math. Pures et Appl. Cité Scient. Bat M2, 59665 Villeneuve d'Ascq, France

examine the abstract inhomogeneous Cauchy problem

$$u' \;:=\; \frac{d}{dt}\,u \;=\; A\,u \;+\; f(t)\,, \qquad u(0) \;=\; u_0\,, \tag{0.1}$$

for some vector $u_0 \in \mathbb{D}$ and an $\mathbb{X}$-valued function $f(t)$. As a typical example, let $\mathbb{X}$ be a space of functions in a (complex) variable z, and let A be a differential operator in the variable z. In this situation, the inhomogenuity will also depend upon z, and we shall always assume that it is holomorphic in both variables t and z. Even in general, we restrict ourselves to the case when $f(t)$ is holomorphic in a sectorial region[1] G, having an asymptotic power series expansion $\hat f(t)$ as $t \to 0$ in G.

Since we shall only be interested in solutions of (0.1) that are also holomorphic at least in a subregion of G and have asymptotic power series expansions $\hat u(t)$ as $t \to 0$, this Cauchy problem is equivalent to the integral equation[2]

$$u(t) \;=\; g(t) \;+\; A \int_0^t u(\tau)\; d\tau\,, \qquad g(t) \;=\; u_0 \;+\; \int_0^t f(\tau)\, d\tau\,.$$

For $\mathbb{D} = \mathbb{X}$ and a bounded operator A, the above Cauchy problem always has a unique solution that can be obtained as

$$u(t) \;=\; \mathrm{e}^{tA}\, u_0 \;+\; \int_0^t \mathrm{e}^{(t-\tau)A}\, f(\tau)\, d\tau \;=\; g(t) \;+\; A \int_0^t \mathrm{e}^{(t-\tau)A}\, g(\tau)\, d\tau\,,$$

where e^{tA} is defined by means of the exponential series. This series converges for all values $t \in \mathbb{C}$, and hence the solution is holomorphic in G with values in the Banach space $\mathbb{X}$. However, if the operator is unbounded, which is typically the case in most applications, say, to partial differential equations, then the series for e^{tA} will fail to converge (strongly) for all $t \neq 0$, and hence we cannot use the above formula to find a solution of our Cauchy problem. This fact has been the starting point of the theory of semigroups, giving a meaning to the divergent series for e^{tA}, at least for positive real values of t. However, semigroups shall not be our main line of approach in this article. Instead, we shall let ourselves be guided by the following elementary observation: Suppose that both $g(t)$ and $u(t)$ are $\mathbb{X}$-valued functions that are holomorphic in a disc about the origin. Then both functions can be represented by convergent power series that we choose to denote as

$$g(t) \;=\; \sum_{j=0}^{\infty} \frac{t^j}{j!}\; g_j\,, \qquad u(t) \;=\; \sum_{j=0}^{\infty} \frac{t^j}{j!}\; u_j\,.$$

[1] A sectorial region G is defined as an open subset of a sector $S_{d,\alpha,r} = \{t : 0 < |t| < r,\ 2\,|d - \arg t| < \alpha\}$ such that for every $\beta < \alpha$ there exists a ρ with $0 < \rho < r$ for which the closed sector $\overline{S} = \{t : 0 < |t| \le \rho,\ 2\,|d - \arg t| \le \beta\}$ fits into G. Note that by definition a closed subsector does not contain the origin! The number d, resp. α, shall sometimes be refered to as the *bisecting direction*, resp. the *opening*, of G. In applications one will mostly consider the case of $d = 0$; here, however, d is allowed to be an arbitrary real number.

[2] A proof of this equivalence uses that by assumption the operator A is closed, and hence mild solutions that are differentiable are also classical ones; for details refer to, e. g., [1] or [12].

For any closed operator A, one may proceed as in the proof of Theorem 1 to show that this $u(t)$ is a solution of (0.1) if, and only if,

$$u_j \;=\; g_j \;+\; A\,u_{j-1} \qquad \forall\, j \ge 0\,, \tag{0.2}$$

with the usual interpretation of $u_{-1} = 0$. Therefore, given any vectors $g_j \in \mathbb{X}$ and $u_j \in \mathbb{D}$ for which (0.2) holds, one obtains a *formal solution* of (0.1). However, as shall be seen later, in many interesting situations the series for $u(t)$ obtained in this manner *shall fail to converge for any* $t \neq 0$, even when that for $g(t)$ has a positive radius of convergence. Nonetheless, the recently developped theory of multisummation can still give a meaning to some, but not all, of these divergent power series solutions, and this is what this article is all about!

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1 Examples and the general setting

Here we give a first list of problems that, after suitable reformulation, are special cases of (0.1):

1. For any Banach space $\mathbb{X}$ and operators $A_0, \ldots, A_\kappa$, the inhomogeneous κth order Cauchy problem

$$\sum_{j=0}^{\kappa} A_j\, u^{(j)} \;=\; f(t)\,, \qquad u^{(\nu)}(0) \;=\; u_\nu \qquad \forall\, \nu = 0, \ldots, \kappa - 1\,,$$

   under the assumption that A_κ has an inverse R, can be rewritten in the usual way as a first order system on the Cartesian product $\mathbb{X}^\kappa$. Hence the results obtained here apply to higher order equations as well. However, in concrete cases it may not be advisable to rewrite a higher order equation as a system, since some identities may be easier to obtain for the original equation than for the corresponding system.

2. Let $\mathbb{X}$ be any function space over $\mathbb{C}$, in which functions that are infinitely often differentiable are dense; e. g., $\mathbb{X}$ may be the set $\mathcal{O}_c(D_r)$ of all functions that are holomorphic in the disc D_r of radius $r > 0$ about the origin, and continuous up to its boundary, equipped with the norm

$$\|f\| \;=\; \sup_{|z|\le r} |f(z)|\,.$$

   If the operator $A = \partial_z^2$ is the second derivate, then (0.1) is nothing but the inhomogeneous Cauchy problem for the (complex) heat equation in one spatial dimension. The right hand side $f(t)$ then has to be considered as a function $f(t,z)$ of two variables, while u_0 is a given function $u_0(z)$. In the homogeneous case, (0.1) has a unique formal power series solution

$$\hat{u}(t,z) \;=\; \sum_{j=0}^{\infty} \frac{t^j}{j!}\, u_j(z)\,, \tag{1.1}$$

where the coefficient $u_j(z)$ equals the $2j$th derivative of $u_0(z)$. For general $u_0(z)$, this series fails to converge for any $t \neq 0$. The summability properties of it have been studied in an article of *Lutz, Miyake, and Schäfke* [8], that by now may be considered as the initialization of a number of papers aiming at generalizations of their result. For a more detailed review of this and other results, see Section 7.

3. As a generalization of Example 2, one can consider the same problem in higher spatial dimensions by going to a space of (holomorphic) functions in several variables [5, 9, 10]. Another example of interest is the equation $(\partial_t^\nu - \partial_z^\mu)\, u = 0$, which has been studied in [11]. As was indicated in Example 1, this equation may be rewritten in the form (0.1).

4. For $\mathbb{X}$ as above, let $p(x,y) = \sum_{j=0}^{\kappa} x^{\kappa-j}\, p_j(y)$ be an arbitrary polynomial in two variables and complex coefficients. Then we consider the problem

$$p(\partial_t, \partial_z)\; u \;=\; f(t,z)\,, \tag{1.2}$$

with the right hand side given as a (convergent) power series in t, with coefficients that depend upon z, and which we choose to denote as

$$f(t,z) \;=\; \sum_{j=0}^{\infty} \frac{t^j}{j!}\, f_j(z)\,.$$

A series (1.1) is a formal solution of (1.2) if, and only if, its coefficients satisfy

$$\sum_{\nu=0}^{\kappa} p_\nu(\partial_z)\, u_{j-\nu}(z) \;=\; f_{j-\kappa}(z) \qquad \forall\, j \geq \kappa\,. \tag{1.3}$$

Hence, we may think of determining $u_j(t)$ from this relation, but to do so we have, in general, to solve an inhomogeneous ordinary (linear) differential equation. If μ denotes the order of this equation, then to obtain a unique solution, we choose initial constants u_{jn}, for $0 \leq n \leq \mu - 1$, or equivalently, a polynomial $q_j(z)$ of degree at most $\mu - 1$. Since (1.3) also leaves the initial terms $u_0(z), \ldots, u_{\kappa-1}(z)$ undetermined, we may altogether prescribe a function $u_i(t,z)$ of the form

$$u_i(t,z) \;=\; \sum_{j=0}^{\kappa-1} \frac{t^j}{j!}\, u_j(z) \;+\; \sum_{j=\kappa}^{\infty} \frac{t^j}{j!}\, q_j(z)$$

with arbitrary functions $u_0(z), \dots, u_{\kappa-1}(z)$ in our Banach space $\mathbb{X}$, and polynomials $q_j(z)$, assuming for simplicity that the infinite series converges. A solution of (1.2) then will be a sum of $u_i(t,z)$ and another (unknown) function $u_r(t,z)$, which vanishes at the origin accordingly. Bringing the function $u_i(t,z)$ over to the other side, we can rewrite (1.2) as an equation for $u_r(t,z)$. In the space $\mathbb{Y}$ of functions that are having a zero of order μ at the origin (which is a closed subspace of $\mathbb{X}$), the operator $p_0(\partial_z)$ becomes invertible, and therefore (1.2), according to Example 1, can be rewritten as an abstract Cauchy problem (0.1), provided that we restrict ourselves to right hand sides that are functions with values in $\mathbb{Y}$. For a discussion of existing results on the (multi-)summability of formal solutions of (1.2), refer to Section 7.

5. Linear partial differential in more than two variables and/or with variable coefficients can also be reformulated in the form (0.1), provided that the coefficients do not depend upon t, and some results for such equations shall also be discussed in Section 7.

In view of the above examples, it is natural to think of the following setting as *the standard one* for investigations of convergence resp. multisummability of formal power series solutions of pde. However, most of the results of this paper are valid for other Banach spaces resp. operators as well!

- $\mathbb{X} = \mathcal{O}_c(D_r)^\kappa$, where $\mathcal{O}_c(D_r)$ (as above) denotes the set of functions holomorphic in the disc D_r of radius $r > 0$ and continuous up to its boundary, and κ is a natural number. As norm on $\mathbb{X}$ we may use, e. g.,

$$\left\|(f_1, \dots, f_\kappa)^T\right\| \;=\; \sum_{j=0}^{\kappa} \sup_{|z|\le r} |f_j(z)|\,.$$

- The operator A is a matrix of differential operators, i. e.,

$$A \;=\; [\,a_{\nu\mu}(\partial_z)\,]_{\nu,\mu=1}^{\kappa}\;,$$

  with differential polynomials $a_{\nu\mu}(\partial_z) \in \mathbb{C}[\partial_z]$. The domain $\mathbb{D}$ of A is the subspace of (vector-)functions that are holomorphic in D_r, and so that all derivatives up to the order of the differential operator are continuous up to the boundary. On this domain, one can verify that all differential operators are closed.

It is clear that all but the first one of the above examples may be investigated in this standard situation, but in other examples to follow one better considers a different space and/or operator.

2 Multisummability – a brief review

In this section we briefly recall some definitions and results on k-summability resp. multisummability of formal power series with coefficients in the Banach space $\mathbb{X}$. For more details about this topic, refer to [3].

Let $(x_j)_{j=0}^{\infty}$ be an infinite sequence of elements from a Banach space $\mathbb{X}$, and define a corresponding (formal) power series by means of $\hat{x}(t) = \sum_j t^j x_j$. We then say as follows:

- For $s \geq 0$, we say that the power series $\hat{x}(t)$ *is of Gevrey order* s, provided that constants $C, K > 0$ exist for which

$$\|x_j\| \;\leq\; C\, K^j\, \Gamma(1+sj) \qquad \forall\, j \geq 0\,.$$

  If no such s exists, then $\hat{x}(t)$ is said to be *of infinite Gevrey order.* Note that for $s = 0$, the above inequality is equivalent to saying that the power series has a positive radius of convergence, while otherwise $\|x_j\|$ may grow, roughly speaking, like $(j!)^s$. In any case, $\hat{x}(t)$ is of Gevrey order s if, and only if, its *formal Borel transform of order* $k = 1/s$, which by definition is the power series

$$y(t) \;=\; \sum_{j=0}^{\infty} \frac{t^j}{\Gamma(1+sj)}\; x_j\,,$$

  converges for every $t \in \mathbb{C}$ with $K\,|t| < 1$, with K as above. Observe that traditionally the order of the (formal) Borel transformation is defined as the reciprocal of s rather than s itself!

- For a sectorial region G, an $\mathbb{X}$-valued function $x(t)$ that is holomorphic in G, and a formal power series $\hat{x}(t)$ as above, we write $x(t) \cong \hat{x}(t)$ in G, provided that for every closed subsector $\overline{S}$ in G and every $N \geq 0$ there exists a constant $C_N(\overline{S})$ such that

$$\Big\| x(t) \;-\; \sum_{j=0}^{N-1} t^j\, x_j \Big\| \;\leq\; C_N(\overline{S}) \qquad \forall\, t \in \overline{S}\,. \tag{2.1}$$

  We express this fact in words by saying that $x(t)$ is *asymptotic to* $\hat{x}(t)$ in G. For some $s \geq 0$, if constants C and K exist that may depend upon $\overline{S}$ but not upon N, such that $C_N(\overline{S}) \leq C\, K^N\, \Gamma(1+sN)$ holds for all $N \geq 0$, then we say that $\hat{x}(t)$ is *the Gevrey-asymptotic of* $f(t)$ *of order* s, writing $x(t) \cong_s \hat{x}(t)$ in G. Note that $x(t) \cong_s \hat{x}(t)$ in G implies that the series $\hat{x}(t)$ is of Gevrey order s in the sense defined above, and in particular for $s = 0$ one obtains that $\hat{x}(t)$ has a positive radius of convergence, that $f(t)$ is holomorphic at the origin, and that $\hat{x}(t)$ is nothing but its power series expansion.

- For $k > 0$ and a real number d, the series $\hat{x}(t)$ is said to be *k-summable in the direction* d, provided that it is of Gevrey order $s = 1/k$ (such that its formal Borel transformation $y(t)$ of order k is holomorphic in a disc of positive radius about the origin), and in addition, there exist numbers $\delta, C, K > 0$ for which $y(t)$ can be continued into the sector $S_{d,\delta} = \{t : |d - \arg t| < \delta/2\}$ and satisfies the estimate

$$\|y(t)\| \;\leq\; C\; \exp\,[K\,|t|^k] \qquad \forall\, t \in S_{d,\delta}\,. \tag{2.2}$$

If this is so, then the function $x(t)$ defined by the integral

$$x(t) \;=\; t^{-k} \int_0^{\infty(d)} \mathrm{e}^{-(\tau/t)^k} \; y(\tau)\; dt^k \tag{2.3}$$

is referred to as the *k-sum of $\hat{x}(t)$ in the direction d*, or for short as the sum of $\hat{x}(t)$, in case k and d are clear from the context. Note that the integral (2.3) converges for all t with $\mathrm{Re}\,(K - t^{-k}\mathrm{e}^{ikd}) < 0$, which describes a sectorial region in the t-plane of opening π/k and bisecting direction $\arg t = d$. Since instead of the ray $\arg\tau = d$, we may choose a different ray of integration, provided that we stay in the sector $S_{d,\delta}$, we can even holomorphically continue the sum $x(t)$ into a sectorial region G_d of opening larger than π/k. In this region we have $x(t) \cong_s \hat{x}(t)$, and owing to the large opening of G_d, this property alone determines $x(t)$ uniquely in terms of the formal series with which we started (for a proof of this fact, refer to [3]).

- Aside from *J. Ecalle's* definition, there are several equivalent characterizations of multisummability. Here, we shall be content with the following one: *A series $\hat{x}(t)$ is multisummable if, and only if, numbers $k_1 > \ldots > k_q > 0$ and arbitrary real numbers $d_1, \ldots, d_q$ satisfying*

$$2\,|d_j \;-\; d_{j-1}| \;\le\; \pi\,(1/k_{j-1} \;-\; 1/k_j) \qquad \forall\, j = 2, \ldots, q \tag{2.4}$$

exist, for some integer $q \ge 2$, such that $\hat{x}(t) = \hat{x}_1(t) + \ldots + \hat{x}_q(t)$, with $\hat{x}_j(t)$ being k_j-summable in the direction d_j. Rouhgly speaking, this means that a series is multisummable provided that it can be decomposed into a sum of finitely many series that are k_j-summable, for (distinct) values $k_j > 0$. Note that the terms $\hat{x}_j(t)$ in this decomposition are not uniquely defined by $\hat{x}(t)$. However, if $x_j(t)$ denotes the sum of $\hat{x}_j(t)$, then $x(t) := x_1(t) + \ldots + x_q(t)$ is, according to the restriction (2.4), holomorphic in a sectorial region whose opening exceeds π/k_1, and *one can show that $x(t)$ is, in fact, independent of the decomposition of $\hat{x}(t)$ into the sum of terms $\hat{x}_j(t)$, hence in this sense it is uniqely defined* by means of $\hat{x}(t)$, so that we can, and shall, refer to it as *the sum of $\hat{x}(t)$*.

Remark 1: Note that both Gevrey order and summability of a series may depend upon the Banach space $\mathbb{X}$ in the following sense: Let $\mathbb{Y}$ be another Banach space such that $\mathbb{X}$ can be continuously embedded into $\mathbb{Y}$ and, considered as a subset of $\mathbb{Y}$, is dense in, but not equal to, $\mathbb{Y}$. E. g., we may take $\mathbb{X} = L_{p_1}(I)$ and $\mathbb{Y} = L_{p_2}(I)$, with a compact interval I and $1 \le p_2 < p_1 \le \infty$. Then the two norms $\|\cdot\|_{\mathbb{X}}$ and $\|\cdot\|_{\mathbb{Y}}$, restricted to $\mathbb{X}$, cannot be equivalent. Hence, for every $j \ge 0$ there exists $x_j \in \mathbb{X}$ with $\|x_j\|_{\mathbb{Y}} = 1$, but $\|x_j\|_{\mathbb{X}} \ge \Gamma(1 + j^2)$, and in the special case mentioned above, these x_j may be chosen as holomorphic on some disc containing I, or even polynomials, since the set of these functions is dense. Then, the power series $\hat{x}(t) = \sum_j t^j\, x_j$ is convergent in $\mathbb{Y}$ for $|t| < 1$, hence is trivially k-summable in every direction d, for every value $k > 0$. On the other hand, in $\mathbb{X}$ the series $\hat{x}(t)$ is of infinite Gevrey order and therefore cannot be multisummable. Whether one can even find a series $\hat{x}(t)$ that in $\mathbb{Y}$ is not convergent but, say, 1-summable in a direction d, while it fails to be multisummable in $\mathbb{X}$, is not obvious and seems to be an open problem. □

Remark 2: For later use, we wish to point out that the above sequence (x_j) can be chosen to be linearly independent. To see this, observe first that the span of the x_j has infinite dimension, since any two norms on a finitely dimensional vector space are equivalent. Therefore, we can choose a subsequence (x_{j_k}) which is linearly independent. Since $j_k \geq k$, we have that $\|x_{j_k}\|_{\mathbb{X}} \geq \Gamma(1+j_k^2) \geq \Gamma(1+k^2)$, while $\|x_{j_k}\|_{\mathbb{Y}} = 1$, for every $k \geq 0$. □

Remark 3: Observe that in the examples given above one wishes to analyze k-summability of a formal solution $\hat{u}(t,z)$ that is a power series in t with coefficients that depend holomorphically on z. As it turns out, it is natural to always write $\hat{u}(t,z)$ as a series of the form (1.1), and therefore its formal Borel transform of order k is equal to

$$y(t,z) \;=\; \sum_{j=0}^{\infty} \frac{t^j}{j!\,\Gamma(1+j/k)}\; u_j(z)\,.$$

Instead of this series, it is more convenient to study another one that is obtained from (1.1) through an application of *J. Ecalle's deceleration operator* and is given by

$$\tilde{y}(t,z) \;=\; \sum_{j=0}^{\infty} \frac{t^j}{\Gamma(1+sj)}\; u_j(z)\,,$$

with $s = 1 + 1/k$. As has been shown in [3], convergence and properties of analytic continuation and growth of both functions are identical, hence we have that a series of the form (1.1) is k-summable in a direction d if, and only if, $\tilde{y}(t,z)$ can be continued into a, typically small, sector with bisecting direction d and satisfies a growth estimate analogous to (2.2). □

For later use we prove the following result for k-summability that is a version of [3, Theorem 34] for closed operators:

Lemma 1 *Let a formal series $\hat{x}(t) = \sum_j t^j\, x_j$ have coefficients in $\mathbb{D}$, so that a formal, i. e. termwise, application of A is defined and results in the series $\hat{y}(t) = A\,\hat{x}(t) = \sum_j t^j\, A\, x_j$. If both series are k-summable in a direction d, and if $x(t)$ and $y(t)$ denote their sums, then both are holomorphic in a sectorial region G of opening larger than π/k and bisecting direction d. Moreover, the values of $x(t)$ are in $\mathbb{D}$ for every $t \in G$, and*

$$y(t) \;=\; A\,x(t) \qquad \forall\; t \in G\,.$$

Proof: The statements on the holomorphy of $x(t)$ and $y(t)$ are clear by definition of k-summability. In order to prove the identity on G, note that the domain

$\mathbb{D}$ of the closed operator A is a Banach space with respect to the graph norm $\|x\|_A = \|x\| + \|A\,x\|$. From the definition of k-summability we obtain that the series $\hat{x}(t)$, when regarded in this new Banach space $\mathbb{D}$, is again k-summable in the direction d. Since A is a bounded operator from $\mathbb{D}$ to $\mathbb{X}$, we may apply [3, Theorem 34] to complete the proof. □

3 Formulation of the problems to be discussed

While our main interest is in solutions of an abstract Cauchy problem (0.1), it shall in view of Remark 3 be convenient to study a more general integral equation instead: For a fixed real parameter $s > 0$, let $g_s(t)$ be a given function that is holomorphic in a sectorial region G and has values in the space $\mathbb{X}$. Moreover, we also assume that $g_s(t) \cong \hat{g}_s(t)$ as $t \to 0$ in G, with some power series that we choose to denote as

$$\hat{g}_s(t) \;=\; \sum_{j=0}^{\infty} \frac{t^j}{\Gamma(1+sj)}\, g_j\,, \qquad g_j \in \mathbb{D} \quad j \ge 0\,. \tag{3.1}$$

For these data, we wish to investigate existence and/or uniqueness of solutions of the integral equation

$$u_s(t) \;=\; g_s(t) \;+\; A \int_0^t \frac{(t^{1/s} - \tau^{1/s})^{s-1}}{\Gamma(s)} \;\; u_s(\tau)\; d\tau^{1/s}\,,$$

where $u_s(t)$ is supposed to be another function that is holomorphic in G, and has an asymptotic expansion $\hat{u}_s(t)$ which is natural to denote as

$$\hat{u}_s(t) \;=\; \sum_{j=0}^{\infty} \frac{t^j}{\Gamma(1+sj)}\, u_j\,. \tag{3.2}$$

The notation suggests that the coefficients u_j are independent of s, and this is shown in Theorem 1 as a consequence of the closedness of A. Introducing the operator

$$A_s\, u(t) := A \int_0^t \frac{(t^{1/s} - \tau^{1/s})^{s-1}}{\Gamma(s)}\; u(\tau)\; d\tau^{1/s} \;=\; t\,A \int_0^1 \frac{(1-x)^{s-1}}{\Gamma(s)}\; u(tx^s)dx\,,$$

acting on functions that are holomorphic in G and at least integrable at the origin, and so that the values of the integral are in the domain $\mathbb{D}$ of A, we may rewrite the above integral equation in compact form as

$$u_s(t) \;=\; g_s(t) \;+\; A_s\, u_s(t)\,. \tag{3.3}$$

It is not difficult to see that, for fixed $t \in G$, the operator A_s has a limit as $s \to 0$ which is $A_0\, u(t) := t\,A\,u(t)$. Hence equation (3.3) makes good sense for $s = 0$ as well, becoming equal to

$$u_0(t) \;=\; g_0(t) \;+\; t\,A\,u_0(t)\,.$$

This equation is especially interesting in the context of singular perturbations of ordinary differential equations; to see this, compare the examples in Section 4.

It is not clear at this moment whether (3.3) has any solution $u(t)$ that satisfies our requirements, but if it does, then the coefficients u_j of its asymptotic expansion are uniquely defined, owing to te closedness of the operator A, and are in fact *independent of* s:

Theorem 1 *For* $s \geq 0$*, suppose that* $g_s(t)$ *and* $u_s(t)$*, as described above, satisfy* (3.3)*. Then* (0.2) *holds, hence in particular all* u_j *are in* $\mathbb{D}$*, and are independent of* s*, for* $j \geq 0$*.*

Proof: Suppose that $f(t)$ and $g(t)$ both are holomorphic in G, with derivatives that are integrable at the origin, hence $f(t) = f_0 + \int_0^t f'(\tau)\, d\tau$, and analogously for $g(t)$. Using [1, Proposition 1.1.6], we conclude that if $f(t) = A\, g(t)$ for $t \in G$, then $f'(t) = A\, g'(t)$ for $t \in G$ follows, owing to the closedness of A. Hence for fixed $j \geq 0$ and $s > 0$, equation (3.3) may be differentiated j-times to show

$$\begin{aligned} u_s^{(j)}(t) &= g_s^{(j)}(t) \;+\; A \int_0^1 \frac{(1-x)^{s-1}}{\Gamma(s)}\, \partial_t^j\; t\, u_s(tx^s)\, dx \\ &= g_s^{(j)}(t) \;+\; A \int_0^1 \frac{(1-x)^{s-1}}{\Gamma(s)} \left[t\, x^{sj}\, u_s^{(j)}(tx^s) \;+\; j x^{s(j-1)}\, u_s^{(j-1)}(tx^s)\right] dx. \end{aligned}$$

The theory of asymptotic expansions implies that for $t \to 0$ in G we have

$$u_s^{(j)}(t) \;\to\; \frac{j!}{\Gamma(1+sj)}\, u_j\,, \quad g_s^{(j)}(t) \;\to\; \frac{j!}{\Gamma(1+sj)}\, g_j\,,$$

and the same for $j-1$ instead of j (provided that $j \geq 1$). This, together with the closedness of A, completes the proof for $s > 0$. The case $s = 0$ can be proven analogously. □

Now suppose that sequences $(g_j)_0^\infty$ from the space $\mathbb{X}$ and $(u_j)_0^\infty$ from $\mathbb{D}$ are given, such that (0.2) holds. Then (3.3) holds formally, i. e. to say termwise, for the series $\hat{u}_s(t)$ and $\hat{g}_s(t)$ in place of $u_s(t)$ and $g_s(t)$. Hence it is natural to ask the following non-trivial questions for every fixed $s \geq 0$:

- Let a sectorial region G and a function $g_s(t)$ that is holomorphic in G be given, and assume that $g_s(t) \cong \hat{g}_s(t)$ in G. Can we then find another function $u_s(t)$ that satisfies the equation (3.3) and is asymptotic to the unique formal solution $\hat{u}_s(t)$ in G, and if so, is this solution even unique?
- In the situation of the previous item, assume that $g_s(t) \cong_\sigma \hat{g}_s(t)$ in G, for some $\sigma \geq 0$. If a solution $u_s(t)$ exists, do we have $u_s(t) \cong_\sigma \hat{u}_s(t)$ in G?
- In the situation described in the first item, assume that $\hat{g}(t)$ is multi-summable and that $g(t)$ is its sum. Then does the same hold for $\hat{u}(t)$ and $u(t)$?

While it is not clear that the formal solution of (3.3) will be multisummable, it follows from our next result that if so, then under a natural assumption on A, its sum is a solution of (3.3). In this context, we shall use the following terminology:

- We say that A is *summability-preserving*, if for every formal series $\hat{x}(t) = \sum_j t^j \, x_j$, with $x_j \in \mathbb{D}$ we have that k-summability of $\hat{x}(t)$ in a direction d implies the same for the series $A\, \hat{x}(t) = \sum_j t^j \, A\, x_j$.

Bounded operators are always summability-preserving, as has been shown in [3, Theorem 35]. An unbounded operator, in general, will not have this property, as it may even map a summable series to one of infinite Gevrey order. However, it follows from the definition of k-summability and Cauchy's integral formula for derivatives that a differential operator, in our standard situation, has this property. Whether or not a closed operator always is summability-preserving, is an open question.

Theorem 2 *Suppose that $\hat{u}_s(t)$ and $\hat{g}_s(t)$ are so that* (3.3) *holds formally, and let $k > 0$ and $d \in \mathbb{R}$ be given.*

(a) *If both $\hat{u}_s(t)$ and $\hat{g}_s(t)$ are k-summable in the direction d, then their sums $u_s(t)$ and $g_s(t)$ are holomorphic in a sectorial region G of opening larger than π/k and bisecting direction d, and* (3.3) *holds for all $t \in G$.*

(b) *If A is summability-preserving, then k-summability of $\hat{u}_s(t)$ in the direction d implies the same for $\hat{g}_s(t)$.*

Proof: It follows from the definition that k-summability of $\hat{u}_s(t)$ in the direction d implies the same for the series

$$\hat{x}_s(t) \;:=\; \sum_{j=1}^{\infty} \frac{t^j}{\Gamma(1+sj)} \; u_{j-1} \,,$$

and the respective sums $u_s(t)$ and $x_s(t)$ are related by

$$x_s(t) \;=\; \int_0^t \frac{(t^{1/s} - \tau^{1/s})^{s-1}}{\Gamma(s)} \; u_s(\tau) \; d\tau^{1/s} \,.$$

Formally we have that $\hat{u}_s(t) = \hat{g}_s(t) + A\, \hat{x}_s(t)$, hence (b) follows. For (a), use Lemma 1. □

Part (b) of the last theorem says that for a formal solution of (3.3) to be k-summable in a direction d, the summability (in the same sense) of the formal series $\hat{g}_s(t)$ is a necessary condition. Whether or not it is sufficient, too, at least for some very special operators A, shall be discussed later.

4 Examples – Part II

The following additional examples show that equation (3.3), for the case $s = 0$, is related to singular perturbations of linear ordinary differential equations:

6. For a non-zero complex constant a and a parameter ε, consider the simple inhomogeneous ODE
$$\varepsilon\, x' \;=\; a\, x \;-\; f(z),$$
with $f(z)$ holomorphic near the origin. This is the easiest example of a *singular perturbation problem.* Clearly, this equation can be rewritten as (3.3), with $s = 0$, by changing ε/a into t and setting $A = \partial_z$, and then has the formal solution
$$\hat{x}(t,z) \;=\; a^{-1} \sum_{j=0}^{\infty} t^j \; f^{(j)}(z)\,.$$
As we shall see in Section 7, results for the multisummability of the formal solution of this problem are strongly analogous to the ones for Example 1.

7. Instead of the previous example, consider
$$z^{r+1}\, \varepsilon\, x' \;=\; a\, x \;-\; f(z),$$
with a (non-zero) natural number r. This is an equation with an irregular-singular point at the origin, but is still simple enough to be solved explicitly. Nonetheless, the results on the multisummability of its formal solution change drastically, compared to those for the previous example.

8. In [6], the following linear system of ODE has been investigated:
$$z^{r+1}\, \varepsilon\, x' \;=\; A(z,\varepsilon)\, x \;-\; f(z,\varepsilon),$$
where the $n \times n$ matrix $A(z,\varepsilon)$ and the vector $f(z,\varepsilon)$ both are holomorphic near the origin of $\mathbb{C}^2$, and where $A(0,0)$ is invertible. So again, when $A(z,\varepsilon) = A(z)$ is independent of ε, such systems can be rewritten in the form (0.1).

The examples given here and earlier show that (3.3) is particularly interesting in the cases $s = 0$ and $s = 1$. Even for other integer values of $s \ge 0$, one may rewrite (3.3) in the form
$$u_s(t^s) \;=\; g_s(t^s) \;+\; A \int_0^t \frac{(t-\tau)^{s-1}}{\Gamma(s)} \quad u_s(\tau^s)\; d\tau\,,$$
and then differentiate s times to see that it is equivalent to a higher order differential equation for $u_s(t^s)$ that can be brought into the form (0.1). This shall be not used here, however.

5 Bounded operators

In this section, we consider a bounded operator A on the domain $\mathbb{D} = \mathbb{X}$. In this situation, the following result is easily obtained:

Theorem 3 *For $\mathbb{D} = \mathbb{X}$ and a bounded operator A, let g_j and u_j for $j \geq 0$ be given, and assume that* (0.2) *holds. Then for any $s \geq 0$, if $g_s(t)$ is holomorphic in a sectorial region G, and $g_s(t) \cong \hat{g}_s(t)$ in G, there exists exactly one solution $u_s(t)$ of* (3.3)*, and this solution is holomorphic in G in case of $s > 0$, resp. in $G_r := G \cap \{|t| < r\}$ with sufficiently small $r > 0$ for $s = 0$, and $u_s(t) \cong \hat{u}_s(t)$ in G resp. G_r. Moreover, if the asymptotic of $g_s(t)$ is of some Gevrey order $\sigma \geq 0$, then the same holds true for that of $u_s(t)$. In particular, given $k > 0$, and $d \in \mathbb{R}$, the series* (3.2) *is k-summable in the direction d if, and only if, the series* (3.1) *is so summable.*

Proof: Let G and $g_s(t)$ be as in the theorem, and assume that $s > 0$; the (much simpler) proof for $s = 0$ is well-known and will be left to the reader. Define the following operator-valued functions:

$$T_s(t;A) \;=\; \sum_{j=0}^{\infty} \frac{t^j}{\Gamma(1+sj)}\, A^j\,, \qquad U_s(t;A) \;=\; \sum_{j=1}^{\infty} \frac{t^{j-1/s}}{\Gamma(sj)}\, A^j. \tag{5.1}$$

The function $T_s(t;A)$ is entire, of exponential order $1/s$ and finite type, and is *Mittag-Leffler's function* [3, p. 233] extended to operators, while $U_s(t;A)$ is related to $T_s(t;A)$ by the identity

$$U_s(t^s;A) \;=\; \frac{d}{dt}\, T_s(t^s;A) \qquad \forall\; t \neq 0\,.$$

The purpose of these functions here is as follows: Define

$$\left.\begin{aligned} u_s(t) \;&=\; g_s(t) \;+\; \int_0^t U_s\big((t^{1/s}-\tau^{1/s})^s;A\big)\; g_s(\tau)\; d\tau^{1/s} \\ &=\; T_s(t;A)\, u_0 \;+\; \int_0^t T_s\big((t^{1/s}-\tau^{1/s})^s;A\big)\; f_s(\tau)\; d\tau^{1/s} \end{aligned}\right\}\quad \forall\; t \in G \tag{5.2}$$

with the function $f_s(t)$ related to $g_s(t)$ by means of the identity

$$f_s(t^s) \;=\; \frac{d}{dt}\, g_s(t^s) \qquad \forall\; t^s \in G\,.$$

Hence in particular we have

$$f_s(t) \;\cong\; \sum_{j=1}^{\infty} \frac{t^{j-1/s}}{\Gamma(sj)}\, g_j \qquad \text{in } G\,.$$

Then $u_s(t)$ is holomorphic in G and can be shown to satisfy (3.3). Uniqueness of the solution follows in the usual fashion by estimating the integral on the right of (3.3). Finally, the asymptotic expansion is obtained by defining $u_{s,j_0}(t)$ and $f_{s,j_0}(t)$ as in the proof of Theorem 1, then verifying that $t^{j_0}\, u_{s,j_0}(t)$ solves the integral equation (3.3), but with $t^{j_0}\,(g_{s,j_0}(t) + \Gamma^{-1}(1+sj_0) A\, u_{j_0-1})$ replacing $g_s(t)$. Estimating the resolvent formula (5.2) for this case, one can show boundedness of $u_{s,j_0}(t)$ as $t \to 0$ in G. □

6 Unbounded operators

In this section, we try to generalize results from the previous section to unbounded operators. To do this, we consider a fixed closed operator A, which we assume to be *holomorphicity-preserving* in the following sense: *If $g(t)$ is holomorphic in a region $G \subset \mathbb{C}$ and has values in the domain $\mathbb{D}$ of A, then $A\, g(t)$ is also holomorphic in G.* Note that if A is summability-preserving, it is holomorphicity-preserving, too: Expand $g(t) = \sum_j (t-t_0)^j\, g_j$, with $t_0 \in G$, then for a summability-preserving A the series $\sum_j (t-t_0)^j\, A\, g_j$ will have a positive radius of convergence, and closedness of A implies $\sum_j (t-t_0)^j\, A\, g_j = A \sum_j (t-t_0)^j\, g_j$.

In what follows, we let $\mathbb{D}_\infty$ denote a subspace of $\mathbb{D}$ on which all iterates A^j, $j \in \mathbb{N}$, are defined, and we use the following terminology:

- For $s \ge 0$, a vector $x \in \mathbb{D}_\infty$ is said to have *Gevrey order s* (at most), provided that constants $C, K > 0$ exist for which $\|A^j\, x\| \le C\, K^j\, \Gamma(1+sj)$ for every $j \ge 0$. The set of all such vectors clearly is a subspace of $\mathbb{D}_\infty$ and shall be denoted as $\mathbb{D}_s$. Note that vectors x may not have a (finite) Gevrey order, hence $\cup_s \mathbb{D}_s$ may be smaller than $\mathbb{D}_\infty$. Also observe that the constants C and K are allowed to depend upon the vector x. Moreover, make sure not to mix up the two terms of one vector resp. a formal power series to be of Gevrey order s.

- For $s \ge 0$, a subset $X \subset \mathbb{D}_s$ is said to have *uniform Gevrey order s*, provided that constants $C, K > 0$ exist for which we have $\|A^j\, x\| \le \|x\|\, C\, K^j\, \Gamma(1+sj)$ for every $j \ge 0$ and every $x \in X$. Note that a finite-dimensional subspace of $\mathbb{D}_s$ is always of uniform Gevrey order!

- For an unbounded operator A, the series (5.1) for $T_s(t;A)$ and $U_s(t;A)$ are strictly formal, since no norm is defined for their partial sums. However, on the domain $\mathbb{D}_s$ we can still define, generally unbounded, operators that we shall denote by the same symbols $T_s(t;A)$ and $U_s(t;A)$, setting

$$T_s(t;A)\, x \;=\; \sum_{j=0}^{\infty} \frac{t^j}{\Gamma(1+sj)}\, A^j\, x\,, \quad U_s(t;A)\, x \;=\; \sum_{j=1}^{\infty} \frac{t^{j-1/s}}{\Gamma(sj)}\, A^j\, x \quad (6.1)$$

  since these series converge for $x \in \mathbb{D}_s$ and $|t|\, K < 1$, with K as above, depending on x.

- The operator $T_s(t;A)$ which has been defined above maps vectors $x \in \mathbb{D}_s$ to functions that are holomorphic in a disc about the origin, with a radius that may depend upon x. The function $U_s(t;A)\,x$, however, has a singularity at the origin which generally is a branch point. Since we shall always restrict the variable t to sectorial regions which by definition do not contain the origin and at the same time are defined via a restriction of $\arg t$, we still can regard $U_s(t;A)\,x$ as a holomorphic function in every sectorial region G of sufficiently small radius.

- Let $x \in \mathbb{D}_s$ be given. If $d \in \mathbb{R}$ and values $k, \delta > 0$ exist for which the function $T_s(t;A)\,x$ can be holomorphically continued into the sector $S_{d,\delta}$, and if for suitable constants C, K, different from the ones that occured above, one has

$$\big\|T_s(t;A)\,x\big\| \;\le\; \|x\|\,C\;\exp\,[K\,|t|^k] \qquad \forall\; t \in S_{d,\delta}\,, \tag{6.2}$$

  then we say that *the vector x is of exponential growth* (at most) k in the direction d. The set of all $x \in \mathbb{D}_s$ that are of this growth shall be denoted as $\mathbb{D}_{s,k,d}$.

- A subset $X \subset \mathbb{D}_{s,k,d}$ is said to have *uniform exponential growth* k in the direction d, provided that (6.2) holds with the same constants δ, C, K for all $x \in X$.

- For arbitrary $x \in \mathbb{D}_\infty$, we use the notation $\hat{T}_s(t;A)$ to denote the formal operator

$$x \;\longmapsto\; \hat{T}_s(t;A)\,x \;=\; \sum_{j=0}^{\infty} \frac{t^j}{\Gamma(1+sj)}\;A^j\;x$$

  mapping $\mathbb{D}_\infty$ into the set of formal power series.

- For $s \ge 0$, $k > 0$ and $d \in \mathbb{R}$, we say that a vector $x \in \mathbb{D}_\infty$ is (s,k)-*summable in the direction d* provided that the power series $\hat{T}_s(t;A)\,x$ is k-summable in the direction d in the sense defined in Section 2. It follows from results in [3] that this holds if, and only if, the vector x is in the set $\mathbb{D}_{\sigma,k,d}$ for $\sigma = s + 1/k$. A set $X \subset \mathbb{D}_\infty$ is said to be *uniformly (s,k)-summable in the direction d*, if it is of uniform Gevrey order σ and has uniform exponential growth of order k.

Using this terminology, we can now prove the following analogues to Theorem 3:

Theorem 4 *Suppose that the series* (3.1) *converges for sufficiently small values of $|t|$, and that its coefficients are within a set of vectors that have uniform Gevrey order s in the sense defined above. Then the formal solution $\hat{u}_s(t)$ of* (3.3) *is convergent, too, and its sum is given by* (5.2).

Proof: According to the definition of uniform Gevrey order, the coefficients u_j, given by (0.2), can be estimated as

$$\|u_j\| \;\le\; \sum_{\ell=0}^{j} \|A^{j-\ell}\, g_\ell\| \;\le\; C \sum_{\ell=0}^{j} K^{j-\ell}\, \Gamma(1+s(j-\ell))\, \|g_\ell\|\,,$$

for suitable constants C, K, independent of j. The Beta integral formula [3, p. 228] implies that $\Gamma(1+sj)\,\Gamma(1+s\ell) \le \Gamma(1+sj+\ell)$ for all $j, \ell \ge 0$, and this, together with convergence of $\hat g_s(t)$, shows absolute convergence of the double series

$$\hat u_s(t) \;=\; \sum_{j=0}^{\infty} \frac{t^j}{\Gamma(1+sj)} \sum_{\ell=0}^{j} A^{j-\ell}\, g_\ell \;=\; \sum_{j,\ell=0}^{\infty} \frac{t^{j+\ell}}{\Gamma(1+s(j+\ell))}\; A^j\, g_\ell\,.$$

The sum of this double series can then be seen to be equal to (5.2). □

Theorem 5 *For $s \ge 0$, let $g_s(t)$ be holomorphic in a sectorial region G, and assume $g_s(t) \cong \hat g_s(t)$ in G. Moreover, for every closed subsector $\overline{S} \subset G$, assume that the values $g_s(t)$, for $t \in \overline{S}$, are uniformly of Gevrey order s. Then* (5.2) *defines a solution of* (3.3) *that is holomorphic in a subregion $G_1 \subset G$ of the same opening, and $u_s(t) \cong \hat u_s(t)$ in G_1, with $\hat u_s(t)$ given by* (0.2).

Proof: Let $\overline{S} \subset G$ be given. By assumption there exist C, K such that

$$\|A^j\, g_s(\tau)\| \;\le\; \|g_s(\tau)\|\, C\, K^j\, \Gamma(1+sj) \qquad \forall\, j \ge 0\,, \quad \tau \in \overline{S}\,.$$

Hence, the series for $U_s((t^{1/s} - \tau^{1/s})^s; A)\, g_s(\tau)$ converges for every $t \in \overline{S}$ with $|t|\, K < 1$ and every $\tau = x\, t$ with $0 \le x \le 1$, and convergence is locally uniform in t. Consequently, we may use (5.2) to define a function $u_s(t)$ that is holomorphic in $\overline{S} \cap \{|t| < K^{-1}\}$. Using the same arguments as in the proof of Theorem 3, one may then complete the proof. □

The following result on k-summability is trivial when the series $\hat g_s(t)$ is a constant vector, i. e., when all its coefficients but the constant one vanish. For $s = 1$, this means that we are dealing with a homogeneous initial value problem. Hence roughly speaking, the theorem can be applied to an inhomogeneous equation in cases where the homogeneous situation is well understood.

Theorem 6 *For $s \ge 0$, $k > 0$, and $d \in \mathbb{R}$, let $\hat g_s(t)$ be k-summable in the direction d, and let $g_s(t)$ be its sum, while $g_\sigma(t)$, $\sigma = s + 1/k$, denotes its Borel transform of order k, which then is holomorphic in a sector $S_{d,\delta}$, for some sufficiently small $\delta > 0$. Moreover, assume that the set of vectors $\{g_\sigma(\tau) : \tau \in S_{d,\delta}\}$ is uniformly (s,k)-summable in the direction d. Then the formal solution $\hat u_s(t)$ of* (3.3) *is k-summable in the direction d.*

Proof: The assumptions made (including the one that A is holomorphicity-preserving) say that the series

$$\sum_{j=0}^{\infty} \frac{t^j}{\Gamma(1+\sigma j)} \, A^j \, g_\sigma(\tau) \ , \quad \tau \ \in \ S_{d,\delta}$$

converges for $|t| < r$, with some $r > 0$ independent of τ, and can be continued with respect to t into a small sector bisected by the ray $\arg t = d$. Making the opening of this sector, resp. the value δ, smaller, we may without loss in generality assume that the function $T(t, \tau)$ defined by this series is holomorphic in $S_{d,\delta} \times S_{d,\delta}$ and, according to the definition of uniform (s, k)-summability, can be estimated as

$$\big\|T(t,\tau)\big\| \ \le \ \|g_\sigma(\tau)\| \, C \, \exp[K|t|^k] \, ,$$

with suitable values $C, K > 0$. With help of Cauchy's integral formula for the derivative, one can show the same estimate, in a somewhat smaller polysector that for simplicity will again be denoted as $S_{d,\delta} \times S_{d,\delta}$, and with different constants C, K, for the function

$$U(t,\tau) = \sigma \, t^{1-1/\sigma} \, \partial_t \, T(t,\tau) \, ,$$

and definining

$$u_\sigma(t) \ = \ g_\sigma(t) \ + \ \int_0^t \ U((t^{1/\sigma} - \tau^{1/\sigma})^\sigma, \tau) \ d\tau^{1/\sigma} \ ,$$

one obtains a function that is holomorphic in $S_{d,\delta}$ and satisfies an estimate analogous to (2.2). By means of termwise integration of the power series expansion of the integrand, one can show that $u_\sigma(t)$ is in fact holomorphic at the origin, and $\hat{u}_\sigma(t)$ is its (convergent) power series representation. This, however, proves k-summability in the direction d of the series $\hat{u}_s(t)$. □

7 Comparison with existing results

For non-linear systems of ordinary differential equations, it has been proven that all formal power series solutions, under very weak assumptions on the form of the system, are multisummable; for three independent proofs, see [7, 13, 2]. This makes the method of multisummation to be an almost perfect tool to handle divergent power series solutions of ODE, and investigating the so-called *Stokes phenomenon*. The situation changes, however, if one wants to treat solutions of partial differential equations, or of singular perturbations of ODE: While for certain relatively special situations one has proven the multisummability of formal solutions, for other very elementary situations, a formal solution is not multisummable at all. For a list of papers in this direction, refer to [4]; here we shall mainly concentrate on the following simple situation that nonetheless is characteristic for other PDE:

- Let $\mathbb{X}$ be a Banach space of functions of one variable that we shall denote as z, although the domain D in which z varies may well be a real interval.
- Let the norm $\|\cdot\|$ on $\mathbb{X}$ have the property that uniform convergence of a sequence (x_n) of functions implies norm-convergence, and that on the other hand every norm-convergent sequence contains a subsequence that converges pointwise (to the same limit function). Note that these assumptions hold, e. g., for $\mathcal{L}_p$-spaces on a bounded interval.
- Let the domain D be bounded, and let the set $\mathbb{D}_\infty = \mathcal{O}(D_r)$ of functions, holomorphic on $D_r = \{|z| < r\} \supset D$, be a dense subset of $\mathbb{X}$.
- Let the operator A map $\phi \in \mathbb{D}$ to its second derivative. In other words, let the equation (0.1) be the (inhomogeneous) heat equation.

In this setting, given $\phi \in \mathbb{D}_\infty$, the homogeneous equation $u_t = u_{zz}$, $u(0,z) = \phi(z)$, has the formal solution

$$\hat{u}(t,z) \;=\; \sum_{j=0}^{\infty} \frac{t^j}{j!}\, \phi^{(2j)}(z)\ .$$

In [8], the authors investigated the case where $\phi(z)$ is holomorphic near the origing and showed that $\hat{u}(t,z)$ is 1-summable in a direction d if, and only if, ϕ can be contiued into the union of two small sectors bisected by rays $\arg z = d/2$ and $\arg z = \pi + d/2$ and is of exponential growth at most 2 there. This result fits into our abstract setting by choosing $\mathbb{X} = \mathcal{O}_c(D_{r_1})$, $0 < r_1 < r$, but we shall show that the same result holds for *any* $\mathbb{X}$ that satisfies the assumptions made above: For the time being, *assume* that $\hat{u}(t,z)$ is 1-summable in a direction d. Then the same holds for the series $\partial_z\, \hat{u}(t,z)$ as well, and using other properties of multisummable series, we find that the series

$$\hat{y}(t,z) \;=\; \sum_{j=0}^{\infty} \frac{t^j}{\Gamma(1+j/2)}\, \phi^{(j)}(z)$$

is 2-summable in the directions $d/2$ and $\pi + d/2$. This holds if, and only if, the series

$$v(t,z) \;=\; \sum_{j=0}^{\infty} \frac{t^j}{\Gamma(1+j)}\, \phi^{(j)}(z)$$

is norm-convergent for sufficiently small values of $|t|$, and the resulting function can be continued with respect to t into sectors $S_{d/2,\delta}$ and $S_{\pi+d/2,\delta}$, with sufficiently small $\delta > 0$, and satisfies an estimate analogous to (2.2) for $k = 2$. Using Cauchy's formula for derivatives, we find for the partial sums $v_n(t,z)$ of this series the following integral representation:

$$v_n(t,z) \;=\; \frac{1}{2\pi i} \oint_{|w|=r_1} \phi(w)\, \frac{1 - (\frac{t}{w-z})^{n+1}}{w - z - t}\, dw\,,$$

with $0 < r_1 < r$. Since norm-convergence implies pointwise convergence of a subsequence of these $v_n(t,z)$ to the same limit $v(t,z)$, we obtain that for t and z in a sufficiently small disc about the origin

$$v(t,z) \;=\; \frac{1}{2\pi i} \oint_{|w|=r-} \phi(w)\, \frac{1}{w-z-t}\, dw \;=\; \phi(z+t)\;.$$

In view of this identity, the above statements on continuation and growth of $v(t,z)$ are equivalent to the function $\phi(t)$ being holomorphic in the two sectors $S_{d/2,\delta}$ and $S_{\pi+d/2,\delta}$ together with a growth estimate (2.2) for $k = 2$. Hence 1-summability of the formal solution of the homogeneous Cauchy problem for the heat equation is indeed independent of the choice of the Banach space $\mathbb{X}$, provided that we restrict ourselves to spaces that satisfy the (natural) assumptions made above. This fact makes it at least appear natural to expect the same independence for other differential operators!

As a concluding remark, we wish to say that on one hand, the existing results on multisummability of formal solutions of PDE may be formulated in a functional analytic setting. On the other hand, the fact that they in some sense are independent of the choice of the underlying Banach space supports the assumption that to obtain further results, one may not really have to use functional analytic methods, but can continue to use direct, more straightforward analysis. However, this is not a proven fact but just a feeling that the authors have!

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