Transformation of reducible equations to Birkhoff standard form

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Abstract

Bolibruch proved in 1994 that irreducible systems can always be transformed, using analytic transformations, into Birkhoff standard form, i.e. into an equation with polynomial coefficient matrix. Here, we discuss the same question for reducible ones, and we also allow meromorphic transformations.

0 Introduction

Let an n–dimensional system of ordinary differential equations of the form

$$z\,x' = A(z)\,x, \quad A(z) = z^r \sum_{k=0}^{\infty} A_k\, z^{-k} \tag{0.1}$$

be given, where the series may converge for $|z| > R$, say, and r is a non–negative integer which we refer to as the *Poincaré rank* of (0.1). In 1913, G.D. Birkhoff [5] raised the following question: Can one always find an *analytic transformation*

$$x = T(z)\,y, \quad T(z) = \sum_{k=0}^{\infty} T_k\, z^{-k} \tag{0.2}$$

(with T_0 invertible, and the series converging for sufficiently large $|z|$), such that the transformed equation

$$z\,y' = B(z)\,y, \quad B(z) = T^{-1}(z)[A(z)\,T(z) - z\,T'(z)] \tag{0.3}$$

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has a polynomial $B(z)$ as its coefficient matrix? Every such equation (0.3) will then be called a *Birkhoff standard form* (B.s.f. for short) for (0.1) (w.r. to analytic equivalence), and we shall say that then (0.1) is *analytically equivalent to* B.s.f.

Birkhoff himself in [5] showed that the answer to his question is positive under the additional assumption that the *monodromy matrix* of (0.1) is diagonalizable, but seemed to believe that the same would hold generally. However, in 1959 Gantmacher [9] and Masani [12] independently produced examples of equations (0.1) (in the smallest non-trivial dimension $n = 2$) in which no such transformation to Birkhoff standard form exists. These counterexamples being triangular, this led to the following (harder) problem:

Calling (0.1) *reducible* if an analytic transformation exists for which the transformed equation is (lower) *triangularly blocked*, (with square diagonal blocks of arbitrary dimensions), is it so that every *irreducible* equation is analytically equivalent to B.s.f?

This question was answered positively, first for dimension $n = 2$ by Jurkat, Lutz and Peyerimhoff [11], then for $n = 3$ by Balser [2], and finally for any dimension by Bolibruch [6], [7]. In this article, we shall consider reducible equations and address the following two questions:

1. Under which additional assumptions can a reducible system (0.1) be analytically equivalent to B.s.f?
2. Allowing *meromorphic* transformations $x = T(z)\,y$, where $T(z)$ and its inverse are assumed to be meromorphic near infinity, can *every* (reducible) system (0.1) be (meromorphically) transformed into Birkhoff standard form – note, however, that here we do not want to allow the Poincaré rank of (0.3) to become larger than that of (0.1) (if one *does not restrict* the Poincaré rank, then a variant of Birkhoff's result shows the answer to be positive). If such a transformation exists, we shall say that (0.1) is *meromorphically equivalent* to B.s.f.

For dimensions $n = 2$ resp. $n = 3$, Jurkat, Lutz and Peyerimhoff [11] resp. Balser [1] have shown the answer to question 2 to be positive. For general dimension, but under the additional assumption of the leading matrix A_0 of (0.1) having distinct eigenvalues, H.L. Turrittin [13] also obtained a positive answer, but in general the answer to 2 is still open and will also not be answered in this article. Instead, we shall obtain a number of sufficient conditions under which such a meromophic transformation exists. In this discussion we shall restrict ourselves to reduced equations, since Bolibruch's result shows that for irreducible ones analytic transformations already suffice.

1 Analytic transformations

From now on, we assume an equation (0.1) of the form

$$A(z) = \begin{bmatrix} A_{11}(z) & \mathrm{O} & \cdots & \mathrm{O} \\ A_{21}(z) & A_{22}(z) & \cdots & \mathrm{O} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1}(z) & A_{m2}(z) & \cdots & A_{mm}(z) \end{bmatrix} \tag{1.1}$$

given, with square diagonal blocks of sizes s_j, $1 \leq j \leq m$, and $m \geq 2$. We shall call such an equation *a reduced system.* Moreover, we always assume that *the diagonal blocks are irreducible*, hence by Bolibruch's result [6], [7] we can find a lower triangularly blocked analytic transformation $T(z)$ such that the transformed equation (0.3) is again reduced (of the same type as (0.1)), but its diagonal blocks are polynomials and its off diagonal blocks are polynomials both in $1/z$ and z. *So for simplicity of notation, we shall assume that* (0.1) *already has diagonal blocks which are polynomials in z and it has off diagonal blocks which are polynomials in $1/z$ and z.*

We now wish to discuss existence of an analytic transformation

$$T(z) = \begin{bmatrix} I & \mathrm{O} & \cdots & \mathrm{O} \\ T_{21}(z) & I & \cdots & \mathrm{O} \\ \vdots & \vdots & \ddots & \vdots \\ T_{m1}(z) & T_{m2}(z) & \cdots & I \end{bmatrix} \tag{1.2}$$

such that the transformed equation is in B.s.f. (and reduced). To do so, we formulate the following condition upon the diagonal blocks of (1.1):

E) For every ν, μ with $1 \leq \nu < \mu \leq m$, and every integer number $k \geq 1$, let $A_{\nu\nu}(0)$ and $A_{\mu\mu}(0) + kI$ have no eigenvalue in common.

This condition is a natural generalization (to a block triangular case) of a well-known classical nonresonance condition for Fuchsian systems of ODE. To clarify the meaning of this condition, we now prove the following auxiliary result on inhomogenuous equations:

Lemma 1 *Let square matrices $A_1(z)$ and $A_2(z)$ of polynomials of degree $\leq r$, not necessarily of the same dimensions, be given. Then the following statements are equivalent:*

1) *For every matrix $C(z)$ (of the appropriate size) of polynomials in $1/z$ and z, having poles at infinity of orders at most r, we can find a polynomial $B(z)$ of degree at most r, such that the inhomogenuous equation*

$$zT'(z) = A_2(z)\,T(z) - T(z)\,A_1(z) + C(z) - B(z) \tag{1.3}$$

has a solution $T(z)$ which is a polynomial in $1/z$ with $T(\infty) = 0$.

2) *For every natural number $k \geq 1$, the matrices $A_1(0)$ and $A_2(0) + kI$ do not have an eigenvalue in common.*

Proof: To conclude **1)** from **2)**, consider any

$$C(z) = \sum_{k=-k_0}^{r} C_k z^k$$

and proceed by induction with respect to k_0: For $k_0 = 0$, we may take $T(z) \equiv 0$, $B(z) = C(z)$. For $k_0 \geq 1$, choose a matrix T such that

$$(A_2(0) + k_0\, I)T - TA_1(0) = -C_{-k_0}$$

(note that according to [9] this equation has a unique solution T provided **2)** is satisfied). Then we have

$$\tilde{C}(z) = z^{-k_0}[TA_2(z)\,T - T\,A_1(z) + k_0 T] + C(z) = \sum_{k=-k_0+1}^{r} C_k z^k,$$

hence by induction hypothesis there exist $\tilde{T}(z)$ and $B(z)$ such that (1.3) holds for $\tilde{C}(z)$ in place of $C(z)$, and taking $T(z) = \tilde{T}(z) + z^{-k_0}T$ we then obtain (1.3). Conversely, if **2)** is violated for some $k = k_0$, one can choose $C(z) = z^{-k_0}C$ in such a way that **1)** fails. □

From the above Lemma, we now obtain

Theorem 1 *Let $A(z)$ be as in* (1.1)*, with polynomial diagonal blocks. Then a transformation* (1.2)*, transforming* (0.1) *into B.s.f., exists if the eigenvalue condition* **E)** *holds.*

Proof: If (1.2) exists, then the transformed equation obviously is again reduced (of the same type as (1.1)). So it can be easily seen that existence of (1.2) is equivalent to the following problem: For every ν, μ with $1 \leq \nu <$

$\mu \leq m$, and every $C(z) = z^r \sum_0^{k_0} C_k\, z^{-k}$ (of the appropriate size), can one find a polynomial $B(z)$ such that the inhomogenuous equation

$$zT'(z) = A_{\mu\mu}(z)\, T(z) - T(z)\, A_{\nu\nu}(z) + C(z) - B(z)$$

has a convergent power series solution $T(z)$ in inverse powers of z? This, however, follows from Lemma 1. □

Remark 1 *From Lemma 1 we see that condition* **E)** *in Theorem 1 is necessary in the following sense: For given diagonal blocks of* (1.1) *for which* **E)** *is violated, one may choose the off–diagonal blocks of* (1.1) *so that the conclusion of the theorem is false – observe that in case* (1.1) *is analytically equivalent to B.s.f., then the corresponding transformation $T(z)$ has to be a polynomial in $1/z$, because the origin is a regular singular point of the equation satisfied by $T(z)$.*

Remark 2 *The above theorem gives an answer to the question under which condition a triangularly blocked equation can be transformed to Birkhoff standard form by means of a* triangularly blocked *analytic transformation (in which case the resulting equation is also triangularly blocked). (For other sufficient conditions of such a type see* [8]*). It is, however, possible that a triangularly blocked equation can be analytically transformed to a Birkhoff standard form which no longer is triangularly blocked (because the transformation used is not triangularly blocked either). Whether in such a case another B.s.f. esists which is triangularly blocked seems an open question. So because of this, Theorem 1 is not really the final answer to the question 1 in the Introduction, but just gives a condition under which the answer is positive, and this condition is also necessary in the sense of the previous remark.*

We now give conditions under which **E)** can be made to hold under analytic transformations: To do this, let an arbitrary system (1.1) be given. For $\nu = 1, \ldots, m$, let t_ν denote the real part of the trace of the matrix $A_{\nu\nu}(0)$ (observe that by assumption $A_{\nu\nu}(z)$ is a polynomial in z, so this makes good sense). Note that, while the numbers t_ν are invariant with respect to analytic transformations, they can be changed modulo one by meromorphic transformations (if we restrict to transformations which preserve the structure of the equation). In detail, this follows from

Proposition 1 *For an arbitrarily given meromorphic transformation $T(z)$, let $k = k_T$ be such that*

(1.4) $$\det T(z) = z^k (c + \mathrm{O}(1/z)), \quad z \to \infty$$

with $c \neq 0$. *Then for systems* (0.1) *and* (0.3), *we have*

$$\text{trace}(A(z)) = \text{trace}(B(z)) - k + \mathrm{O}(1/z)). \tag{1.5}$$

Proof: First, let $T(z)$ be an analytic transformation (hence $k = 0$). In this case, $B(z) = T^{-1}(z)A(z)T(z) + \mathrm{O}(1/z)$, hence (1.5) follows, using that the trace of a matrix is invariant under similarity. In the general case, it is well known that every meromorphic transformation can be factored as a product $T(z) = T_1(z)\, z^K T_2(z)$, where $T_1(z)$, $T_2(z)$ are analytic transformations (moreover, $T_1(z)$ is a polynomial in $1/z$) and K is a diagonal matrix of integer diagonal entries (this factorization is referred in [10] to as the *Sauvage lemma*). Using this, we may restrict to the case of $T(z) = z^K$, (hence $k = \text{trace}(K)$), and then the proof follows, again using invariance of the trace under similarity. □

We shall now prove two lemmas; the first of them serves as a tool in the proof of the second one, while the second one shall clarify the role of the numbers t_ν (defined above) for what we have in mind:

Lemma 2 *Let an arbitrary irreducible equation* (0.1) *and integers* k_j *satisfying* $k_{j+1} \geq k_j + r(n-1)$ *be arbitrarily given. Let* $\mu_1, \ldots, \mu_n$ *be the eigenvalues of the matrix* $(1/2\pi i) \ln G$ *(where* G *is the monodromy matrix of* (0.1)*), enumerated in any prescribed ordering, and normalized by requiring their real parts to be in the half–open interval* $[0, 1)$*. Then there exist integers* $d_1, \ldots, d_n$ *which, for a suitable permutation* σ*, satisfy* $0 \leq d_{\sigma(j+1)} - d_{\sigma(j)} \leq r$*, such that* (0.1) *can be analytically transformed to B.s.f.* $B(z)$*, with* $B(0)$ *in lower triangular Jordan canonical form and having eigenvalues equal to* $\mu_j + k_j + d_j$*,* $1 \leq j \leq n$*.*

Proof: Follows from an analysis of the proof of Theorem 1 in [6]. □

Lemma 3 *Suppose that for some equation* (1.1) *we have (with* t_ν *as defined above)*

$$\frac{t_\nu}{s_\nu} + \frac{r s_\nu (s_\nu - 1)}{2} + \frac{s_\nu - 1}{s_\nu} \leq \frac{t_{\nu+1}}{s_{\nu+1}} - \frac{r s_{\nu+1}(s_{\nu+1} - 1)}{2} - \frac{s_{\nu+1} - 1}{s_{\nu+1}} \tag{1.6}$$

for $\nu = 1, \ldots, m-1$ *(recall that* s_ν *is the dimension of the corresponding diagonal block). Then* (1.1) *is analytically equivalent to B.s.f.*

Proof: It follows from Lemma 2, taking $k_j = jr(s_\nu - 1)$, that there is a diagonally blocked analytic transformation for which the constant term of the νth diagonal block of the transformed equation has eigenvalues $\mu_j + jr(s_\nu - 1) + d_j$, $1 \le j \le s_\nu$, with real parts of μ_j in $[0, 1)$ and d_j as described in Lemma 2. Moreover, their sum is an analytic invariant, according to Proposition 1. This implies

$$t_\nu = \mathrm{Re}\left(\sum_{i=1}^{s_\nu} \mu_i\right) + \frac{r}{2} s_\nu (s_\nu - 1)(s_\nu + 1) + \sum_{i=1}^{s_\nu} d_i.$$

Let d and D denote the minimal and the maximal value of the numbers d_j respectively. Let in turn μ and $\tilde{\mu}$ denote the numbers with the minimal and the maximal real parts respectively from the numbers μ_j. Then

$$D - \frac{1}{s_\nu} \sum_{i=1}^{s_\nu} d_i \le \frac{r(s_\nu - 1)}{2}, \qquad \frac{1}{s_\nu} \sum_{i=1}^{s_\nu} d_i - d \le \frac{r(s_\nu - 1)}{2},$$

$$\mathrm{Re}\left(\tilde{\mu} - \frac{1}{s_\nu} \sum_{i=1}^{s_\nu} \mu_i\right) \le \frac{s_\nu - 1}{s_\nu}, \qquad \mathrm{Re}\left(\frac{1}{s_\nu} \sum_{i=1}^{s_\nu} \mu_i - \mu\right) < \frac{s_\nu - 1}{s_\nu}.$$

Hence we may assume without loss in generality that the corresponding assertions already hold for the eigenvalues of $A_{\nu\nu}(0)$, which then implies that the difference between the maximal value for the real part of any eigenvalue and the average t_ν / s_ν is bounded by $r s_\nu (s_\nu - 1)/2 + (s_\nu - 1)/s_\nu$, while the difference between t_ν / s_ν and the minimal value of the real parts is strictly smaller than the same number $r s_\nu (s_\nu - 1)/2 + (s_\nu - 1)/s_\nu$. Hence we see that (1.6) implies that **E)** holds, so the proof is completed using Theorem 1. □

Roughly speaking, the proof of Lemma 3 is based upon choosing numbers k_j with minimal differences. The following theorem gives another type of sufficient condition under which a system (1.1) is analytically eqivalent to B.s.f, based on an application of Lemma 2, but choosing k_j with *sufficiently large* differences. (This theorem for the case of two blocks is contained in Corollary 1 of [8] and for any number of blocks it is an immediate corollary of Theorem 1 and Remark 2 in [8], but here we present a simpler proof).

Theorem 2 *Let an equation* (1.1)*, with polynomial diagonal blocks* $A_{\nu\nu}(z)$*, be given, and assume that for* $2 \le \nu \le m$ *each matrix* $A_{\nu\nu}(0)$ *has at least one eigenvalue which is incongruent modulo one to all eigenvalues of the matrices* $A_{11}(0), \ldots, A_{\nu-1,\nu-1}(0)$*. Then* (1.1) *is analytically equivalent to B.s.f.*

Proof: Note that the eigenvalues of $A_{\nu\nu}(0)$, reduced modulo one such that their real parts are in $[0, 1)$, coincide with the eigenvalues of the matrix

$(1/2\pi i)\ln G_\nu$ (where G_ν is the monodromy matrix of the νth diagonal block of (1.1), and let these be denoted by $\mu_1^{(\nu)}, \dots, \mu_{s_\nu}^{(\nu)}$, enumerated such that for $2 \le \nu \le m$, the eigenvalue $\mu_1^{(\nu)}$ is distinct from the eigenvalues corresponding to the previous blocks (observe that by assumption such an eigenvalue exists). Applying Lemma 2 (to each diagonal block) with $k_2, \dots, k_{s_\nu}$ sufficiently large (relative to k_1), one can show that (1.1) is analytically equivalent to an equation for which condition **E)** holds. □

2 Meromorphic transformations

In view of Lemma 3 and Proposition 1, we shall now address the question whether in case (1.6) fails for a given equation, we can find a *meromorphic* transformation to a likewise blocked equation (of the same Poincaré rank) for which the corresponding inequality holds. For this purpose we will consider a fixed formal fundamental solution $H(z)$ of (1.1) of the following form:

$$H(z) = \hat{F}(z)\, z^L \ \exp[Q(z)], \tag{2.1}$$

with

- a lower triangularly blocked formal (matrix) power series (in $1/z$) $\hat{F}(z)$, whose formal determinant is not the zero series,
- a diagonal matrix $Q(z)$ of polynomials *in roots of* z, such that (in the block structure of $A(z)$) each diagonal block of $Q(z)$ is *closed under analytic continuation*; this means that for every diagonal entry $q(z)$ of $Q(z)$ all (finitely many) analytic continuations $q(z\exp[2\nu\pi i])$ belong to the same diagonal block as $q(z)$, so that we can find a diagonally blocked matrix $R = \mathrm{diag}[R_1, \dots, R_m]$, with each block being a permutation matrix, for which
$$Q(z\mathrm{e}^{2\pi i}) = R^{-1}Q(z)R,$$
and
- a constant, lower triangularly blocked matrix L for which $\exp[2\pi i L] = DR$, with R as above, and D commuting with $Q(z)$ (and being lower triangularly blocked), and we choose L so that its eigenvalues have real parts in the interval $[0, 1)$.

Remark 3 *A reduced system always has such a formal fundamental solution: According to* [4]*, every such system possesses a formal fundamental* *solution $H(z) = \Psi(z)\exp[Q(z)]$, with $Q(z)$ as above and $\Psi(z)$ being a formal logarithmic matrix having the same lower triangular block structure as $A(z)$, and combining results from* [4] *and* [3]*, one can see that $H(z)$ can be chosen to be of the form required above.*

With help of such a formal fundamental solution we define *the formal meromorphic normal form* of (0.1) to be the system

$$zx' = C(z)\,x, \quad C(z) = L + z^{L+I} Q'(z)\, z^{-L}. \tag{2.2}$$

It follows from the structure of the formal solution that $C(z e^{2\pi i}) = C(z)$, and some elementary estimate, using the assumption upon the eigenvalues of L, then shows that (2.2) has Poincaré rank equal to the smallest natural number which is larger than or equal to the maximal (rational) degree of the elements in $Q(z)$. This, however, shows that its rank cannot be larger than r (which was the Poincaré rank of (0.1)).

We shall call a lower triangularly blocked meromorphic transformation $T(z)$ *admissible* (w.r.to (2.2)), if it does not increase the Poincaré rank when applied to (2.2)). For the diagonal blocks $T_{\nu\nu}(z)$ of $T(z)$, we define $k_\nu = k_{T_{\nu\nu}}$ as in Proposition 1. Then we have

Proposition 2 *Let $A(z)$ be as in* (1.1)*, with polynomial diagonal blocks. Then to every admissible meromorphic transformation $T(z)$ we can find another meromorphic transformation $\tilde{T}(z)$, lower triangularly blocked, transforming* (1.1) *into $B(z)$ (having Poincaré rank at most r), with polynomial diagonal blocks $B_{jj}(z)$ such that*

$$\mathrm{trace}(B_{jj}(0)) = \mathrm{trace}(L_{jj}) - k_j, \quad 1 \le j \le m.$$

Proof: We may write the formal fundamental solution $H(z)$ of (1.1) in the form $H(z) = \hat{F}(z)\,T(z)\,G(z)$, $G(z) = T^{-1}(z) z^L \exp[Q(z)]$. For sufficiently large integer N, we can factor $\hat{F}(z)\,T(z) = \tilde{T}(z)\,\hat{F}_N(z)$, with a meromorphic transformation $\tilde{T}(z)$ (which, in fact, has a terminating expansion in z^{-1}) and a formal power series in z^{-1} of the form $\hat{F}_N(z) = I + \mathrm{O}(z^{-N})$, both factors being lower triangularly blocked. The transformation $x = \tilde{T}(z)y$ then produces an equation with coefficient matrix $B(z)$ (as in (0.3)) which is also lower triangularly blocked, but will, in general, not have polynomials for its diagonal blocks. Moreover, the Poincaré rank of (0.3)) can be checked to equal that of the *formal meromorphic normal form* (because of definition of admissibility of $T(z)$), hence is the minimal rank that can occur within the equivalence class w.r. to formal meromorphic transformations. Therefore, it cannot be larger than that of (1.1). From the fact that (0.3)) has a formal fundamental solution of the form $\tilde{H}(z) = \hat{F}_N(z)\,G(z)$, one can compute, using Proposition 1:

$$\text{trace}(B_{jj}(z)) = zp_j(z) + \text{trace}(L_j) - k_j + \mathrm{O}(z^{-1}), \tag{2.3}$$

with a polynomial $p_j(z)$. The diagonal blocks of (0.3)) are again irreducible (note that, according to [4], irreducibility with respect to meromorphic transformations coincides with irreducibility with respect to analytic ones). So applying Bolibruch's result to the diagonal blocks of (0.3)), we may find a diagonally blocked analytic transformation for which the transformed equation then has polynomial diagonal blocks. However, this analytic transformation does not change the leading terms in (2.3), which completes the proof. □

We shall now give sufficient conditions for the existence of an admissible transformation for which (1.6) is satisfied:

Theorem 3 *Let an equation* (1.1)*, with polynomial diagonal blocks, be given, and let* (2.1) *be a formal fundamental solution of* (1.1)*, for which the matrix* L *is diagonally blocked. Then* (1.1) *is meromorphically equivalent to B.s.f.*

Proof: Observe that the transformation $T(z) = \text{diag}[z^{k_1} I_{s_1}, \ldots, z^{k_m} I_{s_m}]$ is admissible, for every choice of integers $k_1, \ldots, k_m$, and apply Proposition 2. □

Since it may not be so easy to verify that L is diagonally blocked, it is worthwhile to give the following sufficient condition under which this is going to happen:

Corollary to Theorem 3 *Let an equation* (1.1)*, with polynomial diagonal blocks*

$$A_{\nu\nu}(z) = z^r A_{\nu\nu} + \ldots,$$

be given, and assume that the spectra of the matrices $A_{\nu\nu}$ *are all distinct. Then* (1.1) *is meromorphically equivalent to B.s.f.*

Proof: Note that the eigenvalues of $A_{\nu\nu}$, divided by r, equal the leading terms of the polynomials in $Q(z)$, and use this to conclude that L then must be diagonally blocked. □

Aside from the situation where L is diagonally blocked, there are many more cases in which L has a form such that admissible transformations of the form $T(z) = z^K$, with a diagonal matrix K of integer diagonal entries, exist for which the transformed equation satisfies (1.6), and consequently is meromorphically equivalent to B.s.f. This is so, whenever we can find matrices $K = \text{diag}[K_1, \ldots, K_m]$ for which the numbers $k_j = \text{trace}(K_j)$ have sufficiently large differences, so that the transformation $T(z) = z^K$ is admissible, i.e. the Poincaré rank of

$$z^{-K} C(z) z^K - K = z^{-K} [L - K + z^{L+I} Q'(z)\, z^{-L}] z^K$$

is not larger than that of $C(z)$. We shall not go into more detail about this here, since it is not so easy to generally describe when this is going to happen.

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