# Systems of linear ordinary differential equations – a review of three solution methods

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**Abstract**

This is a review and comparison of three different methods for solving systems of linear ordinary differential equations with variable coefficients.

## Introduction

Systems of ordinary differential equations (ODE for short) are of great interest, both in mathematics and physics. If their coefficient matrix is time-dependent, then their solutions only occasionally can be computed in explicit form in terms of *known functions* such as the exponential function (of a matrix), or other so-called *higher transcendental functions* including *Bessel's* or the *hypergeometric function*. In this article we collect and describe methods that are popular among mathematicians and/or physicists for computing the solutions of such a system. In some exceptional cases these methods may lead to explicit solution formulas, while in general they end with representations of solutions as infinite series that may or may not converge, but still give useful insight into the behaviour of the solutions. As illustrating examples we shall frequently refer to two simple but nonetheless nontrivial systems of the following very special form:

1. For $d \times d$ constant matrices $\Lambda$ and $A$, with $\Lambda$ being diagonalizable, we shall follow *K. Okubo* [24] and refer to [1]

$$(\Lambda - t)\, x'(t) \;=\; A\, x(t) \qquad (0.1)$$

as *the hypergeometric system* in dimension $d \geq 2$. This system may not have any direct application in physics or other areas but has, partially in more general form, been frequently investigated by mathematicians. The reason for its popularity with the latter group is that it is complicated enough to make its solutions *new higher transcendental functions*, while on the other hand it is simple in the following sense: The eigenvalues of $\Lambda$, i. e., the points where $\Lambda - t$ fails to be invertible, as well as the point $t = \infty$, are *regular singularities* of (0.1), hence it is what is called *a Fuchsian system.* The name for this system refers to the fact that for $d = 2$ a fundamental solution can be computed in terms of the hypergeometric function (and other elementary ones); for this, refer to a book of *W. Balser* [2]. For $d \geq 3$, however, it is believed, although perpaps not rigorously proven, say by differential Galois theory, that its solutions only occasionally are *known functions.*

2. For $\Lambda$ and $A$ as above, we shall call

$$x'(t) \;=\; (\Lambda + t^{-1}\, A)\; x(t) \qquad (0.2)$$

*the confluent hypergeometric system* in dimension $d \geq 2$. It is related to (0.1) by means of Laplace transformation, but also by a confluence of all but one singularity of the hypergeometric system, as was shown by *R. Schäfke* [28]. Having an irregular singularity at $t = \infty$, and a regular one at the origin, (0.2) may appear more complicated than the previous one, but owing to their close relation, it is fair to say that they are of the same degree of transcendency in the sense that if we can solve either one of them, then we can solve the other one as well. For $d = 2$, a fundamental solution of the system (0.2) can be computed in terms of the *confluent hypergeometric function.*

Mathematicians that have investigated either one of the two systems, analyzing the behaviour of their solutions and/or evaluating their Stokes constants, include *G. D. Birkhoff* [7], *H. W. Knobloch* [15], *K. Okubo* [22, 23], *M. Kohno* [16, 18, 17], *R. Schäfke* [26, 27], *Balser, Jurkat, and Lutz* [4, 5], *Kohno and Yokoyama* [19], *T. Yokoyama* [30, 31], and *M. Hukuhara* [14].

Aside from these two examples, we shall consider a general linear system of ODE, denoted as

$$x'(t) \;=\; H(t)\; x(t)\,, \qquad (0.3)$$

with a matrix $H(t)$ whose entries are functions defined in some domain $D$ that is either a real (open) interval or an open and connected subset of the complex

[1] We shall adopt the convention and write $\Lambda - t$ instead of $\Lambda - t\, I$, with $I$ being the identity matrix of appropriate dimension.

numbers. If necessary, we shall require stronger assumptions on $H(t)$, such as analyticity, but for the time being we shall make do with continuity. Note that some of the results to be presented here carry over to, or even have been developped for, the case when $H(t)$ is not a matrix but a more general, perhaps even unbounded, operator in a Banach space. While we shall not attempt to treat such a general and considerably more difficult situation here, we mention as a simple example the situation when $X$ is a Banach or Hilbert space of functions $f(x)$, with functions that are arbitrarily often differentiable being dense in $X$, and instead of a matrix $H(t)$ we consider the operator $\partial_x^2$, assigning to $f$ its second derivative. In this case, instead of a system (0.3) of ODE we deal with the heat or diffusion equation $\partial_t \, u(t,x) = \alpha \, \partial_z^2 \, u(t,x)$, where $\alpha > 0$ is the diffusion constant. Given an initial condition $u(0,x) = \phi(x) \in X$ which is arbitrarily often differentiable, one can formally obtain a solution as

$$u(t,x) \;=\; \sum_{k=0}^{\infty} \frac{(\alpha\, t)^k}{k!} \, \partial_x^{2k} \, \phi(x) \;=\; \mathrm{e}^{\alpha t \partial_x^2} \, \phi(x) \; . \tag{0.4}$$

Here, the operator under consideration is independent of time, which is an easy situation when dealing with linear systems of ODE, since for any constant matrix $H$ the series $\mathrm{e}^{tH} = \sum_k (t^k/k!) \, H^k$ converges. However, owing to the fact that $\partial_x^{2k} \, \phi(x)$ in general is of magnitude $(2k)!$, the series (0.4) may diverge for every $t \neq 0$. Very recently, it has been shown by *Lutz, Miyake, and Schäfke* [20] that for many, but not all, functions $\phi(x)$ the series is summable in a sense to be discussed later; for this and other results in this direction, the reader may also refer to a paper by *W. Balser* [3], as well as to the literature listed there. If the diffusion constant $\alpha$ is allowed to be purely imaginary, say: $\alpha = i\, \tilde{\alpha}$, then instead of the diffusion equation one obtains the Schrödinger equation for a free particle. The results from the papers quoted above easily carry over to this situation as well.

According to the general theory, we may regard equation (0.3) as solved, if we can find $d$ linearly independent solution vectors, since it is well known that the set of all solutions is a linear space of dimension $d$. Such solutions can be arranged into a $d \times d$ *fundamental matrix* $X(t)$, and if $X(t)$ is any matrix whose columns solve (0.3), then it is fundamental if, and only if, its determinant is non-zero *at least at one point* $t_0$, implying $\det X(t) \neq 0$ *at all* $t \in D$.

The methods that we are going to describe in the following three sections are quite different at first glance, but are all based upon the following simple observation:

- Suppose that the integral

$$Q(t) \;=\; \int_{t_0}^{t} H(\tau) \; d\tau \tag{0.5}$$

(which exists for any $t_0 \in D$, owing to continuity of $H(\tau)$) gives rise to a function that commutes with $H(t)$. Then one can verify that the matrix

$$X(t) \; = \; \mathrm{e}^{Q(t)} \; := \; \sum_{n=0}^{\infty} \frac{1}{n!} \, Q(t)^n \; ,$$

with the series being absolutely convergent for all $t \in D$, gives a fundamental solution of (0.3), which is normalized by the fact that $X(t_0) = I$.

This assumption is certainly satisfied whenever $H(t)$ is a constant matrix $H$, in which case the fundamental solution is $X(t) = \mathrm{e}^{(t-t_0)\, H}$. So the difficulty in computing a fundamental solution for (0.3) is caused by the fact that, in general, the commutator $[Q(t), H(t)] := Q(t)\, H(t) - H(t)\, Q(t)$ is not going to vanish. E. g., in case of (0.2) we have

$$[Q(t), H(t)] \; = \; \big(1 \; - \; t_0/t \; - \; \log|t/t_0|\big) \; [\Lambda \, , \, A] \, ,$$

which vanishes if, and only if, $\Lambda$ and $A$ commute, and this is a relatively rare situation: If $\Lambda$ is a diagonal matrix with all distinct diagonal entries, then it commutes with $A$ if, and only if, $A$ also is diagonal. However, if they do commute, then the matrix $X(t) = |t/t_0|^A \, \mathrm{e}^{(t-t_0)\Lambda}$ is a fundamental solution. In the case of complex $t$, an even simpler one is given by

$$X(t) \; = \; t^A \; \mathrm{e}^{t\Lambda} \; ,$$

for any choice of the branch of $t^A = \mathrm{e}^{(\log t)\, A}$.

We also wish to mention that the commutator $[Q(t), H(t)]$ certainly vanishes if for arbitrary values $t_1, t_2 \in D$ we have $[H(t_1), H(t_2)] = 0$, and if this is so, we say that $H(t)$ *satisfies the commutator condition*. We do not have a proof nor a counterexample for the commutator condition being necessary for $[Q(t), H(t)] \equiv 0$ to hold.

Roughly speaking, the methods to be discussed treat a general system (0.3) as a perturbation of a second one for which the commutator condition is satisfied. In the first two approaches, the transition between the two systems is by introducing a parameter $\lambda$ and analyzing the dependence of a fundamental solution upon $\lambda$, while the third method is best understood as finding linear transformations linking the solution spaces of the two systems. In all approaches, *power series* either in $\lambda$ or $t$ are used. While one at first may proceed in a *formal manner*, one eventually is forced to ensure convergence of these series. We shall indeed see in the last section that in some situations power series occur who do not converge, but it will be indicated briefly that even then one can use a technique of *summation* to still make good use of these divergent series.

# 1 The exponential ansatz of Magnus

Since a fundamental solution $X(t)$ of (0.3) has a non-zero determinant, we may define $Q(t) = \log X(t)$, or equivalently write $X(t) = \mathrm{e}^{Q(t)}$, with whatever determination of the multi-valued logarithm of a matrix. This, however, leaves the question whether one may compute $Q(t)$ without presuming $X(t)$ to be known. That this can be done, even in situations more general than (0.3), has been shown in an article by *W. Magnus* [21]. However, observe that the suggestive idea of saying that

$$\frac{d}{dt}\,\log X(t) \;=\; X'(t)\,X(t)^{-1} \;=\; H(t)$$

implying $\log X(t) = \int_{t_0}^t H(\tau)d\tau$, may not hold except when $H(t)$ satisfies the commutator condition, which brings us back to what was discussed above! Hence we need a more sophisticated approach, and to facilitate computation, it is best to slightly generalize (0.3), introducing a (complex) parameter $\lambda$ and write

$$x' \;=\; \lambda\,H(t)\,x\,. \tag{1.1}$$

A fundamental solution $X(t;\lambda)$ then depends upon $t$ as well as $\lambda$, and we wish to represent it as

$$X(t;\lambda) \;=\; \mathrm{e}^{\lambda Q(t;\lambda)} \;=\; \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}\,Q(t;\lambda)^k\,, \qquad Q(t;\lambda) \;=\; \sum_{j=0}^{\infty} \lambda^j\,Q_j(t)\,, \tag{1.2}$$

with coefficient matrices $Q_j(t)$ to be determined, and convergence of the second series to be investigated later. While the computation to follow can be facilitated by using some well-known identities, e. g., for computing the derivative of an exponential matrix, we shall follow a more direct approach, leading to an identity from which one can recursively compute the matrices $Q_j(t)$: For every natural number $k \ge 2$, we set

$$Q(t;\lambda)^k \;=\; \sum_{j=0}^{\infty} \lambda^j\,Q_{jk}(t)\,, \quad Q_{jk}(t) \;=\; \sum_{\nu=0}^{j} Q_{j-\nu}(t)\,Q_{\nu,k-1}(t)\,.$$

Setting $Q_{j1}(t) = Q_j(t)$ and interchanging the order of summation, we conclude

$$X(t;\lambda) \;=\; I \;+\; \sum_{\mu=1}^{\infty} \lambda^\mu \sum_{j=0}^{\mu-1} \frac{1}{(\mu-j)!}\,Q_{j,\mu-j}(t)\,,$$

and in order that this expression is a solution of (1.1), with $X(t_0;\lambda) = I$, we need to have $Q_0(t) = \int_{t_0}^t H(\tau)\,d\tau$, and for $\mu \ge 1$

$$\sum_{j=0}^{\mu} \frac{1}{(\mu+1-j)!}\,Q_{j,\mu+1-j}(t) \;=\; \int_{t_0}^{t} H(\tau) \sum_{j=0}^{\mu-1} \frac{1}{(\mu-j)!}\,Q_{j,\mu-j}(\tau)\;d\tau\,. \tag{1.3}$$

Suppose that for some $\mu \ge 1$ we would already know $Q_0(t), \ldots, Q_{\mu-1}(t)$ – this is certainly correct for $\mu = 1$. Then we also know $Q_{jk}(t)$ for all $j = 0, \ldots, \mu - 1$

and all $k \geq 1$. Hence we may use (1.3) to explicitly find the next matrix $Q_\mu(t) = Q_{\mu 1}(t)$. We leave it to the reader to verify that

$$Q_1(t) = \frac{1}{2} \int_{t_0}^{t} \int_{t_0}^{t_1} [H(t_1), H(t_2)] \, dt_2 \, dt_1 \,,$$

$$\begin{aligned} Q_2(t) &= \frac{1}{6} \int_{t_0}^{t} \int_{t_0}^{t_1} \int_{t_0}^{t_2} \Big( [[H(t_1), H(t_2)], H(t_3)] \\ &\qquad + \big[[H(t_3), H(t_2)], H(t_1)\big] \Big) dt_3 \, dt_2 \, dt_1 \,. \end{aligned}$$

Similarly, the other coefficients can be computed in terms of higher order commutators – for details, and a different proof of these identities, refer to articles by *Dahmen and Steiner* [9] resp. *Dahmen, Scholz, and Steiner* [8].

Note that all $Q_k(t)$ with $k \geq 1$ vanish whenever $H(t)$ satisfies the commutator condition, and there are other situations possible when Magnus' series for $Q(t; \lambda)$ may terminate. In general, however, we have to deal with investigating convergence of the power series $Q(t; \lambda)$, in particular for the value of $\lambda = 1$. While we shall postpone the discussion of the general case to later, we conclude this section with the following easy but instructive example showing that we cannot always expect convergence at $\lambda = 1$:

- Suppose that

  $$H(t) = \begin{bmatrix} a & 0 \\ t & 0 \end{bmatrix}, \qquad a \neq 0 \,.$$

  In this case, the fundamental solution $X(t; \lambda)$, with $X(0; \lambda) = I$, of (1.1) can be verified to be

  $$X(t; \lambda) = \begin{bmatrix} e^{\lambda a t} & 0 \\ \dfrac{1 + (\lambda a t - 1)\, e^{\lambda a t}}{\lambda a^2} & 1 \end{bmatrix}.$$

  This matrix has a removable singularity at $\lambda = 0$. Using the theory of logarithms of a matrix, one finds that $X(t; \lambda) = e^{Q(t;\lambda)}$ with

  $$Q(t; \lambda) = \log X(t; \lambda) = \begin{bmatrix} \lambda a t & 0 \\ \dfrac{t\left(1 + (\lambda a t - 1)\, e^{\lambda a t}\right)}{a\left(e^{\lambda a t} - 1\right)} & 0 \end{bmatrix}.$$

  Again, the singularity of $Q(t; \lambda)$ at $\lambda = 0$ is removable, and hence an expansion as in (1.2) holds for sufficiently small values of $|\lambda|$. For $t \neq 0$, however, $Q(t; \lambda)$ has a first order pole at $\lambda = 2\pi i/(at)$, so that the radius of convergence is smaller than 1 whenever $|t| > 2\pi/|a|$. We shall analyze this effect in more detail in the following section.

## 2 The Feynman-Dyson series, and more general perturbation techniques

Here we briefly mention that the fundamental solution of (1.1) can also be represented by a convergent power series

$$X(t;\lambda) \;=\; \sum_{k=0}^{\infty} \lambda^k \, X_k(t) \tag{2.1}$$

with $X_0(t) = I$ and

$$X_k(t) \;=\; \int_{t_0}^{t} H(\tau)\, X_{k-1}(\tau)\, d\tau\,, \quad k \geq 1\,.$$

By repeated insertion of this recursion relation into itself, one can also write $X_k(t)$ as an $n$-fold integral, and after some manipulation one can obtain a form that is referred to as the *Feynman-Dyson series* [12] which contains time-ordered products of the matrix $H(t)$. This shall not be discussed here, but we should like to say that the series in its original form is intimately related to the *Liouville-Neumann method*, see, e. g., [29], which is also used in the proof of *Picard-Lindelöf's Theorem* on existence and uniqueness of solutions to initial value problems. From estimates given there one can show that in our situation the series converges for every $\lambda$, hence $X(t;\lambda)$ *is an entire function of* $\lambda$. Knowing this, one can conclude that the matrix $Q(t;\lambda) = \log X(t;\lambda)$, studied in the previous section, is holomorphic at least in a sufficiently small disc about the origin, so that Magnus' series in (1.2) has indeed a positive radius of convergence. From the theory of logarithms of a matrix one knows that $Q(t;\lambda) = \log X(t;\lambda)$, regarded as a function of $\lambda$, may become singular once two eigenvalues of $X(t;\lambda)$ differ by a non-zero multiple of $2\pi i$, and this may or may not happen for values of $\lambda$ in the unit disc, as is seen in the example given at the end of the previous section. Therefore, the radius of convergence of Magnus' series may, for any fixed $t \neq t_0$ be smaller than 1, in which case the series fails to converge at $\lambda = 1$. As a way out of this dilemma, one may use explicit summation methods providing continuation of holomorphic functions to compute $Q(t;\lambda)$ outside of the circle of convergence of Magnus' series, but we shall not discuss this here in detail.

A similar approach as above works when investigating a system of the form

$$x' \;=\; \big(H_0(t) \;+\; \lambda H(t)\big)\, x\,, \tag{2.2}$$

where $H_0(t)$ satisfies the commutator condition, so that for $\lambda = 0$ a fundamental solution $X_0(t)$ (which in case of $H_0(t) \equiv 0$ may be taken as the identity matrix) of (2.2) is known. For example, in case of the confluent hypergeometric system (0.2), we may choose $H_0(t) = \Lambda + t^{-1} D$, with $D$ being a diagonal matrix consisting of the diagonal elements of $A$. In this case $D$ and $\Lambda$ commute, so that $X_0(t) = t^D \, \mathrm{e}^{t\Lambda}$. The series (2.1) then is a solution of (2.2) if, and only if,

$$X_k'(t) \;=\; H_0(t)\, X_k(t) \;+\; H(t)\, X_{k-1}(t) \qquad \forall\, k \geq 1\,.$$

With the standard technique of *variation of constants* one obtains the recursion

$$X_k(t) \;=\; X_0(t)\Big[\,C_k \;+\; \int_{t_0}^{t} X_0^{-1}(\tau)\; X(\tau)\; H_{k-1}(\tau)\; d\tau\;\Big] \qquad \forall\; k \geq 1\,,$$

with constant matrices that can be chosen arbitrarily. To have $X(t_0; \lambda) = I$, one should pick $C_k = 0$. However, if one wishes to obtain a fundamental solution with a prescribed behaviour as $t \to \infty$, say, then other choices for $C_k$ are more appropriate. In the *Diplomarbeit* of *C. Röscheisen* [25], this technique has been used for the system (0.2) to obtain fundamental solutions that, in sectors in the complex plane, have a certain asymptotic behaviour.

The approach discussed so far is referred to as a *regular perturbation* of linear systems, since (2.2), for $\lambda = 0$, is still a linear system of ODE. Other cases arise when the parameter $\lambda$ also occurs in front of the derivative $x'$, in which case one speaks of a *singular perturbation.* Such cases have been analyzed, e. g., in an article of *Balser and Mozo* [6], and it has been shown there that one meets power series that are divergent for every $\lambda \neq 0$; but can be summed using the techniques to be discussed later.

# 3 Power series methods

The methods discussed in the previous sections have the advantage of being applicable to systems where the coefficient matrix $H(t)$ is fairly general. What we shall do here is restricted to cases when $H(t)$ is a meromorphic function for $t \in D$, meaning that $D$ is an open and connected subset of the complex numbers $\mathbb{C}$, and $H(t)$ is either holomorphic or has a pole at any point $t_0 \in D \subset \mathbb{C}$. As we shall see, it is natural in this context to distinguish three different cases:

## 3.1 Regular points

If $H(t)$ is holomorpic at a point $t_0 \in D$, then $t_0$ is referred to as a *regular point* of (0.3). In this case, we can expand $H(t)$ into its power series about $t_0$, and hence for some $\rho > 0$ we have

$$H(t) \;=\; \sum_{k=0}^{\infty} (t-t_0)^k\; H_k\,, \qquad |t-t_0| < \rho\,, \tag{3.1}$$

with coefficient matrices $H_k$ that we assume known. *Assuming* that the fundamental solution $X(t)$ can also be represented by a power series, we write analogously

$$X(t) \;=\; \sum_{k=0}^{\infty} (t-t_0)^k\; X_k\,,$$

and inserting into (0.3) and comparing coefficients, we obtain that

$$(k+1)\, X_{k+1} \;=\; \sum_{j=0}^{k}\; H_{k-j}\, X_j \qquad \forall\; k \geq 0\,.$$

Selecting $X_0 = I$, the remaining coefficients $X_k$ are determined by this identity, and a direct estimate shows that the power series so obtained converges for $|t - t_0| < \rho$, and its sum indeed is the fundamental solution of (0.3) normalized by $X(t_0) = I$. This argument shows that theoretically we can compute a fundamental solution of (0.3) by a power series ansatz, provided that the coefficient matrix $H(t)$ is holomorphic in the dics $|t - t_0| < \rho$, and moreover, we obtain holomorphy of $X(t)$ in the same disc! We can even do better than this: If we choose any curve from $t_0$ to any other point $t \in D$, we can cover the curve with discs that remain in $D$, and by successive reexpansion of $X(t)$ compute its continuation to the point $t$. However, note that examples show that continuation along a closed curve may not end with the same fundamental solution with which we started!

## 3.2 Singularities of first kind

An important issue in the theory of ODE is to analyze how solutions behave when the variable $t$ tends to a singularity $t_0$ of the coefficient matrix $H(t)$. Even if we succeed in calculating a fundamental solution in closed form, or by means of a convergent power series about a regular point, this may still be a difficult problem: An explicit formula for $X(t)$ may be so complicated that we cannot find out whether or not $X(t)$ grows, or stays bounded, or even goes to 0 as $t \to t_0$; the power series, even when $t_0$ is a point on the boundary of its circle of convergence, will not immediately say much about the behaviour of $X(t)$ at $t_0$ anyway. So this is why other ways of representing $X(t)$ are still to be desired. This can relatively easily be done at a singularity of first kind, meaning any point $t_0$ where $H(t)$ has at most a first order pole: Suppose that

$$H(t) \;=\; (t - t_0)^{-1-r} \sum_{k=0}^{\infty} (t - t_0)^k \; H_k \,, \qquad |t - t_0| < \rho \,, \tag{3.2}$$

then one refers to $r$ as the *Poincaré rank* of (0.3) at $t_0$, and a singularity of first kind is characterized by $r = 0$. In addition, we assume for simplicity that the matrix $H_0$ satisfies the following *eigenvalue condition*:

(E) If $\lambda$ and $\mu$ are two distinct eigenvalues of $H_0$, then $\lambda - \mu$ is not an integer.

In this situation, a fundamental solution $X(t)$ exists that has a representation of the form

$$X(t) \;=\; \Big( \sum_{k=0}^{\infty} (t - t_0)^k \; X_k \Big) \; (t - t_0)^{H_0} \,. \tag{3.3}$$

Choosing $X_0 = I$, the remaining coefficients are uniquely determined by the identity

$$X_k \, (H_0 + k) \;-\; H_0 \, X_k \;=\; \sum_{j=0}^{k-1} H_{k-j} \, X_j \qquad \forall \; k \geq 1 \,, \tag{3.4}$$

since the eigenvalue assumption made above ensures that the left hand side, which is nothing but a system of linear equations in the entries of $X_k$, has a unique solution. Again, estimating coefficients implies that the power series in (3.3) converges for $|t-t_0| < \rho$, and this representation immediately explains how $X(t)$ behaves as $t \to t_0$, since we see that $X(t)\,(t-t_0)^{-H_0}$ even is holomorphic at $t_0$.

Another way of looking at the above result is as follows: The convergent power series $T(t) = \sum_{k=0}^{\infty}(t-t_0)^k\,X_k$, when used as a transformation $x = T(t)\,y$, changes (0.3) to the system $y' = (t-t_0)^{-1}\,H_0\;y$, whose fundamental solution is $Y(t) = (t-t_0)^{H_0}$. In general, if $T(t)$ is any invertible matrix, then the linear transformation $x = T(t)\,y$ takes (0.3) to the new system

$$y' \;=\; \tilde{H}(t)\,y\,, \qquad \tilde{H}(t) \;=\; T^{-1}(t)\,\big(\,H(t)\,T(t) \;-\; T'(t)\,\big)\,,$$

and one may hope that the system so obtained can be solved easier than the original one, perhaps since the commutator relation discussed in the introduction is satisfied. The same approach works with a singularity of first kind when the eigenvalue assumption (E) is violated, leading to an analogous result. For more details on this, refer to the book of *F. R. Gantmacher* [13], or that of *W. Balser* [2]. As we shall see in the following subsection, this idea even can be used for systems with singularity of higher Poincaré rank.

Applying this result to the hypergeometric system (0.1), which for diagonal $\Lambda$ has regular singularities at all diagonal elements of $\Lambda$, plus an additional one at $t = \infty$, we see that in principle we may compute fundamental solutions at each singularity, and then by successive reexpansion even find out how these matrices are connected with one another. These *connection formulas* have important applications and have therefore been studied much in the literature.

A related method is commonly used to solve the Schrödinger equation

$$i\,U'(t) \;=\; H(t)\,U(t)$$

in quantum mechanics and quatum field theory, where the Hamiltonian $H(t) = H_0(t) + \lambda\,H_1(t)$ can be split into a free part $H_0(t)$ and an interacting part $H_1(t)$. (Here $H(t), H_0(t), H_1(t)$ denote hermitian matrices or self-adjoint operators.) With $U = U_0\,U_1$, $i\,U_0' = H_0\,U_0$, one obtains the Schrödinger equation in the *Dirac interaction picture* [10, 11]

$$i\,U_1' \;=\; \lambda\,\tilde{H}_1(t)\,U_1\,, \qquad \tilde{H}_1(t) := U_0^{-1}(t)\,H_1(t)\,U_0(t)\,.$$

## 3.3 Singularities of second kind

The confluent hypergeometric system (0.2) has a regular singularity at the origin, hence we may compute a fundamental solution of the form (3.3), with $t_0 = 0$. Owing to absence of other finite singularities, the power series in this representation converges for every $t \in \mathbb{C}$. However, it is not obvious how the solutions so obtained behave as $t \to \infty$. By means of a change of variable $t = 1/\tau$, system (0.2) becomes equal to $y' = -(\tau^{-2}\,\Lambda + \tau^{-1}\,A)\,y$, with $y'$ denoting the derivative of $y$ with respect to $\tau$. This new system has a singularity of Poincaré rank $r = 1$ at the origin, and this is why we say that (0.2) has the same rank at

$\infty$. Hence, the methods of the previous subsection do not apply. Nonetheless, it is natural to look for a transformation $x = T(t)\, y$ which changes (0.2) into a new system that may be solved directly, and since we want the fundamental solution $Y(t)$ of the transformed system to have the same behaviour at $\infty$ as that of the original equation, we wish to represent $T(t)$ as a power series in $t^{-1}$, denoted as

$$T(t) \;=\; \sum_{k=0}^{\infty} t^{-k}\, T_k\,. \tag{3.5}$$

Since $T^{-1}(t)$ should also be such a power series, we require in addition that the matrix $T_0$ be invertible. For simplicity we require that $\Lambda$ is not only diagonalizable, but is indeed a diagonal matrix, whose diagonal entries are all distinct. Then we may even restrict to $T_0 = I$, and it can be shown *that a transformation as above exists, for which the transformed system has the form*

$$y' \;=\; (\Lambda + t^{-1}\, D)\, y\,,$$

with a diagonal matrix $D$ that is equal to the diagonal entries of the original matrix $A$ in (0.2). So in a sense the matrix $T(t)$ is a *diagonalizing transformation* for the confluent hypergeometric system.

Even for a general system of arbitrary Poincaré rank, say, at the point $\infty$ it is well known that a transformation (3.5) exists for which the transformed system satisfies the commutator condition needed to compute its fundamental solution – however, in all but some exceptional situations, the series in (3.5) fails to converge for every $t$. However, there is a relatively recent theory of *multisummability* that allows to still make use of this series and compute a fundamental solution with help of finitely many integral transformations. We cannot go into any details about this, but refer to the books by *W. Balser* [1, 2] for details. In the first one, one can even find a proof for the fact that all power series that arise as formal solutions even for nonlinear equations are multisummable. Unfortunately, this is no longer the case with series that solve even very simple partial differential equations; e. g., the series (0.4) solving the heat equation fails to be multisummable for certain initial conditions.

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