Gevrey order of formal power series solutions of inhomogeneous partial differential equations with constant coefficients

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Abstract

In an earlier paper, the first author showed that certain normalized formal solutions of homogeneous linear partial differential equations with constant coefficients are multisummable, with a multisummability type that can be determined from a Newton polygon associated with the PDE. In this article, some of the results obtained there are extended in several directions: First of all, arbitrary formal solutions of inhomogeous PDE are considered, and it is shown that, in some sense, they can be computed completely explicitly. Secondly, the Gevrey order of these formal solutions is determined. Finally, formal power series are discussed that, in general, do not satisfy a PDE with constant coefficients, but instead may be considered as solutions of singularly perturbed ODE, or integro-differential equations of a certain form.

Introduction

In [3, 6], the first author introduced and studied *normalized formal solutions* of a Cauchy problem for general homogeneous linear partial differential equations in two variables having constant coefficients. Multisummability of these formal power series was then investigated in [4]. In detail, it has been shown that, under the assumption that the initial condition used is holomorphic near the origin, one can determine a multisummability type corresponding naturally to the PDE under consideration. The normalized formal solution then is multisummable in a given multidirection, provided that the initial condition can be continued into finitely many (small) sectors, and in every such sector is at most of a certain exponential growth that, in general, depends upon the sector. The multisummability type, the location of the sectors, and the corresponding

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exponential growth orders can all be computed from a *characteristic equation* corresponding to the PDE.

In this article, we shall mainly be interested in *inhomogeneous partial differential equations* in two complex variables of the form

$$p(\partial_t, \partial_z)\, u \;=\; \hat{f}(t,z)\,. \tag{0.1}$$

Here and later, ∂_t resp. ∂_z always stand for partial derivation with respect to t resp. z, and $p(t,z) \in \mathbb{C}[t,z]$ is a given polynomial in two complex variables with coefficients in the complex number field $\mathbb{C}$. The right hand side of (0.1) will be an arbitrary formal power series in two variables, which in short-hand notation will be expressed as $\hat{f}(t,z) \in \mathbb{C}[[t,z]]$. Our main goal is to study the set of all formal power series $\hat{u}(t,z) \in \mathbb{C}[[t,z]]$ that solve (0.1). While this problem obviously is symmetric in the two variables t and z, our treatment will, in some sense, prefer the one variable t over the other. For this reason, and to be able to compare with results in earlier articles of the first autor's, we shall expand $p(\partial_t, \partial_z)$ and $\hat{f}(t,z)$ as

$$p(\partial_t, \partial_z) \;=\; \partial_t^{\kappa}\, p_0(\partial_z) \;-\; \sum_{\nu=1}^{\kappa} \partial_t^{\kappa-\nu}\, p_\nu(\partial_z)\,, \quad \hat{f}(t,z) \;=\; \sum_{j=0}^{\infty} \frac{t^j}{j!}\, \hat{f}_{j*}(z) \tag{0.2}$$

where $p_\nu(z) \in \mathbb{C}[z]$, and $\hat{f}_{j*}(z) = \sum_{n=0}^{\infty} f_{jn}\, z^n/n! \in \mathbb{C}[[z]]$. A formal power series $\hat{u}(t,z) = \sum_{j=0}^{\infty} t^j\, \hat{u}_{j*}(z)/j!$, with $\hat{u}_{j*}(z) = \sum_{n=0}^{\infty} u_{jn}\, z^n/n! \in \mathbb{C}[[z]]$, is a formal solution of (0.1) if, and only if, the coefficients satisfy the inhomogeneous ODE

$$p_0(\partial_z)\, \hat{u}_{j*}(z) \;=\; \hat{f}_{j-\kappa,*}(z) \;+\; \sum_{\nu=1}^{\kappa} p_\nu(\partial_z)\, \hat{u}_{j-\nu,*}(z) \qquad \forall\; j \geq \kappa\,.$$

The first coefficients $\hat{u}_{0*}(z), \ldots, \hat{u}_{\kappa-1,*}(z)$ may be chosen arbitrarily, while (if $d > 0$ denotes the degree of $p_0(z)$) for each $j \geq \kappa$ we may arbitrarily select coefficients u_{jn} for $0 \leq n \leq d-1$ and then compute the remaining ones from this identity. So the entries $\hat{u}_{0*}(z), \ldots, \hat{u}_{\kappa-1,*}(z)$ and u_{jn} for $0 \leq n \leq d-1$ may be considered as the *initial data*[1] of a formal solution. It is obvious that the choice of these data influences the Gevrey order of $\hat{u}(t,z)$ (for the definition used here, see p. 18), and in fact may cause this order to be infinite! However, even if we choose all initial data to vanish, and the inhomogenuity $\hat{f}(t,z)$ to converge, the formal solution still may be divergent, which in our terminology corresponds to a Gevrey order $s = (s_1, s_2)$ with at least one $s_j \geq 1$. So it is the main concern of this article to analyse the Gevrey order s of $\hat{u}(t,z)$ in dependence of the initial data and the inhomogenuity.

The homogeneous equation corresponding to (0.1) has been studied in [3, 6, 4], and Gevrey order resp. multisummability of a particular solution, referred to as the *normalized formal solution*, has been investigated. The work done

[1] In fact, it may be more appropriate to refer to the values u_{jn} for $0 \leq n \leq d-1$ as boundary data, but for simplicity we choose not to do this here.

here, even when specialized to the homogeneous situation, considerably improves some of the results obtained in the articles quoted above: We show that the set of all formal solutions of (0.1) can be *explicitly parametrized* in terms of another formal power series in two variables, denoted as $\hat\psi(t,z)$. The coefficients of this series can, in turn, be found in terms of the coefficients f_{jn} of the right hand side of (0.1) and the initial conditions that the formal solution satisfies. As the most interesting situations, we shall study the two cases when all series $\hat f_{j*}(z) = \sum_{n=0}^\infty f_{jn}\, z^n/n!$ converge in a disc about the origin of a (finite) radius that is independent of j, resp. are entire functions of an exponential growth that does not depend upon j. For both cases, we shall find the Gevrey order of the formal solution, generalizing corresponding results from [1, 8] for the heat equation. To achieve the results described above, it is vital to use a general approach and study formal series that involve two so-called *moment functions*. In the special case when these two moment functions both are equal to factorials, such series occur as formal solutions of (0.1). In another case, they are formal solutions of an ordinary differential equation whose coefficients depend upon a parameter, while in general they may be considered as formal solutions of a *moment-PDE* that shall be described in Section 3.

Remark 1: As a standard example for our investigations, we shall consider the PDE

$$(\partial_t^\kappa \;-\; a\,\partial_z^\mu)\,u \;=\; \hat f(t,z)\,.$$

Here, a is a non-zero complex parameter, and κ and μ are (non-zero) natural numbers. So in this case we have $p(t,z) = t^\kappa - a\,z^\mu$, hence $p_0(z) \equiv 1$, $p_\kappa(z) = z^\mu$, and the remaining $p_\nu(z)$ vanishing identically. For $a = 1$, the corresponding homogeneous equation has been investigated by *M. Miyake* [19], and the case of $\kappa = 1,\ \mu = 2,\ a = 1$ shall frequently be referred to as *the complex heat equation.* Note that for our investigations it suffices to treat the case of $a = 1$, since an easy change of variable may be used to come to this situation. However, it is important to observe the general situation to see that for this article there is no difference between the heat equation and the Schrödinger equation (in one spatial variable). □

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1 Moment functions

In [2, 5], generalized integral operators resp. moment summability methods have been introduced and studied. For convenience of the reader we shall briefly review the definitions and results needed here.

A pair of functions $e(z)$ and $E(z)$ shall be called *kernel functions of order* $k > 1/2$, provided that they have the following properties:

1. The function $e(z)$ is holomorphic in $S_{k,+} = \{z \in \mathbb{C} \setminus \{0\} : 2k\,|\arg z| < \pi\}$, and $z^{-1}\,e(z)$ is integrable at the origin, meaning that a function $f(z)$ with

$f'(z) = z^{-1}\, e(z)$ is continuous at the origin in the sense that a value f_0 exists so that for every proper subsector S of $S_{k,+}$ and every $\varepsilon > 0$ there exists a $\delta > 0$ with $|f(z) - f_0| < \varepsilon$ whenever $z \in S$ and $|z| < \delta$ holds – also compare [2, p. 61]. For positive real $z = x$, the values $e(x)$ are assumed to be positive real numbers. Moreover, we demand that for every $R, \varepsilon > 0$ there exist constants $C, K > 0$ such that

$$|e(z)| \;\le\; C\, \exp[-(|z|/K)^k] \quad \forall\, z \text{ with } 2k|\arg z| \;\le\; \pi - \varepsilon\,,\ |z| \ge R\,. \quad (1.1)$$

In the literature, this property of $e(z)$ is referred to as being *exponentially flat of order k in $S_{k,+}$*.

2. The function $E(z)$ is entire and *of exponential growth at most k*, meaning that for constants $C, K > 0$, not necessarily the same as above, we have

$$|E(z)| \;\le\; C\, \exp[K\, |z|^k] \qquad \forall\, z \in \mathbb{C}\,. \quad (1.2)$$

Moreover, in $S_{k,-} = \{z \in \mathbb{C} \setminus \{0\} : 2\,|\pi - \arg z| < \pi(2 - 1/k)\} = \mathbb{C} \setminus \overline{S}_{k,+}$, the function $z^{-1}\, E(1/z)$ is required to be integrable at the origin.

3. The two functions $e(z)$ and $E(z)$ are *linked* as follows: In terms of the kernel function $e(z)$, we define the *corresponding moment function* $m(u)$ by

$$m(u) \;=\; \int_0^\infty x^{u-1}\, e(x)\, dx, \quad \text{Re } u \ge 0\,.$$

Note that the integral converges absolutely and locally uniformly for these u, so that $m(u)$ is holomorphic for Re $u > 0$ and continuous up to the imaginary axis, and the values $m(x)$ are positive real numbers for $x \ge 0$. Using this function, we require that $E(z)$ has the power series expansion

$$E(z) \;=\; \sum_{n=0}^{\infty} \frac{z^n}{m(n)} \qquad \forall\, z \in \mathbb{C}\,. \quad (1.3)$$

Under these assumptions, it follows that the moments $m(n)$ are *of the same order* as $\Gamma(1 + n/k)$ in the sense that constants $C_\pm > 0$ exist for which we have

$$C_- \quad \le \quad \left[\frac{m(n)}{\Gamma(1 + n/k)}\right]^{1/n} \quad \le \quad C_+ \qquad \forall\, n \ge 1\,. \quad (1.4)$$

In particular, this shows that the order of a pair of kernel functions is uniquely defined, and that the entire function $E(z)$ is exactly of exponential growth k, or in other words, is of exponential order k and finite non-zero type.

As an example, let $e(z) = k\, z^k \exp[-z^k]$. Then $m(u) = \Gamma(1 + u/k)$, and $E(u)$ is the well-known *Mittag-Leffler function* $E_{1/k}(z) = \sum_0^\infty z^n/\Gamma(1 + n/k)$ of index $s = 1/k$ [2, p. 233]. In some sense these functions are, in fact, the most important pair of kernel functions of this order and shall therefore be referred to as *the standard ones of order k*, but many more interesting examples can be derived from the general theory developped in [2, 5]. E. g., there exists a pair of kernel functions of order $k > 1/2$ for which the corresponding moment function equals

$$m(u) \;=\; \frac{\Gamma(1+s_1 u)\cdot\ldots\cdot\Gamma(1+s_\nu u)}{\Gamma(1+\sigma_1 u)\cdot\ldots\cdot\Gamma(1+\sigma_\mu u)}\;,$$

with arbitrary positive real numbers s_j, σ_j such that $\sum s_j = 1/k + \sum \sigma_j$. All questions asked here, say, about existence of solutions of moment-PDE, or Gevrey order or even summability of formal solutions, are independent of the particular choice of moment functions of fixed order. Therefore, we shall in proofs later on restrict to the standard kernels.

With help of kernel functions $e(z)$ and $E(z)$, we define a pair of integral operators as follows:

4. Let $S = S(d,\alpha) = \{z \in \mathbb{C} \setminus \{0\} \,:\, 2\,|d - \arg z| < \alpha\}$ be a sector of infinite radius, bisecting direction d, and opening α. Moreover, let f be holomorphic in S, integrable at the origin and *of exponential growth of order at most k in S*, meaning that for every $\varepsilon > 0$ there exist constants $C, K > 0$ with $|f(z)| \le C\,\exp[K|z|^k]$ for $2\,|d - \arg z| \le (\alpha - \varepsilon)$. Then for $2\,|d - \tau| < \alpha$, the integral

$$(T_m f)(z) \;=\; \int_0^{\infty(\tau)} e(u/z)\, f(u)\, \frac{du}{u} \qquad (1.5)$$

   converges absolutely and locally uniformly for z in a sectorial region with bisecting direction τ and opening π/k and can, by variation of τ, be continued into a sectorial region $G = G(d, \alpha + \pi/k)$ of opening $\alpha + \pi/k$ and bisecting direction $\arg z = d$. In this region, the function $T_m f$ is holomorphic and bounded at the origin.

5. If G is as in item 4, and f is holomorphic in G and bounded at the origin, then we define

$$(T_m^- f)(u) \;=\; \frac{1}{2\pi i}\int_{\gamma_k(\tau)} E(u/z)\, f(z)\, \frac{dz}{z} \qquad (1.6)$$

   with $2\,|\tau - d| < \alpha$, and $\gamma_k(\tau)$ denoting the path from the origin along $\arg z = \tau - (\varepsilon+\pi)/(2k)$ to some $z_1 \in G$ of modulus r, then along the circle $|z| = r$ to the ray $\arg z = \tau + (\varepsilon+\pi)/(2k)$, and back to the origin along this ray, for $\varepsilon, r > 0$ so small that $\gamma_k(\tau)$ fits into G. In other words, the path $\gamma_k(\tau)$ is the positively oriented boundary of a sector in G with bisecting direction τ, finite radius, and opening slightly larger than $\pi/(2k)$. The dependence of the path on ε and r will be inessential and therefore is not displayed. One can show that the function $T_m^- f$, for $S = S(d,\alpha)$ as in (1), is holomorphic in S, bounded at the origin, and of exponential growth at most k.

In the case of $e(z) = k\, z^k\, \exp[-z^k]$, the two integral operators coincide with the version of Laplace resp. Borel operators defined and studied in [2]. For kernels corresponding to the moment function $m(u) = \Gamma(1 + s\,u)/\Gamma(1 + \sigma\,u)$ for $1/k = s - \sigma > 1/2$, the two operators are *Jean Ecalle's acceleration* resp. *deceleration operators*. Even in general, they have many properties in common with those classical operators; for this, refer to [2, 5]. It is convenient to say that the operators T_m, T_m^-, as well as the moment function $m(u)$, corresponding to kernels of order $k > 1/2$, are also of this order. One can also introduce kernels, resp. operators, of any order $k \in (0, 1/2)$, but these shall not be needed here.

Remark 2: In [2, p. 88] it has been shown that the moment function $m(u)$ admits the following representation in terms of $E(z)$:

$$\frac{1}{m(u)} \;=\; \frac{1}{2\pi i} \int_\gamma E(z)\; z^{-u-1}\; dz \qquad \forall\; u \quad \text{with} \quad \operatorname{Re} u \ge 0\,,$$

with a path γ as in Hankel's formula for the reciprocal Gamma function: From infinity along the negative real axis to the point -1, then on the unit circle in the mathematically positive direction, and then back to infinity, again following the negative real axis. For $u = n \in \mathbb{N}_0$, the integrals along the two radial parts of γ cancel one another, so that for arbitrary $r > 0$ and $n \in \mathbb{N}_0$

$$\frac{1}{m(n)} \;=\; \frac{1}{2\pi i} \int_{|z|=r} E(z)\; z^{-n-1}\; dz\,, \tag{1.7}$$

which is, of course, nothing but the Cauchy formula for the coefficients of the power series expansion of $E(z)$. Note that the integral on the right vanishes, due to Cauchy's Theorem, when n is a negative integer. This is why it is natural here to interprete $1/m(n) = 0$ for $-n \in \mathbb{N}$, since then we shall be able to use (1.7) for all integers n. □

Remark 3: Observe that a kernel function $e(z)$ determines the corresponding moment function $m(u)$, and this one in turn determines the corresponding $E(z)$ by means of (1.3). Whether a given moment function $m(u)$ determines the kernel $e(z)$ is by no means obvious and shall not be discussed here at all. Therefore, whenever we shall later on speak of a *given moment function*, we implicitly mean to say that $e(z)$ is given, too. □

It has been shown in [2] that for every pair of integral operators as defined above, one can define a so-called *moment summability method*, and using this, one can introduce what is called a *theory of multisummability* in the sense of *Jean Ecalle* [10, 11, 15, 16, 14, 13] – this, however, shall not be used here.

2 Some notation, and a basic result

In what follows, we shall use the following notation that is different from that of the first author's previous articles [6, 4, 7], but more suitable for our purposes here:

1. Throughout, we shall consider a fixed polynomial $q(t,z) \in \mathbb{C}[t,z]$ in two complex variables. As shall be made clear in (3.1) this polynomial is linked to $p(t,z)$ occuring in (0.1) by $q(t,z) = t^\kappa\, z^\mu\, p(t^{-1}, z^{-1})$. Hence in the example described in Remark 1, we have $q(t,z) = z^\mu - a\, t^\kappa = p(z,t)$, and as was said there it suffices to understand the situation of $a = 1$, to which we shall restrict from now on.

2. We shall always expand $q(t,z)$ as
$$q(t,z) \;=\; \sum_{j=0}^{\kappa} t^j\, q_{j*}(z) \;=\; \sum_{n=0}^{\mu} q_{*n}(t)\, z^n \;=\; \sum_{j=0}^{\kappa}\sum_{n=0}^{\mu} t^j\, q_{jn}\, z^n\,, \qquad (2.1)$$
with polynomials $q_{j*}(z) \in \mathbb{C}[z]$, $q_{*n}(t) \in \mathbb{C}[t]$, resp. constants $q_{jn} \in \mathbb{C}$. *In addition, we shall restrict ourselves to the case of $\mu, \kappa \ge 1$, and assume that $q_{0*}(z)$ and $q_{\kappa *}(z)$, as well as $q_{*\mu}(t)$, do not vanish identically.* These assumptions are motivated by the main application of our results to the PDE (0.1).

3. While $q_{0*}(z)$, by assumption, does not vanish identically, it may have a root at the origin, whose order shall be denoted by $a_0 \ge 0$. The cases when a_0 is larger than the order of the root of at least one of the other $q_{j*}(z)$ will be of particular interest, and this is so for $q(t,z) = z^\mu - t^\kappa$, since here $a_0 = \mu$ while $q_{\kappa *}(z)$ does not vanish at the origin. We do, however, not make any assumption on the orders of roots right now!

4. As we shall see later, it is important to introduce expansions of $r(t,z) := q_{0*}(z)/q(t,z)$ in the following region in $\mathbb{C}^2$: Let $R_0 > 0$ be such that the polynomial $q_{0*}(z)$ does not vanish in the ring $0 < |z| < R_0$. For every pair of radii R_i, R_a with $0 < R_i < R_a < R_0$, let
$$m_0 \;=\; \min_{R_i \le |z| \le R_a} |q_{0*}(z)|\,, \quad M_\nu \;=\; \max_{R_i \le |z| \le R_a} |q_{\nu *}(z)|\,,$$
hence $m_0 > 0$. Also note that $M_\kappa > 0$, since by assumption $q_{\kappa *}(z)$ does not vanish identically. Accordingly, there exists a unique $R_1 > 0$, depending upon R_i and R_a and determined by $\sum_{\nu=1}^{\kappa} R_1^\nu\, M_\nu = m_0$. As a result,
$$|q(t,z)| \;\ge\; m_0 \;-\; \sum_{j=1}^{\kappa} |t|^\nu\, M_\nu \;>\; 0$$
for all $(t,z) \in G(R_i, R_a) := \{|t| < R_1\} \times \{R_i < |z| < R_a\}$. Hence $r(t,z)$ is holomorphic in $G(R_i, R_a)$ and can be expanded as
$$r(t,z) \;=\; \sum_{j=0}^{\infty} t^j\, r_{j*}(z) \;=\; \sum_{n=-\infty}^{\infty} r_{*n}(t)\, z^n \;=\; \sum_{j=0}^{\infty}\sum_{n=-n_j}^{\infty} t^j\, r_{jn}\, z^n \quad (2.2)$$
with the series converging absolutely in the region described above, and integer numbers n_j that are upper bounds for the pole order of $r_{j*}(z)$ and

shall, for simplicity of notation, *be assumed to be non-negative and weakly increasing.* For our standard example $q(t,z) = z^\mu - t^\kappa = p(z,t)$, expansion (2.1) is nothing but the geometric series $r(t,z) = (1 - t^\kappa/z^\mu)^{-1} = \sum_{j=0}^\infty t^{j\kappa} z^{-j\mu}$, hence $r_{j*}(z) = z^{-j\mu/\kappa}$ whenever j is divisible by κ, and zero otherwise.

5. The functions $r_{j*}(z)$ that were introduced above are rational and recursively determined by the identities

$$r_{0*}(z) \;\equiv\; 1\,, \qquad \sum_{\nu=0}^{\kappa} q_{\nu*}(z)\, r_{j-\nu,*}(z) \;=\; 0 \qquad \forall\, j \geq 1\,, \tag{2.3}$$

interpreting $r_{j*}(z) \equiv 0$ for $j \leq -1$. To obtain a good estimate for the numbers n_j in (2.2), let $a_\nu \geq 0$ denote the order of the zero of the polynomial $q_{\nu*}(z)$ at the origin, interpreting $a_\nu = \infty$ if $q_{\nu*}(z)$ vanishes identically. Solving (2.3) for $r_{j*}(z)$, one can see that $n_0 = 0$, while $n_j \;\leq\; \max\{0\,,\; n_{j-\nu} + a_0 - a_\nu\,,\; 1 \leq \nu \leq \kappa\}$, setting $n_{j-\nu} = -\infty$ whenever $\nu > j$. By induction, this may be seen to imply

$$n_j \;\leq\; j\,d \qquad \forall\, j \geq 0\,, \tag{2.4}$$

with a (non-negative rational) number d that is given by

$$d \;=\; \max\{0\,,\; \nu^{-1}\,(a_0 - a_\nu)\,,\; 1 \leq \nu \leq \kappa\} \;\geq\; 0\,, \tag{2.5}$$

and clearly $d \leq a_0$. This estimate shall play a prominent role in Section 5. Observe for later that in our main example described in Remark 1 we find $d = \mu/\kappa$, so in particular $d = 2$ for the complex heat equation.

6. The functions $r_{*n}(t)$ satisfy the inhomogeneous difference equation

$$\forall\, n \in \mathbb{Z}\,: \qquad \sum_{m=0}^{\mu} q_{*m}(t)\, r_{*,n-m}(t) \;=\; \begin{cases} q_{0n} & (0 \leq n \leq \mu) \\ 0 & \text{(otherwise)} \end{cases}$$

but this identity cannot be used to compute the $r_{*n}(t)$ recursively. Instead, one can use the standard formula for the coefficients of a Laurent series to obtain an integral representation for $r_{*n}(t)$; this, however, will not be needed here.

7. In what follows, we shall frequently consider formal Laurent series $\hat\psi(t,z)$ of the following special form

$$\hat\psi(t,z) = \sum_{j=0}^{\infty} t^j\, \hat\psi_{j*}(z) = \sum_{n=-\infty}^{\infty} \hat\psi_{*n}(t)\, z^n = \sum_{j=0}^{\infty} \sum_{n \geq -d\,j} t^j\, \psi_{jn}\, z^n \tag{2.6}$$

with d as in (2.5). Hence such series are formal power series in t with coefficients $\hat\psi_{j*}(z)$ that are formal Laurent series in z, whose (formal) pole orders are at most $d\,j$. The set of such series shall be denoted as $\mathbb{C}_d[[t,z]]$, and it is easily seen that this set forms an algebra with respect to the standard operations for formal series. Note, however, that $\mathbb{C}_d[[t,z]]$, except for $d = 0$, is not closed with respect to formal differentiation. For simplicity of notation, we shall set $\psi_{jn} = 0$ for $n < -d\,j$ and write $\hat\psi_{j*}(z) = \sum_n \psi_{jn}\, z^n$. As is natural, we shall refer to the series

$$\hat\psi_{pp}(t,z) \;=\; \sum_{j=0}^{\infty} \sum_{n\,<\,0} t^j\, \psi_{jn}\, z^n$$

as *the principal part* of $\hat\psi(t,z)$.

8. Since $r(t,z)$, when expanded as in (2.2), may be regarded as an element of $\mathbb{C}_d[[t,z]]$, we obtain a mapping $\hat\psi(t,z) \mapsto \hat u(t,z) := r(t,z)\,\hat\psi(t,z)$ from $\mathbb{C}_d[[t,z]]$ into itself, and since $1/r(t,z) = 1 + \sum_1^\kappa t^j\, q_{j*}(z)/q_{0*}(z)$ can be verified to be in $\mathbb{C}_d[[t,z]]$ as well, this mapping is, in fact, bijective. We shall write

$$\left.\begin{aligned} \hat u(t,z) &= r(t,z)\,\hat\psi(t,z) \;=\; \sum_{j=0}^{\infty} t^j\, \hat u_{j*}(z) \\ &= \sum_{n=-\infty}^{\infty} \hat u_{*n}(t)\, z^n \;=\; \sum_{j=0}^{\infty} t^j \sum_n u_{jn}\, z^n \\ u_{jn} &= \sum_{\nu=0}^{j} \sum_m r_{\nu m}\, \psi_{j-\nu,n-m} \end{aligned}\right\} \qquad (2.7)$$

bearing in mind that in the last line the inner sum extends over such m with $-\nu\, d \le m \le n + (j-\nu)\, d$ only, since for other m the term in the sum vanishes. Therefore, the sum always is finite and is, in fact, equal to 0 whenever $n < -d\,j$. We may rewrite the first line of (2.7) as $q(t,z)\,\hat u(t,z) = q_{0*}(z)\,\hat\psi(t,z)$ and compare coefficients to obtain the formula

$$\sum_{\nu=0}^{\kappa} q_{\nu*}(z)\,\hat u_{j-\nu,*}(z) \;=\; q_{0*}(z)\,\hat\psi_{j*}(z) \qquad \forall\ j \ge 0 \qquad (2.8)$$

interpreting $\hat u_{-\nu*}(z) \equiv 0$ for $\nu \ge 1$. This identity may be solved for $\hat u_{j*}(z)$, thus providing a recursion relation for the formal Laurent series $\hat u_{j*}(z)$. Expanding with respect to z, we conclude from (2.8) for $j \ge 0$ and $n \in \mathbb{Z}$, with a_0 as defined above, that the numbers u_{jn} and ψ_{jn} are related by the following infinitely many linear equations:

$$\sum_{\nu=0}^{\kappa} \sum_m q_{\nu m}\, u_{j-\nu,n-m} \;=\; \sum_m q_{0m}\, \psi_{j,n-m} \qquad (2.9)$$

interpreting $u_{j-\nu,n-m} = 0$ as well as $\psi_{j,n-m} = 0$ whenever $n < -d\,j + m$, or when $j - \nu$ resp. j is negative. For our standard example $q(t, z) = z^\mu - t^\kappa = p(z, t)$, this mapping resp. its inverse is given by the relations

$$u_{jn} \;=\; \sum_{0 \le \nu \le j/\kappa} \psi_{j-\nu\kappa, n+\nu\mu}\,, \qquad \psi_{jn} \;=\; u_{jn} \;-\; u_{j-\kappa,n+\mu}$$

for all integers j, n, interpreting $u_{jn} = \psi_{jn} = 0$ for $n < -j\,d$.

9. Let two *moment functions* m_1, m_2 in the sense of Section 1 be given. For an arbitrary series of the form

$$\hat f(t,z) \;=\; \sum_{j=0}^{\infty} \sum_{n} t^j \, f_{jn} \, z^n \;\in\; \mathbb{C}_d[[t,z]]$$

we define a formal *power* series in two variables by

$$\left.\begin{aligned} \hat f(t,z;m_1,m_2) \quad &= \quad \sum_{j=0}^{\infty} \frac{t^j}{m_1(j)} \sum_{n=0}^{\infty} f_{jn} \, \frac{z^n}{m_2(n)} \\ = \sum_{j=0}^{\infty} \frac{t^j}{m_1(j)} \, \hat f_{j*}(z;m_2) \;&=\; \sum_{n=0}^{\infty} \hat f_{*n}(t;m_1) \, \frac{z^n}{m_2(n)} \,. \end{aligned}\right\} \qquad (2.10)$$

Observe that in view of (1.7) we interprete $1/m_2(n) = 0$ for negative integers n; therefore, *in the definition of $\hat f(t,z;m_1,m_2)$ the summation with respect to n, unlike for $\hat f(t,z)$, is restricted to $n \ge 0$.* In this manner, we obtain a surjective linear map from $\mathbb{C}_d[[t,z]]$ onto $\mathbb{C}[[t,z]]$, which we shall call *the moment transformation.*

10. The moment transformation $\hat f(t,z) \mapsto \hat f(t,z;m_1,m_2)$ introduced in the previous item is nothing but a formal, i. e. termwise, application of the operator $T^-_{m_1}$, associated with the moment function m_1, to $\hat f(t,z)$ when regarded as a power series in t whose coefficients are formal Laurent series in z, followed by a formal application of $T^-_{m_2}$, associated with m_2, to its coefficients. It shall sometimes be convenient to denote this operation as

$$\hat f(t,z) \longmapsto \hat f(t,z;m_1,m_2) \;=\; \hat T^-_{m_1,t} \circ \hat T^-_{m_2,z} \; \hat f(t,z)\,. \qquad (2.11)$$

It shall be natural to extend this map to cases when the moments $m_1(n)$ and/or $m_2(n)$ are identically equal to 1 for $n \ge 0$. While for $m_1(n) \equiv 1$ the corresponding $\hat T^-_{m_1,t}$ is the identity operator on the set of formal power series in t, note that for $m_2(n) \equiv 1$ the operator $\hat T^-_{m_2,z}$ acts on formal Laurent series in z and is, in fact, the *truncation operator* removing the principal part; this operator shall be denoted by $\hat T^-_z$ for simplicity. In particular, if $m_1(n) \equiv m_2(n) \equiv 1$, then the series $\hat f(t,z;m_1,m_2)$ equals the power series part $\hat T^-_z \, \hat f(t,z)$ of $\hat f(t,z)$, and therefore *is, in general, different from* the series $\hat f(t,z)$.

11. For functions m_1, m_2, we define formal *moment-differential* operators $\partial_{m_1,t}$ and $\partial_{m_2,z}$, acting termwise on formal power series, denoted as in (2.10), by setting

$$\partial_{m_1,t} \, \frac{t^j}{m_1(j)} \, \hat{f}_{j*}(z;m_2) \;=\; \frac{t^{j-1}}{m_1(j-1)} \, \hat{f}_{j*}(z;m_2)\,,$$

$$\partial_{m_2,z} \, \hat{f}_{*n}(t;m_1) \, \frac{z^n}{m_2(n)} \;=\; \hat{f}_{*n}(t;m_1) \frac{z^{n-1}}{m_2(n-1)}\;,$$

with the understanding that the right hand sides vanish when $j = 0$ resp. $n = 0$. Observe that for $m_1(u) = \Gamma(1+u)$, the operator $\partial_{m_1,t}$ coincides with (termwise) partial differentiation ∂_t. More generally, if $m_1(u) = \Gamma(1+s\,u)$ for some fixed $s > 0$, then $\partial_{m_1,t}$ is intimately related with, although not equal to, fractional derivation with respect to t. For $m_1(n) \equiv 1$, the corresponding operator is equal to division of a power series by t, after elimination of terms that are independent of this variable. Analogous statements hold true for the other operator.

Except for the case of $j = 0$, resp. $n = 0$, the operator $\partial_{m_1,t}$, resp. $\partial_{m_2,z}$, introduced above can be inverted, and we can therefore define integer powers of them in the usual fashion. Altogether, we obtain for $\nu, \mu \in \mathbb{N}_0$ and $\hat{f}(t,z;m_1,m_2)$ as in (2.10)

$$\partial^{\nu}_{m_1,t}\,\partial^{\mu}_{m_2,z} \, \hat{f}(t,z;m_1,m_2) \;=\; \sum_{j=0}^{\infty} \frac{t^j}{m_1(j)} \sum_{n=0}^{\infty} f_{j+\nu,n+\mu} \, \frac{z^n}{m_2(n)}\;, \qquad (2.12)$$

while on the other hand

$$\partial^{-\nu}_{m_1,t}\,\partial^{-\mu}_{m_2,z} \, \hat{f}(t,z;m_1,m_2) \;=\; \sum_{j=0}^{\infty} \frac{t^{j+\nu}}{m_1(j+\nu)} \sum_{n=0}^{\infty} f_{jn} \, \frac{z^{n+\mu}}{m_2(n+\mu)}\;, \qquad (2.13)$$

and similarly for $\partial^{\nu}_{m_1,t}\,\partial^{-\mu}_{m_2,z}$ etc. However, observe that $\partial^{-\nu}_{m_1,t}$ is a right-inverse of $\partial^{\nu}_{m_1,t}$, but in general is not a left-inverse.

In the terms introduced above, we now define a linear map from the set $\mathbb{C}_d[[t,z]]$ into the set $\mathbb{C}[[t,z]]$ of formal power series in two variables by

$$\hat{\psi}(t,z) \;\longmapsto\; \hat{u}(t,z) \;\longmapsto\; \hat{u}(t,z;m_1,m_2) \;=\; \hat{T}^{-}_{m_1,t} \circ \hat{T}^{-}_{m_2,z} \hat{u}(t,z)\;, \qquad (2.14)$$

with $\hat{u}(t,z)$ given by (2.7). As we shall show now, this map is surjective, and we shall also find the kernel of this map:

Proposition 1 *Let any formal power series $\hat{\phi}(t,z) = \sum_{j,n\geq 0} t^j\,\phi_{jn}\,z^n$ in two variables be given. Then there exists a formal series $\hat{\psi}(t,z) \in \mathbb{C}_d[[t,z]]$, such that for the corresponding $\hat{u}(t,z)$ given by (2.7) we have*

$$\hat{\phi}(t,z) \;=\; \hat{T}^{-}_{m_1,t} \circ \hat{T}^{-}_{m_2,z} \hat{u}(t,z)\,.$$

More precisely, the principal part of $\hat{\psi}(t,z)$ may be arbitrarily prescribed, and after doing so, the series $\hat{\psi}(t,z)$ is uniquely determined. In particular, we can always choose the principal part to vanish, so that $\hat{\psi}(t,z) \in \mathbb{C}[[t,z]]$. Alternatively, we may also choose the principal part of $\hat{\psi}(t,z)$ in such a way that $\hat{u}(t,z)$ is in $\mathbb{C}[[t,z]]$, i. e., has vanishing principal part.

Proof: Obviously, it is necessary to set $u_{jn} = \phi_{jn}\, m_1(j)\, m_2(n)$ for $j,n \geq 0$, since then by definition

$$\hat{u}(t,z;m_1,m_2) \;:=\; \sum_{j=0}^{\infty} \frac{t^j}{m_1(j)} \sum_{n=0}^{\infty} u_{jn}\, \frac{z^n}{m_2(n)} \;=\; \hat{\phi}(t,z)\,.$$

This leaves to discuss whether it is possible to select coefficients u_{jn} with $j \geq 0$ and $-d\,j \leq n \leq -1$, as well as ψ_{jn}, for $j \geq 0$ and $n \geq -d\,j$, such that the identities (2.9) hold. This may be done by induction with respect to j: For fixed $j \geq 0$, we rewrite (2.9) as saying that

$$\hat{\psi}_{j*}(z) \;-\; \hat{u}_{j*}(z) \;=\; \sum_{\nu=1}^{\kappa} \frac{q_{\nu*}(z)}{q_{0*}(z)}\, \hat{u}_{j-\nu,*}(z)\,, \tag{2.15}$$

with the right hand side vanishing for $j = 0$, resp. being known by induction hypothesis for $j \geq 1$. According to the definition of the number d, the right hand side of this identity is a formal series in $\mathbb{C}_{jd}[[z]]$. Since the power series part of $\hat{u}_{j*}(z)$ is known, this formula determines the power series part of $\hat{\psi}_{j*}(z)$, as well as the principal part of $\hat{\psi}_{j*}(z) - \hat{u}_{j*}(z)$. This observation is sufficient to complete the proof. □

Remark 4: Observe that the proof of the above proposition can also be formulated more explicitly: Given $\hat{u}(t,z;m_1,m_2)$, we conclude for each $j \geq 0$ that the power series part of $\hat{u}_{j*}(z)$ is known, while its principal part, together with the series $\hat{\psi}(t,z)$, can be found as follows: Set

$$\rho_{\nu*}(z) \;=\; \frac{q_{\nu*}(z)}{q_{0*}(z)} \;=\; \sum_{n=a_\nu-a_0}^{\infty} \rho_{\nu n}\, z^n\,, \qquad 0 \,<\, |z| \,<\, R_0\,,$$

and write (2.15) in the form

$$\psi_{jn} \;-\; u_{jn} \;=\; \sum_{\nu=1}^{\kappa} \sum_{m=a_\nu-a_0}^{n+n_{j-\nu}} \rho_{\nu m}\, u_{j-\nu,n-m} \qquad \forall\; j \geq 0\,,\; n \in \mathbb{Z}\,, \tag{2.16}$$

interpreting $\psi_{jn} = u_{jn} = 0$ whenever $n < -n_j$. This formula indeed determines either u_{jn} or ψ_{jn}, after an arbitrary choice of the other quantity, for $-j\,d \leq n < 0$, as well as ψ_{jn} for $n \geq 0$, since then u_{jn} is known. □

3 Formal solutions of moment-PDEs

In what follows, we shall consider a given pair of moment functions m_1, m_2 and wish to investigate formal solutions of an operator equation of the following form:

- With μ, κ, and $q(t,z)$ as in (2.1), we define new polynomials in one, resp. two, variables by setting $p_0(z) = z^\mu\, q_{0*}(z^{-1})$ and $p_\nu(z) = -z^\mu\, q_{\nu*}(z^{-1})$ for $1 \le \nu \le \kappa$, respectively

$$p(t,z) \;=\; t^\kappa\; p_0(z) \;-\; \sum_{\nu=1}^{\kappa} t^{\kappa-\nu}\; p_\nu(z) \;=\; t^\kappa\, z^\mu\; q(t^{-1}, z^{-1})\,. \qquad (3.1)$$

  For a given series $\hat f(t,z;m_1,m_2)$ as in (2.10), we shall investigate the set of formal power series solutions of the equation

$$p\big(\partial_{m_1,t}, \partial_{m_2,z}\big)\; \hat u(t,z;m_1,m_2) \;=\; \hat f(t,z;m_1,m_2)\,. \qquad (3.2)$$

  Regarding $\partial_{m_1,t}, \partial_{m_2,z}$ as *moment-differential operators*, this identity may be viewed as an *inhomogeneous moment-PDE* with constant coefficients.

Remark 5: As the most interesting situation, assume that $m_1(u) = m_2(u) = \Gamma(1+u)$: As was observed before, the moment-differentials $\partial_{m_1,t}, \partial_{m_2,z}$ then become equal to partial derivation, and therefore (3.2) coincides with the inhomogeneous PDE (0.1). □

For the operator equation (3.2), it is natural to ask the following questions:

- How many formal solutions does (3.2) have? More precisely, given any formal power series in two variables, written in the form $\hat u(t,z;m_1,m_2)$, can we identify those of its coefficients u_{jn} that may be chosen arbitrarily, while the remaining ones can be uniquely determined, such that the series formally solves (3.2)? As shall be shown below, the ones that are arbitrary are those u_{jn} with $(j,n) \in \mathbb{I} \subset \mathbb{N}_0^2$, where

$$\mathbb{I} \;=\; \{\,(j,n):\; 0 \le j < \kappa\,\} \cup \{\,(j,n):\; 0 \le n < \mu - a_0\,\}\,. \qquad (3.3)$$

  Note that the set $\mathbb{I}$ has two components, the second of which is empty for $a_0 = \mu$, and this is so in our standard example $p(t,z) = t^\kappa - z^\mu$. This shall play a role below.

- In view of the map (2.14), can we identify those $\hat\psi(t,z)$ for which the corresponding $\hat u(t,z;m_1,m_2)$ is a formal solution of (3.2)? In this case, we can think of these series $\hat\psi(t,z)$ as *a parametrization* of the set of formal solutions. As shall also be shown below, the answer to this question is that we may choose those coefficients ψ_{jn} with $(j,n) \in \mathbb{I}$ arbitrarily, while the remaining ones are determined by the right hand side of (3.2).

To illustrate this concept, we again consider the case $p(t,z) = t^\kappa - z^\mu$, in which case we have

$$\left(\partial^\kappa_{m_1,t} \; - \; \partial^\mu_{m_2,z}\right)\, \hat u(t,z;m_1,m_2) \;=\; \sum_{j,n=0}^\infty \frac{t^j}{m_1(j)}\,(u_{j+\kappa,n} - u_{j,n+\mu})\,\frac{z^n}{m_2(n)}\;.$$

Hence, to obtain a formal solution of (3.2), we may choose u_{jn} with $0 \le j < \kappa$ and $n \ge 0$ arbitrarily, while the remaining ones are determined by the identities

$$u_{j+\kappa,n} \;=\; u_{j,n+\mu} \;+\; f_{jn}\,, \qquad \forall\, j,n \ge 0\,.$$

From (2.9) we conclude that $u_{j,n-\mu} - u_{j-\kappa,n} = \psi_{j,n-\mu}$ for $j \ge 0$, $n \ge \mu$. This implies that $\psi_{jn} = u_{jn}$ can be chosen arbitrarily for all $0 \le j < \kappa$ and $n \ge 0$, while the remaining ones are given by the equations $\psi_{jn} = f_{j-\kappa,n}$ for $j \ge \kappa$, $n \ge 0$. So in this simple situation, we see that the set of formal solutions of (3.2) can be *parametrized by the series $\hat\psi(t,z)$ with arbitrary coefficients ψ_{jn} for all $0 \le j < \kappa$ and $n \ge 0$, while the remaining ones are explicitly given by the inhomogenuity $\hat f(t,z;m_1,m_2)$*. In this case we have that $a_0 \in \mathbb{N}_0$, denoting the order of the zero of $q_{0*}(z)$ at the origin, satisfies $a_0 = \mu$, and therefore the set of pairs (j,n) with $0 \le j < \kappa$ and $n \ge 0$ is equal to $\mathbb{I}$ defined in (3.3). For an analogous result in the general case, we prove the following proposition:

Proposition 2 (Formal solutions) *For any formal series $\hat f(t,z;m_1,m_2)$ as in* (2.10)*, the following statements hold:*

(a) *For every formal power series $\hat\psi(t,z)$, the series $\hat u(t,z;m_1,m_2)$, given by the formal map* (2.14)*, is a formal solution of* (3.2) *if, and only if,*

$$\sum_{m=a_0}^{\mu} q_{0m}\,\psi_{j,n-m} \;=\; f_{j-\kappa,n-\mu} \qquad \forall\, j \ge \kappa\,,\; n \ge \mu\,. \tag{3.4}$$

This condition can be equivalently phrased as saying that

$$\partial^\kappa_{m_1,t}\, p_0(\partial_{m_2,z})\; \hat\psi(t,z;m_1,m_2) \;=\; \hat f(t,z;m_1,m_2)\,.$$

In particular, in order that $\hat u(t,z;m_1,m_2)$ is a formal solution, the coefficients ψ_{jn} of $\hat\psi(t,z)$ with $(j,n) \in \mathbb{I}$ can be arbitrarily prescribed, while the remaining ones are uniquely determined by (3.4)*.*

(b) *Let complex numbers ϕ_{jn} for $(j,n) \in \mathbb{I}$ be given. Then there is precisely one formal solution $\hat u(t,z;m_1,m_2)$ of* (3.2) *that satisfies*

$$u_{jn} \;=\; m_1(0)\,m_2(0)\,\partial^j_{m_1,t}\;\partial^n_{m_2,z}\;\hat u(t,z;m_1,m_2)|_{t=z=0} \;=\; \phi_{jn}$$

for these j,n. The remaining coefficients of $\hat u(t,z;m_1,m_2)$ can be recursively computed from the identities

$$\sum_{m=a_0}^{\mu} q_{0m}\,u_{j,n-m} \;+\; \sum_{\nu=1}^{\kappa}\sum_{m=0}^{\mu} q_{\nu m}\,u_{j-\nu,n-m} \;=\; f_{j-\kappa,n-\mu} \tag{3.5}$$

which hold for all $j \ge \kappa$ and $n \ge \mu$ and determine $u_{j,n-a_0}$ uniquely, owing to $q_{0a_0} \ne 0$.

Proof: Insert $\hat{u}(t,z;m_1,m_2)$ into (3.2), use (2.12), and compare coefficients to show that $\hat{u}(t,z;m_1,m_2)$ is a formal solution if, and only if, (3.5) holds. In view of (2.9), this shows (a). Since by definition of a_0 we have $q_{0a_0} \neq 0$, one can compute $u_{j,n-a_0}$ from (3.5), for every $j \geq \kappa$ and $n \geq \mu$, hence (b) follows, too. □

Remark 6: Observe that in (3.4) the two moment functions m_1, m_2 do not occur. Hence the above proposition shows that the set of formal solutions of (3.2) does not depend upon the choice of the moment functions, in the sense that there is an obvious bijection between the sets corresponding to different pairs of moment functions. □

Remark 7: According to the above proposition, the set of formal solutions of (3.2) may be described in the following two different ways:

(a) For a formal power series $\hat{\psi}(t,z)$, the corresponding $\hat{u}(t,z;m_1,m_2)$ is a formal solution if, and only if (3.4) holds. Hence, given ψ_{jn}, for $(j,n) \in \mathbb{I}$, we can uniquely determine the remaining ψ_{jn} such that (3.4) is satisfied. Consequently, the set

$$\mathcal{P}_\psi \;=\; \{\psi_{jn} : \; (j,n) \in \mathbb{I}\}$$

is one kind of natural parameter set for the formal solutions. As was pointed out before, in the case of $p(t,z) = t^\kappa - z^\mu$ the set $\mathbb{I}$ consists of the pairs (j,n) with $0 \leq j < \kappa$ and $n \geq 0$.

(b) A formal solution $\hat{u}(t,z;m_1,m_2)$ is uniquely determined by an arbitrary selection of u_{jn}, for $(j,n) \in \mathbb{I}$. Consequently, the set

$$\mathcal{P}_u \;=\; \{u_{jn} : \; (j,n) \in \mathbb{I}\}$$

is another natural parameter set, which shall be referred to as *the set of initial conditions.*

For our purposes, it is necessary to observe that the above two parameter sets for the formal solutions are related by the equations (2.9). These relations allow to switch from one system of parameters to the other one in an explicit fashion. To see this, assume that either $\mathcal{P}_\psi$ or $\mathcal{P}_u$ is given, and proceed by induction with respect to j to compute the other parameter set, similarly to the procedure described in Remark 4:

- For $j = 0$, equations (2.9) imply that $\sum_{m=a_0}^{\mu} q_{0m}\,(u_{0,n-m} - \psi_{0,n-m}) = 0$ for every $n \geq a_0$. Since $u_{0,n-m} = \psi_{0,n-m} = 0$ whenever $m > n$, we see that $u_{0n} = \psi_{0n}$ for every $n \geq 0$.
- For $j = 1$, equations (2.9), for $n \leq a_0 - 1$, may be used to recursively compute “auxiliary values” u_{1n} for $n \leq -1$. After that, the same equations may be used for $n \geq a_0$ to find the values of either u_{1n} or ψ_{1n} for $n \geq 0$ (if $\kappa \geq 2$), resp. for $0 \leq n \leq \mu - a_0 - 1$ (if $\kappa = 1$).

- For $j \geq 2$, we proceed very much as for the previous case: First, we compute u_{jn} for $n \leq -1$, then either u_{jn} or ψ_{jn} for $n \geq 0$ (if $\kappa \geq j+1$), resp. for $0 \leq n \leq \mu - a_0 - 1$ (if $\kappa \leq j$).

In our standard example case of $p(t,z) = t^\kappa - z^\mu$, one can verify that (2.9) simplifies to the equations

$$\psi_{jn} \;-\; u_{jn} \;=\; -u_{j-\kappa,n+\mu} \qquad \forall\; j \geq 0\,,\; n \geq -dj\,.$$

Since in this case the set $\mathbb{I}$ consists of pairs (j,n) with $0 \leq j \leq \kappa - 1$, we conclude that for these $u_{j-\kappa,n+\mu} = 0$, and therefore

$$\psi_{jn} \;=\; u_{jn} \qquad \forall\; (j,n) \in \mathbb{I}\,.$$

So here there is no difference in parametrizing formal solution by either u_{jn} or ψ_{jn}, and the same occurs whenever we have $a_0 = \mu$. In general, however, the relations between the two parametersets are non-trivial! Note that a similar parametrization problem occurs in studying the Goursat problem. For details, we refer to an article by *J. Leray* [12]. □

While the notion of formal solutions for moment-PDEs (3.2) gives good sense, it is not clear whether we can also consider functions $u(t,z;m_1,m_2)$ that are holomorphic, say, in a polysector and satisfy (3.2) with the right hand side replaced by a function $f(t,z;m_1,m_2)$ that is holomorphic in the same polysector. In the next section, however, we shall define how the two operators $\partial_{m_1,t}$, $\partial_{m_2,z}$ can be applied to holomorphic functions, and doing so we show that it is possible to discuss solutions of (3.2) that are functions.

4 Toeplitz operators

In this section we shall indicate how the notion of Toeplitz operators may be used in the calculation of formal solutions: Consider the equation (3.2) with the initial data $u_{jn} = 0$ for all $(j,n) \in \mathbb{I}$. Then a formal solution may be written as

$$\hat{u}(t,z;m_1,m_2) \;=\; \partial_{t,m_1}^{-\kappa}\, \partial_{z,m_2}^{a_0-\mu}\, \hat{v}(t,z;m_1,m_2)\,, \tag{4.1}$$

with a uniquely determined formal power series $\hat{v}(t,z;m_1,m_2)$. For an arbitrary formal Laurent power series $\hat{w}(t,z)$, we define the *projection* $\mathbf{Pr}\,\hat{w}(t,z)$ by cutting off all terms containing negative powers of t and/or z. Using the notation of Sections 2 and 3, it follows that $\hat{u}(t,z;m_1,m_2)$ is a formal solution of (3.2) if, and only if, we have

$$\mathbf{Pr}\,\big(p(t^{-1},z^{-1})\hat{u}(t,z)\big) = \hat{f}(t,z)\,. \tag{4.2}$$

Because (4.1) implies that $\hat{u} = t^\kappa z^{\mu-a_0}\,\hat{v}$, this may be rewritten as

$$\mathbf{Pr}\,\big(p(t^{-1},z^{-1})t^\kappa z^{\mu-a_0}\hat{v}(t,z)\big) = \hat{f}(t,z). \tag{4.3}$$

The operator $\hat{v} \mapsto \mathbf{Pr}\,\left(p(t^{-1},z^{-1})t^{\kappa}z^{\mu-a_0}\hat{v}\right)$ shall be named *a Toeplitz operator*, acting on the space of formal power series, and $p(t^{-1},z^{-1})t^{\kappa}z^{\mu-a_0}$ shall be called its *Toeplitz symbol.* Operators of this form, acting on spaces of holomorphic functions, have been introduced and studied in [9, 20, 21].

Remark 8: When specializing to the case of $q_{0*}(z) = 1$, the above Toeplitz operator also appeared in (2.7): In view of the relation $q(t,z) = t^{\kappa}z^{\mu}p(t^{-1},z^{-1})$ we can write the first relation of (2.7) in the form (4.3), with $\hat{v}(t,z)$ and $\hat{f}(t,z)$ replaced by $\hat{u}(t,z)$ and $\hat{\psi(t,z)}$, respectively. □

Example 1 *In case of our standard example of $p(t,z) = t^{\kappa} - z^{\mu}$, the corresponding Toeplitz symbol is equal to $1 - t^{\kappa}z^{-\mu}$. In particular, recall that in this case $a_0 = 0$, according to its definition in* Section 2*.*

Let $k > 0$, $d \in \mathbb{R}$ be given, let $\mathbb{E}$ be a complex Banach space, and let $\mathbb{E}\{t\}_{k,d}$ denote the set of all formal power series $\hat{f}(t)$ with coefficients in $\mathbb{E}$, that are k-summable in the direction d. This is equivalent to the existence of an $\mathbb{E}$-valued function $f(t)$ which is holomorphic in a sectorial region $G = G(d,\alpha)$ of bisecting direction d and an opening $\alpha > \pi/k$, and has $\hat{f}(t)$ as its asymptotic expansion of Gevrey order $1/k$, as $t \to 0$ in G. From Watson's Lemma we obtain that $f(t)$ is uniquely defined by $\hat{f}(t)$ and we shall therefore write $f = \mathbf{S}\,\hat{f}$. For such k-summable series, we define a Toeplitz operator of a slightly more general form than above: Let $\hat{\sigma}(t)$ denote a formal Laurent series with coefficients in $\mathbb{C}$, which is also k-summable in the direction d – by definition this means that the principal part of $\hat{\sigma}(t)$ terminates, while its power series part is in $\mathbb{C}\{t\}_{k,d}$. Accordingly, the product $\hat{\sigma}(t)\,\hat{f}(t)$ is again a formal Laurent series, to which we can apply the projection operator $\mathbf{Pr}$, removing its principal part. In this fashion we obtain a formal Toeplitz operator $\hat{f} \mapsto \mathbf{Pr}\,(\hat{\sigma}\,\hat{f})$. On the other hand, we can also form the product of the functions $\sigma = \mathbf{S}\hat{\sigma}$ and $f = \mathbf{S}\hat{f}$, giving a holomorphic function on the intersection of the two sectorial regions on which the two factors are holomorphic, and then we can define the projection operator $\mathbf{Pr}\,(\sigma\,f)$, subtracting the principal part of the asymptotic expansion, that is to say, of $\hat{\sigma}(t)\,\hat{f}(t)$. In this fashion, we obtain a Toeplitz operator mapping f to $\mathbf{Pr}\,(\sigma\,f)$, acting on a space of functions. For these operators we prove

Proposition 3 *Under the assumptions made above, the formal Toeplitz operator $\mathbf{Pr}\,\hat{\sigma}$ maps $\mathbb{E}\{z\}_{k,d}$ into itself, and we have*

$$\mathbf{S}\,(\mathbf{Pr}\,(\hat{\sigma}\,\hat{f})) \;=\; \mathbf{Pr}\,(\mathbf{S}(\hat{\sigma})\,\mathbf{S}(\hat{f})), \qquad \forall\ \hat{f} \in \mathbb{E}\{z\}_{k,d}. \tag{4.4}$$

Proof: The proof is a direct consequence of results in [2]: Let p denote the (formal) pole order of $\hat{\sigma}$. Then $t^p\,\hat{\sigma}(t) \in \mathbb{C}\{t\}_{k,d}$ follows, and thus we have $(t^p\,\hat{\sigma}(t))\,\hat{f}(t) \in \mathbb{E}\{t\}_{k,d}$ for every $\hat{f}(t) \in \mathbb{E}\{t\}_{k,d}$. It is clear right from the definition of k-summability that we then can subtract any partial sum from $(t^p\,\hat{\sigma}(t))\,\hat{f}(t)$ without affecting its summability, and then an exercise on p.104

of [2] implies that $\mathbf{Pr}\,(\hat\sigma\,\hat f)$ is in $\mathbb{E}\{t\}_{k,d}$. Moreover, (4.4) follows from the properties of the summation operator $\mathbf{S}$. □

The previous result shows that Toeplitz operators have very natural properties on the space of k-summable series, so they should be very useful tools when proving summability of formal solutions of PDE. Furthermore, since a multisummable series can always be decomposed into a finite sum of k_j-summable series, it is possible to extend the above proposition to multisummable series. These investigations, however, are left for future research!

5 Gevrey estimates of formal solutions

In this section we shall find necessary and/or sufficient conditions under which a formal solution $\hat u(t,z;m_1,m_2)$ of a moment-differential equation (3.2) has a given Gevrey order. In view of Proposition 2, it makes good sense to formulate such conditions in terms of the corresponding series $\hat\psi(t,z)$, or alternatively using the *initial data* u_{jn} for $0 \le j < \kappa$ and $n \ge 0$, as well as for $j \ge \kappa$ and $0 \le n < \mu - a_0$. We shall here use the following definition of Gevrey order:

Given $s_1, s_2 \ge 0$, we say that the series (2.7) *has Gevrey order at most* $s = (s_1, s_2)$, provided that the series

$$\hat u(t,z;s) \;=\; \sum_{j=0}^{\infty}\sum_{n=0}^{\infty} \frac{t^j}{\Gamma(1+s_1 j)}\; u_{jn}\; \frac{z^n}{\Gamma(1+s_2 n)} \tag{5.1}$$

converges on some non-empty polydisc about the origin of $\mathbb{C}^2$. Owing to Stirling's formula, or (simpler) upper and lower estimates of the Gamma function, convergence of (5.1) is equivalent to the existence of constants $C, K > 0$ for which

$$|u_{jn}| \;\le\; C\,K^{j+n}\,\Gamma(2+s_1 j+s_2 n) \qquad \forall\; j,n \ge 0\,. \tag{5.2}$$

Observe that here we choose to estimate u_{jn} in this unusual form, since this shall be convenient in later estimates. In particular, the monotonicity of the Gamma-Function for (real) arguments ≥ 2, together with the fact that, owing to some assumption on s_1, s_2 made in Theorem 1, $s_1 j + s_2 n \ge 0$ for all $j \ge 0$ and $n \ge -n_j$, will be of importance.

Assume for the moment that both values s_1, s_2 are positive, and let m_1, m_2 be moment functions of respective orders $k_1 = 1/s_1$, $k_2 = 1/s_2$. Then a series (2.7) is of Gevrey order at most s if, and only if, the series $\hat T^-_{m_1,m_2}\,\hat u(t,z)$ converges on some non-empty polydisc (which, however, need not be the same as that on which $\hat u(t,z;s)$ converges). An analogous statement holds for the cases where $s_1 = 0$ and/or $s_2 = 0$, provided that we then set $m_1(j) \equiv 1$ and/or $m_2(n) \equiv 1$.

To find the Gevrey order of the series (2.7), we recall the estimate of the pole orders n_j obtained in (2.4). Using this, we shall now prove the following result on the Gevrey order of $\hat u(t,z;m_1,m_2)$. Note that related results may be also found in two articles by *M. Miyake* [17, 18].

Theorem 1 *Given a moment-PDE (3.2), let $\hat\psi(t,z)$ be such that the corresponding $\hat u(t,z;m_1,m_2) = \hat T^{-}_{m_1,m_2}\, r(t,z)\,\hat\psi(t,z)$ is a formal solution. Then, for $d \ge 0$ as defined in (2.5), and every $s=(s_1,s_2)$ with $s_1 \ge d\,s_2 \ge 0$, the following statements are equivalent:*

(a) *The series $\hat\psi(t,z)$ is of Gevrey order at most s.*

(b) *The series $\hat f(t,z)$ in (3.2) is of Gevrey order at most s, and (5.2) holds for all $(j,n)\in\mathbb{I}$.*

(c) *The series $\hat u(t,z) = r(t,z)\,\hat\psi(t,z)$ has Gevrey order at most s.*

Proof: First, assume (a), i. e., let constants C, K exist such that (5.2) holds for ψ_{jn} in place of u_{jn}. According to Cauchy's formula for the coefficients of a Laurent series, we conclude existence of constants $C_1, K_1, K_2 \ge 0$ such that

$$|r_{jn}| \;\le\; C_1\, K_1^j\, K_2^n \qquad \forall\; j\ge 0\,,\; n\ge -n_j\,.$$

Estimating (2.7), we see that this implies for $j,n\ge 0$

$$|u_{jn}| \;\le\; C\,C_1 \sum_{\nu=0}^{j}\;\sum_{m=0}^{n+n_{j-\nu}} K_1^{j-\nu} K_2^{n-m} K^{\nu+m}\Gamma(2+s_1\nu+s_2 m)\,.$$

Observing the monotonicity of $\Gamma(x)$ for $x\ge 2$, as well as the definition of d, we see that $\Gamma(2+s_1\nu+s_2m) \le \Gamma(2+s_1\nu+s_2(n+d(j-\nu))) \le \Gamma(2+s_1 j+s_2 n)$, which implies (c). In addition, we can use (3.4) and conclude that (a) also implies (b). Next, if (c) holds, we may equivalently assume (5.2). Then (3.5) implies that (b) holds as well. In order to prove (a), we intend to estimate (2.16): Owing to Cauchy's formula for the coefficients of a Laurent series, we conclude existence of $C_1, K_1 > 0$, not necessarily the same as above, for which $|\rho_{jn}| \le C_1\, K_1^n$ for $j=1,\ldots,\kappa$ and all $n\ge 0$. For some $j\ge 0$, assume that we have shown existence of $C, K_2, K_3 > 0$, such that

$$|u_{\nu n}| \;\le\; C\, K_2^\nu\, K_3^n\; \Gamma(2+s_1\nu+s_2 n) \qquad \forall\; \nu<j\,,\; n\ge -n_\nu\,. \tag{5.3}$$

Observe that this assumption is certainly correct for $j,n\ge 0$, due to condition (c), and is trivially satisfied for $j=0$ and all $n\ge -n_0 = 0$. Using this assumption, we may estimate (2.16) for this value of j and all $n\ge -n_j$ to obtain

$$|\psi_{jn} \,-\, u_{jn}| \;\le\; C\,C_1 \sum_{\nu=1}^{\kappa}\;\sum_{m=a_\nu-a_0}^{n+n_{j-\nu}} K_1^m\, K_2^{j-\nu}\, K_3^{n-m}\, \Gamma(2+s_1(j-\nu)+s_2(n-m))\,.$$

Since $s_1\,\nu - s_2(a_0-a_\nu) \ge s_2(d\,\nu + a_\nu - a_0) \ge 0$, owing to (2.5), we may use the monotonicity of the Gamma function to conclude

$$|\psi_{jn} \;-\; u_{jn}| \;\le\; C\,C_1\,\Gamma(2+s_1 j+s_2 n)\, K_2^j\, K_3^n C_2(n)\,,$$

$$C_2(n) \;=\; \sum_{\nu=1}^{\kappa} K_2^{-\nu} \sum_{m=a_\nu-a_0}^{n+n_{j-\nu}} K_1^m \, K_3^{-m} \,.$$

Without loss of generality, we may assume that $K_1 < K_3$ and that K_2 is very large relative to K_1 and K_3, in which case $C_2(n) \le 1$ follows for all $n \ge -n_j$. Since $\psi_{jn} = 0$ for $n < 0$, we conclude that (5.3) remains true for $\nu = j$ and $-n_j \le n < 0$, hence in fact for all $n \ge -n_j$. Moreover, we conclude that

$$|\psi_{jn}| \;\le\; 2\,C\,K_2^j\,K_3^n\;\Gamma(2+s_1 j+s_2 n) \qquad \forall\; n \ge -n_j \,.$$

In this fashion we find that (c) implies (a). Finally, assume (b). The same proof as above also shows that then (5.2), with ψ_{jn} in place of u_{jn}, follows for all $(j,n) \in \mathbb{I}$. Moreover, (3.4) can be used to show the same estimate for ψ_{jn} with $(j,n) \notin \mathbb{I}$. Thus, the proof is completed. □

6 Application to PDE with constant coefficients

In this section we shall only be concerned with the special moment functions $m_j(u) = \Gamma(1+s_j u)$, for two non-negative real numbers s_1, s_2. To simplify notation we shall, in agreement with (5.1), write $s = (s_1, s_2)$, and instead of $\hat{u}(t,z;m_1,m_2)$ we shall write

$$\hat{u}(t,z;s) = \hat{u}(t,z;s_1,s_2) \quad = \quad \sum_{j=0}^{\infty} \frac{t^j}{\Gamma(1+s_1 j)} \sum_{n=0}^{\infty} f_{jn} \, \frac{z^n}{\Gamma(1+s_2 n)}$$
$$= \sum_{j=0}^{\infty} \frac{t^j}{\Gamma(1+s_1 j)} \, \hat{f}_{j*}(z;s_2) \;=\; \sum_{n=0}^{\infty} \hat{f}_{*n}(t;s_1) \, \frac{z^n}{\Gamma(1+s_2 n)} \,.$$

Using the analogous notation for $\hat{f}(t,z;s)$, we observe that for this particular choice of the moment functions, and specializing to $s_1 = s_2 = 1$, the moment-PDE (3.2) coincides with (0.1), but with the series $\hat{f}(t,z)$ on the right replaced by $\hat{f}(t,z;1,1)$.

In this special case, the following assumptions on the inhomogenuity resp. initial data of (0.1) shall naturally occur in applications:

(A) Assume that the series $\hat{f}(t,z;0,0)$ is of Gevrey order $s = (s_1,1)$, with some value $s_1 \ge d$. Then all series $\hat{f}_{j*}(z;1) = \sum_n f_{jn}\, z^n/n!$ converge for $|z| < \rho$, with a positive and finite value ρ that is independent of j. Moreover, assume that the initial data u_{jn} satisfy (5.2), for $(j,n) \in \mathbb{I}$. This obviously is equivalent to the convergence of the series $\hat{u}_{j*}(z;1)$, $0 \le j \le \kappa - 1$, together with the assumption that the series $\hat{u}_{*n}(t;1)$, $0 \le n < \mu - a_0$, are of the Gevrey order $s_1 - 1$ (as defined in [2], a power series $\hat{f}(z)$ in one variable is said to be of Gevrey order s, provided that its coefficients f_n can be estimated by $C\,K^n\,\Gamma(1+s\,n)$, with suitable constants C, K). In this case, Theorem 1 applies and shows that the formal solution also is of Gevrey order s. This means that when we interprete the formal solution as a power series in t, with coefficients that are holomorphic in a disc about the origin, then this series is of Gevrey order $s_1 - 1$ in the sense of [1]. In case of the heat equation, this agrees with the fact that, even in the homogeneous case, the formal solution in general diverges. Note that examples show that, if the above assumptions are satisfied for some value $s_1 < d$, the formal solution, in general, shall be of order $(d, 1)$, so that one cannot hope to remove the condition $s_1 \geq d\, s_2$ from Theorem 1.

(B) Assume that the series $\hat{f}(t, z)$ is of Gevrey order $s = (s_1, s_2)$, with some value $s_1 \geq d\, s_2$, but for some s_2 that is smaller than 1. Then all series $\hat{f}_{j*}(z; 1)$ have infinite radius of convergence and represent entire functions of exponential order at most $(1 - s_2)^{-1}$. Moreover, assume that the initial data u_{jn} satisfy (5.2), for $(j, n) \in \mathbb{I}$. Then the series $\hat{u}_{j*}(z; 1)$, $0 \leq j \leq \kappa - 1$, also represent entire functions with the same restriction of their order, while the series $\hat{u}_{*n}(t; 1)$, $0 \leq d < \mu - a_0$, are of the Gevrey order $s_1 - 1$, or converge if $s_1 \leq 1$. In this case, Theorem 1 applies and shows that the formal solution, regarded as a power series in t whose coefficients are entire functions, is of order $s_1 - 1$ if this quantity is positive, resp. converges for $s_1 \leq 1$, which can occur for sufficiently small s_2. In particular, for the heat equation such cases have been investigated in [1].

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