Summability of formal power series solutions of a perturbed heat equation

Werner Balser*
Abteilung Angewandte Analysis
Universität Ulm
D- 89069 Ulm, Germany
balser@mathematik.uni-ulm.de

Michèle Loday-Richaud
UFR Sciences
Université d'Angers
F-49045 Angers, France
michele.loday@univ-angers.fr

November 2, 2006

Abstract

In this article we investigate power series solutions of a simple second order partial differential equation in two variables. In contrast to previous articles of various authors, we allow one term in the equation to have a coefficient that is an arbitrary holomorphic function $a(z)$. We show that for arbitrary $a(z)$ the same results concerning 1-summability of the formal solution hold that had been obtained earlier by *Lutz, Miyake, and Schäfke*, resp. by *W. Balser*, for the case of $a(z) \equiv 1$, when the equation becomes the complex heat equation.

Introduction

In an article of *Lutz, Miyake, and Schäfke* [12], the Cauchy problem for the complex heat equation $u_t = u_{zz}$, $u(0, z) = \phi(z)$, with a given function $\phi(z)$ that is holomorphic near the origin, has been investigated. In context with the so-called *Cauchy-Kowalewskaya theory*, it had been observed much earlier that this problem has a unique solution in the space of power series in the variable t with coefficients depending holomorphically upon z. However, the radius of

*Work done while being a Visiting Professor at the University of Angers in May 2003. The first author is grateful to the second for inviting him, and to University of Angers for financial support.

convergence of this power series, in general, is equal to 0: Indeed, one verifies that this power series equals

$$\hat{u}(t,z) \;=\; \sum_{j=0}^{\infty} \frac{t^j}{j!}\,\phi^{(2j)}(z)\,,$$

where we use the notation $\hat{u}$ to emphasize divergence of this series for all $t \neq 0$. This divergence is caused by the fact that the $(2j)$th derivative of $\phi(z)$, in general, is of "magnitude" $(2j)!$ as $j \to \infty$. As their main result, the authors of [12] have shown that this series is 1-summable[1] in a direction $d \in \mathbb{R}$ if, and only if, the initial condition $\phi(z)$ admits holomorphic continuation into (small) sectors bisected by the rays $\arg z = d/2$, resp. $\arg z = \pi + d/2$, and can be estimated by $|\phi(z)| \le C\,\exp(K|z|^2)$ in these sectors, with constants $C, K > 0$. In later articles, *Balser* [1], *Miyake* [14], and *Balser and Miyake* [6] have obtained analogous results for more general PDE in two variables with constant coefficients, under various (restrictive) assumptions on their form. Finally, in [4] arbitrary PDE in two variables and with constant coefficients have been shown to have formal power series solutions that are, in general, multisummable in the sense of *J. Ecalle* [8, 9, 10]. Very recently, a first attempt has been made by *S. Malek* [13] to obtain analogous results for equations with constant coefficients but more than two variables. However, in this situation additional difficulties arise, due to the fact that algebraic functions in several variables exhibit a much more complicated behavior at infinity.

While the results mentioned above all concern PDE with constant coefficients, only few authors have treated the problem of summability of formal solutions for equations with variable coefficients: *M. Hibino* studied first order equations in [11], *S. Ouchi* [15] treated cases that can be viewed best as a perturbation of an ordinary differential equation with an irregular singularity at the origin, and *Pliś and Ziemian* [16] studied inhomogeneous equations with a homogeneous part that fits into the situation of the classical Cauchy-Kowalewski theory, so that divergence of formal solutions is caused by the inhomogenuity only.

In this article, we shall mainly be interested in formal power series solutions of an initial value problem

$$\big(\partial_t \;-\; a(z)\,\partial_z^2\big)\;u \;=\; 0\,, \qquad u(0,z) \;=\; \phi(z)\,, \tag{0.1}$$

where $a(z)$ and $\phi(z)$ are functions that are holomorphic in the disc $\mathbb{D}_\rho$ about the origin of radius $\rho > 0$, while ∂_z, ∂_t stand for partial derivation with respect to z and t, respectively. For $a(z) \equiv 1$, this problem coincides with the one studied in [12], and we shall show that even for general $a(z)$ the formal solution of (0.1) is 1-summable if, and only if, the initial condition $\phi(z)$ has a certain additional property which, however, in general is much harder to verify than for the heat equation case. To obtain this result, it is more natural, both for the sake of

[1]For a detailed discussion of 1-summability, refer to Section 2.

generality as well as for notational convenience, to study an inhomogeneous equation which is best written in an *integrated form* as

$$\big(1 \ - \ a(z)\,\partial_t^{-1}\,\partial_z^2\,\big)\ \hat{u}(t,z) \ = \ \hat{f}(t,z)\,, \tag{0.2}$$

where $\hat{u}(t,z)$, $\hat{f}(t,z)$ are formal power series in t with coefficients that are holomorphic functions of z in $\mathbb{D}_\rho$, and ∂_t^{-1} denotes termwise integration of power series in t. In case that the right hand side is independent of t, this equation is equivalent to the initial value problem (0.1), while in general we can differentiate both sides of (0.2) to see its equivalence with an inhomogeneous initial value problem with right hand side equal to $\partial_t\,\hat{f}(t,z)$ and initial condition $\phi(z) = \hat{f}(0,z)$. For a recent treatment of the inhomogenous case with $a(z) \equiv 1$, see [5].

1 Gevrey order of formal solutions

In order to deal with (0.2), we shall throughout use the following notation:

- Given $\rho > 0$, we shall write $\mathbb{D}_\rho$ for the disc of radius ρ about the origin. By $\mathcal{O}_\rho = \mathcal{O}(\mathbb{D}_\rho)$ we shall denote the space of functions that are holomorphic in $\mathbb{D}_\rho$.

- The set of formal power series in t with coefficients in $\mathbb{C}$, resp. in $\mathcal{O}_\rho$, shall be denoted by $\mathbb{C}[[t]]$, resp. $\mathcal{O}_\rho[[t]]$. Series $\hat{x}(t,z) \in \mathcal{O}_\rho[[t]]$ can then also be viewed as (formal) power series in z with coefficients in $\mathbb{C}[[t]]$, or as power series in two variables t and z, and shall always be written as

$$\hat{x}(t,z) \ = \ \sum_{j=0}^{\infty} \frac{t^j}{j!}\,x_{j*}(z) \ = \ \sum_{n=0}^{\infty} \frac{z^n}{n!}\,\hat{x}_{*n}(t) \ = \ \sum_{j,n=0}^{\infty} \frac{t^j\,z^n}{j!\,n!}\,x_{jn}\,. \tag{1.1}$$

  For formal solutions $\hat{u}(t,z)$, resp. the inhomogenuity $\hat{f}(t,z)$, of (0.2) we shall use a corresponding notation, with x replaced by u, resp. f, everywhere in (1.1).

- We write $\mathcal{O}_\rho[[t]]_1$ for the set of formal power series $\hat{x}(t,z) \in \mathcal{O}_\rho[[t]]$, for which one can find a radius $r \in (0,\rho]$ and constants $C, K > 0$ such that

$$|x_{j*}(z)| \ \le \ C\,K^j\,(2j)! \qquad \forall\ j \ge 0 \quad \forall\ z \in \mathbb{D}_r\,. \tag{1.2}$$

  Observe that then $x_{j*}(z)/j!$ is of "magnitude" at most $j!$, so that $\mathcal{O}_\rho[[t]]_1$ is the space of series of Gevrey order equal to 1 and coefficients in $\mathcal{O}_\rho$.

- For $f \in \mathcal{O}_\rho$, $p \ge 0$, and $r \in (0,\rho)$, we denote the *Nagumo norm of order* p of f as

$$\|f\|_p = \sup_{|z|<r} \big(|f(z)| \cdot (r - |z|)^p\big)\,.$$

Obviously, $\|f\|_p$ also depends upon the choice of r, but to simplify notation, we choose not to display this dependence here. One can verify the following properties of these norms (for a proof of the last and most important property in a slightly more general case, see [7]):

$$\|f+g\|_p \;\le\; \|f\|_p + \|g\|_p,\ \|fg\|_{p+q} \;\le\; \|f\|_p \cdot \|g\|_q,\ \|f'\|_{p+1} \;\le\; \mathrm{e}\,(p+1)\|f\|_p$$

for every r, p as above, and every $f, g \in \mathcal{O}_\rho$. It is worth mentioning that $\|f\|_0$ is nothing but the sup-norm of f on the disc $\mathbb{D}_r$.

Let $\hat f(t,z), \hat u(t,z) \in \mathcal{O}_\rho[[t]]$. Expanding as in (1.1), inserting into (0.2) and comparing coefficients, we obtain that (0.2) holds if, and only if, we have

$$u_{j*}(z) \;=\; f_{j*}(z) \;+\; a(z)\, u''_{j-1*}(z) \qquad \forall\ j \ge 0 \quad \forall\ z \in \mathbb{D}_\rho\,, \tag{1.3}$$

interpreting $u_{-1*}(z) \equiv 0$. Accordingly, we observe that (0.2) defines a bijective map $\hat f(t,z) \longmapsto \hat u(t,z)$ from $\mathcal{O}_\rho[[t]]$ onto itself. Regarding series of Gevrey order equal to 1, we now prove the following theorem:

Theorem 1 *The map $\hat f(t,z) \longmapsto \hat u(t,z)$ is bijective from $\mathcal{O}_\rho[[t]]_1$ onto itself.*

Proof: Let $\hat u(t,z) \in \mathcal{O}_\rho[[t]]_1$, and define $f_{j*}(z)$ by (1.3). Then there exists an $r \in (0,\rho]$ and $C, K > 0$ such that $\|u_{j*}\|_0 \le C\,K^j\,(2j)!$ holds for every $j \ge 0$. Using the properties of Nagumo norms that were listed above, we find that $\|u''_{j-1*}\|_{2j} \le \mathrm{e}^2\, 2j\,(2j-1)\,\|u_{j-1*}\|_{2j-2}$ for $j \ge 1$. Accordingly, one obtains from (1.3)

$$\|f_{j*}\|_{2j} \;\le\; \|u_{j*}\|_{2j} \;+\; \|a\|_0\, \mathrm{e}^2\, 2j\,(2j-1)\,\|u_{j-1*}\|_{2j-2} \qquad \forall\ j \ge 1\,.$$

Since $\|u_{j*}\|_{2j} \le r^{2j}\,\|u_{j*}\|_0$, the right hand side is bounded by $\tilde C\,\tilde K^j\,(2j)!$, with $\tilde C = C\,(1 + \|a\|_0\,\mathrm{e}^2\,K^{-1}r^{-2})$ and $\tilde K = K\,r^2$, so $\hat f(t,z) \in \mathcal{O}_\rho[[t]]_1$ follows. Conversely, if $\hat f(t,z) \in \mathcal{O}_\rho[[t]]_1$, then we find by the same steps as above that for every $r < \rho$ we have

$$\|u_{j*}\|_{2j} \;\le\; \|f_{j*}\|_{2j} \;+\; \|a\|_0\, \mathrm{e}^2\, 2j\,(2j-1)\,\|u_{j-1*}\|_{2j-2} \qquad \forall\ j \ge 1\,.$$

For $g_j = \|f_{j*}\|_{2j}/(2j)!$, define $v_j = g_j + a\,v_{j-1}$, $j \ge 0$, with $v_{-1} = 0$ and $a = \|a\|_0$. Then by induction $\|u_{j*}\|_{2j}/(2j)! \le v_j$ follows. Setting $g(x) = \sum_0^\infty g_j\,x^j$, $v(x) = \sum_0^\infty v_j\,x^j$, we conclude from the recursion for the v_j that $v(x) = g(x)/(1 - a\,x)$ holds formally. However, $\hat f(t,z) \in \mathcal{O}_\rho[[t]]_1$ implies that the series for $g(x)$ has positive radius of convergence, and so the same follows for $v(x)$. This, however, is equivalent to $\hat u(t,z) \in \mathcal{O}_\rho[[t]]_1$. □

2 Summability of the formal solution

In this section we shall show that 1-summability of the formal solution $\hat u(t,z)$ of (0.2) depends upon certain conditions that $\hat f(t,z)$ has to satisfy. To do so, we shall use the notion of summability of series with coefficients in a Banach space. For a general treatment of this theory we shall rely on [2]. Here we shall, for fixed $r > 0$, consider the Banach space of functions that are holomorphic in $\mathbb{D}_r$ and continuous up to its boundary, equipped with the usual sup-norm. We then say that, for some $k > 0$ and $d \in \mathbb{R}$, a series $\hat{x}(t,z) \in \mathcal{O}_\rho[[t]]$ is k-summable in a direction d, if *one can find some* $r \in (0,\rho)$ so that the following two properties hold:

- The *formal Borel transform with respect to the variable t of order k*, i. e. to say, the power series

$$w(t,z) \;=\; \sum_{j=0}^{\infty} u_j(z) \, \frac{t^j}{j! \, \Gamma(1+j/k)} \;, \tag{2.1}$$

  converges absolutely for $|z| \le r$ and $|t| < R$, with $R > 0$ that may depend upon r but is independent of z.

- There exists a $\delta > 0$ so that for every $z \in \overline{\mathbb{D}}_r$ the function $w(t,z)$ can be continued with respect to t into the sector $S_{d,\delta} = \{t : |d - \arg t| < \delta\}$. Moreover, for every $\delta_1 < \delta$ there exist constants $C, K > 0$ so that

$$\|w(t,\cdot)\|_0 \;=\; \sup_{|z|\le r} |w(t,z)| \;\le\; C \, \exp[K\,|t|^k] \qquad \forall\, t \in S_{d,\delta_1} \,. \tag{2.2}$$

If this is so, the Laplace transform of order k of $w(t,z)$, i. e. to say, the function

$$x(t,z) \;=\; t^{-k} \int_0^{\infty(\gamma)} w(\tau,z)\, \mathrm{e}^{-(\tau/t)^k} \, d\tau^k \,,$$

integrating along the ray $\arg \tau = \gamma$ with $|d - \gamma| < \delta$, is called *the k-sum in the direction d of the formal series* $\hat{x}(t,z)$. It follows from the general theory presented in [2] that this sum is holomorphic in $G_r \times \mathbb{D}_r$, with a sectorial region G_r of opening larger than π/k and bisecting direction $\arg t = d$, and $x(t,z)$ is continuous with respect to z up to the boundary of $\mathbb{D}_r$. Observe that the above definition of k-summability agrees with *J.-P. Ramis's* [17] original one when all the functions $u_j(z)$ are constant. In fact, the following lemma characterizes k-summability of a series in two variables in terms of a sequence of series in one variable: For $\hat{x}(t,z) \in \mathcal{O}_\rho[[t]]$, let $\hat{x}_{*n}(t)$ be as defined above; then we have

Lemma 1 *For $k > 0$ and $d \in \mathbb{R}$, the following statements are equivalent:*

(a) *The formal series $\hat{x}(t,z) \in \mathcal{O}_\rho[[t]]$ is k-summable in the direction d.*

(b) *The formal series $\hat{x}_{*n}(t)$ all are k-summable in the direction d. Moreover, there exists a sectorial region G that is independent of n and has opening larger than π/k and bisecting direction d, in which all sums $x_{*n}(t)$ of the series $\hat{x}_{*n}(t)$ are holomorphic, for $n \ge 0$. Finally, for every closed subsector $\overline{S}$ in G there exist constants $C, K > 0$, independent of n, so that for $s = 1 + 1/k$ and every $n, \ell \ge 0$ and $t \in \overline{S}$*

$$|\partial_t^\ell \, x_{*n}(t)| \;\le\; C\, K^{n+\ell}\, n!\, \Gamma(1 + s\,\ell)\,.$$

Proof: This lemma is a consequence of a more general result in [3], but we include its simple proof here for the sake of completeness: Assume (a). Then for some $r \in (0,\rho)$ there exists a sectorial region G that has opening larger than π/k and bisecting direction d, so that the sum $x(t,z)$ of the series $\hat{x}(t,z)$ is holomorphic in $G \times \mathbb{D}_r$. Moreover, for every closed subsector $\overline{S}$ in G there exist constants $C, K > 0$, so that

$$|\partial_t^\ell \, x(t,z)| \ \leq \ C\, K^{n+\ell}\, \Gamma(1+s\,\ell) \qquad \forall\ \ell \geq 0\,,\ t \in \overline{S}\,,\ |z| \leq r\,.$$

Setting $x_{*n}(t) = \partial_z^n x(t,z)|_{z=0}$, we observe

$$\partial_t^\ell \, x_{*n}(t) \ = \ \frac{n!}{2\pi\, i} \oint_{|z|=r} \frac{\partial_t^\ell \, x(t,z)}{z^{n+1}} \, dz\,,$$

and this implies the estimate in (b), with C, K not necessarily the same as above. This shows that each $x_{*n}(t)$ is the k-sum in direction d of some series $\hat{x}_{*n}(t)$, and the definition of $x_{*n}(t)$ implies $\hat{x}_{*n}(t) = \partial_z^n \hat{x}(t,z)|_{z=0}$, hence (b) follows. Conversely, if (b) holds, define $x(t,z) = \sum_{n=0}^\infty x_{*n}(t)\, z^n/n!$. Then we find that for every closed subsector $\overline{S} \subset G$ there exists $r \geq 0$ such that the series converges for $|z| < r$ and $t \in \overline{S}$. Even more than that, we obtain for $t \in S$ that $\partial_t^\ell x(t,z) = \sum_{n=0}^\infty \partial_t^\ell x_{*n}(t)\, z^n/n!$, with the series converging for $|z| < r = 1/K$, and $|\partial_t^\ell x(t,z)| \leq C\, K^\ell\, \Gamma(1+s\,\ell)\,(1-K\,|z|)^{-1}$. Choosing a subsector S of G whose opening is larger than π/k, this implies that $x(t,z)$ is the k-sum of $\hat{x}(t,z) = \sum_{n=0}^\infty \hat{u}_{*n}(t)\, z^n/n!$. □

Using the above lemma, we can now characterize 1-summability of the formal solution:

Theorem 2 *Let $d \in \mathbb{R}$, and assume that $a(0) \neq 0$. Given $\hat{f}(t,z) \in \mathcal{O}_\rho[[t]]$, the formal solution $\hat{u}(t,z)$ of* (0.2) *is 1-summable in the direction d if, and only if, the series $\hat{u}_{*0}(t), \hat{u}_{*1}(t)$ and $\hat{f}(t,z)$ all are 1-summable in the direction d.*

Proof: If the solution $\hat{u}(t,z)$ of (0.2) is 1-summable in the direction d, then we learn from Lemma 1 that all $\hat{u}_{*n}(t)$ are equally summable. Moreover, the general theory implies that partial derivatives of 1-summable series are so summable, too, and so we obtain from (0.2) the 1-summability in the direction d of $\hat{f}(t,z)$. Conversely, assume 1-summability in the direction d of $\hat{u}_{*0}(t), \hat{u}_{*1}(t)$ and $\hat{f}(t,z)$. Set $\hat{u}(t,z) = \hat{u}_{*0}(t) + z\,\hat{u}_{*1}(t) + \partial_z^{-2}\hat{v}(t,z)$. Then $\hat{v}(t,z)$ is a uniquely determined power series from $\mathcal{O}_\rho[[t]]$, satisfying

$$(1 \ - \ b(z)\,\partial_z^{-2}\,\partial_t)\ \hat{v}(t,z) \ = \ \hat{g}(t,z)\,,$$

with $b(z) = 1/a(z)$, $\hat{g}(t,z) = b(z)\,\partial_t\,(\hat{u}_{*0}(t) + z\,\hat{u}_{*1}(t) - \hat{f}(t,z))$. By assumption on $a(z)$, its reciprocal $b(z)$ is holomorphic in a disc about the origin, whose radius shall be denoted by $\tilde{\rho} \leq \rho$. The series $\hat{g}(t,z) \in \mathcal{O}_{\tilde{\rho}}[[t]]$ is 1-summable in the direction d, and the proof shall be completed once we have shown 1-summability of $\hat{v}(t,z)$ in the direction d. To do this, observe that

$$\hat{v}(t,z) \ = \ \sum_{n=0}^\infty \hat{v}_n(t,z)\,, \quad \hat{v}_{n+1}(t,z) = b(z)\,\partial_t\,\partial_z^{-2}\,\hat{v}_n(t,z) \quad \forall\ n \geq 0\,,$$

with $\hat v_0(t,z) = \hat g(t,z)$. According to the theory of 1-summability, each one of the series $\hat v_n(t,z)$ is 1-summable in the direction d, and we denote its sum by $v_n(t,z)$. Then we conclude that $v_{n+1}(t,z) = b(z)\,\partial_t\,\partial_z^{-2}\,v_n(t,z)$ for every $n \ge 0$, and this implies existence of a sectorial region G that is independent of n and has bisecting direction d and opening more than π, and an $r > 0$, also independent of n, so that all the sums $v_n(t,z)$ are holomorphic in $G \times \mathbb{D}_r$ and continuous in z up to the boundary of $\mathbb{D}_r$. Let $\overline{S}$ be a closed subsector of G. From the theory of 1-summability we conclude existence of constants C, K such that $|\partial_t^\ell\, v_0(t,z)| \le C\,K^\ell\,\Gamma(1+2\ell)$ for every $\ell \ge 0$ and $(t,z) \in \overline{S} \times \overline{\mathbb{D}_r}$. With $B = \max\{|b(z)| \,:\, z \in \overline{\mathbb{D}_r}\,\}$ we conclude by induction with respect to n the estimate

$$|\partial_t^\ell\, v_n(t,z)| \;\le\; C\,K^{n+\ell}\,\Gamma(1+2(n+\ell))\,B^n\,\frac{|z|^{2n}}{\Gamma(1+2n)} \qquad \forall\, n \ge 0\,. \tag{2.3}$$

Summation over n then implies

$$|\partial_t^\ell\, v(t,z)| \;\le\; C\,K^\ell\,\Gamma(1+2\ell)\sum_{n=0}^{\infty}\binom{2(n+\ell)}{2n}\,(K\,B\,|z|^2)^n\,.$$

The series on the right hand side converges for sufficiently small values of $|z|$, and can, for fixed z, be bounded by $\tilde K^\ell$, with sufficiently large $\tilde K$, independent of ℓ. So from this estimate we conclude that $v(t,z)$ is the 1-sum in the direction d of some series which, according to the definition of $v(t,z)$, is equal to $\hat v(t,z)$. $\square$

We shall end this article with a discussion of how to apply the above result: The formal series $\hat f(t,z)$ may be considered as known, and while its 1-summability may not be obvious, we shall assume that this is known. In fact, in many applications this series may even converge for sufficiently small values of $|t|$. Hence, what will cause difficulties in a concrete application is that one has to verify 1-summability of the series $\hat u_{*0}(t), \hat u_{*1}(t)$ as well, which are not among the given data. However, the whole formal solution $\hat u(t,z)$, and in particular $\hat u_{*0}(t), \hat u_{*1}(t)$, can (at least theoretically) be computed in terms of $\hat f(t,z)$, and so in a weak sense the summability of $\hat u_{*0}(t), \hat u_{*1}(t)$ can be verified. On the other hand, an *explicit* computation of $\hat u_{*0}(t), \hat u_{*1}(t)$ may not be possible for a general coefficient $a(z)$. However, if $a(z)$ is a constant $a \ne 0$, this computation can be done and implies that

$$\hat u_{*0}(t) \;=\; \sum_{j,k=0}^{\infty}\frac{a^k\,t^{j+k}}{(j+k)!}\,f_{j,2k}\;, \quad \hat u_{*1}(t) \;=\; \sum_{j,k=0}^{\infty}\frac{a^k\,t^{j+k}}{(j+k)!}\,f_{j,2k+1}\;. \tag{2.4}$$

In case of a homogeneous Cauchy problem, i. e., when $f_{jn} = 0$ for every $j \ge 1$, then

$$\hat u_{*0}(t) \;=\; \sum_{k=0}^{\infty}\frac{a^k\,t^k}{k!}\,f_{0,2k}\;, \quad \hat u_{*1}(t) \;=\; \sum_{k=0}^{\infty}\frac{a^k\,t^k}{k!}\,f_{0,2k+1}\,,$$

and 1-summability in a direction d of these series holds if, and only if, the series

$$v_0(t) \;=\; \sum_{k=0}^{\infty} \frac{a^k\, t^k}{(2k)!}\, f_{0,2k}\ , \quad v_1(t) \;=\; \sum_{k=0}^{\infty} \frac{a^k\, t^k}{(2k+1)!}\, f_{0,2k+1}$$

both have positive radius of convergence and the two functions so defined can be continued into a small sector bisected by the ray $\arg t = d$ and are of exponential growth of order at most 1 in this sector. This in turn is equivalent to the functions

$$g_+(z) \;=\; \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!}\, f_{0,2k}\ , \quad g_-(z) \;=\; \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!}\, f_{0,2k+1}$$

being holomorphic in two sectors bisected by rays $\tilde{d} = (d + \arg a)/2$ and the "opposite one" $\pi + \tilde{d}$, and being of growth of order 2 there. Since g_+ is even while g_- is odd, these properties are equivalent to $g(z) = g_+(z) + g_-(z)$ being holomorphic and of order 2 in the same sector, and this function is equal to $f_{0*}(z)$, i. e., to the initial condition for the homogeneous Cauchy problem. For $a = 1$, this is exactly equal to the necessary and sufficient condition for summability of formal solutions of the heat equation, found by *Lutz, Miyake, and Schäfke* [12].

References

[1] W. Balser, *Divergent solutions of the heat equation: on an article of Lutz, Miyake and Schäfke*, Pacific J. of Math., 188 (1999), pp. 53–63.

[2] ——, *Formal power series and linear systems of meromorphic ordinary differential equations*, Springer-Verlag, New York, 2000.

[3] ——, *Summability of power series that are formal solutions of partial differential equations with constant coefficients*, Sovrem. Mat. Fundam. Napravl., 1 (2003), pp. 5–17 (electronic). Differential and functional differential equations. Part 1 (Russian).

[4] ——, *Multisummability of formal power series solutions of partial differential equations with constant coefficients*, J. Differential Equations, 201 (2004), pp. 63–74.

[5] ——, *Power series solutions of the inhomogeneous heat equation*, Sūrikaisekikenkyūsho Kōkyūroku, RIMS Kyoto, 1412 (2005), pp. 151–159.

[6] W. Balser and M. Miyake, *Summability of formal solutions of certain partial differential equations*, Acta Sci. Math. (Szeged), 65 (1999), pp. 543–551.

[7] M. Canalis-Durand, J.-P. Ramis, R. Schäfke, and Y. Sibuya, *Gevrey solutions of singularly perturbed differential and difference equations*, J. reine und angew. Math., 518 (2000), pp. 95–129.

[8] J. Ecalle, *Les fonctions résurgentes I–II*, Publ. Math. d'Orsay, Université Paris Sud, 1981.

[9] ——, *Les fonctions résurgentes III*, Publ. Math. d'Orsay, Université Paris Sud, 1985.

[10] ——, *Introduction à l'Accélération et à ses Applications*, Travaux en Cours, Hermann, Paris, 1993.

[11] M. Hibino, *Divergence property of formal solutions for singular first order linear partial differential equations*, Publ. Res. Inst. Math. Sci., 35 (1999), pp. 893–919.

[12] D. A. Lutz, M. Miyake, and R. Schäfke, *On the Borel summability of divergent solutions of the heat equation*, Nagoya Math. J., 154 (1999), pp. 1–29.

[13] S. Malek, *On the multisummability of formal solutions of partial differential equations.* Preprint, 2002.

[14] M. Miyake, *Borel summability of divergent solutions of the Cauchy problem to non-Kovaleskian equations*, in Partial Differential Equations and Their Applications, C. Hua and L. Rodino, eds., Singapore, 1999, World Scientific, pp. 225–239.

[15] S. Ōuchi, *Multisummability of formal solutions of some linear partial differential equations*, J. Differential Equations, 185 (2002), pp. 513–549.

[16] M. E. Pliś and B. Ziemian, *Borel resummation of formal solutions to nonlinear Laplace equations in 2 variables*, Annales Polonici Mathematici, 67 (1997), pp. 31–41.

[17] J.-P. Ramis, *Les séries k-sommables et leurs applications*, in Complex Analysis, Microlocal Calculus and Relativistic Quantum Theory, D. Iagolnitzer, ed., vol. 126 of Lecture Notes in Physics, Springer Verlag, New York, 1980, pp. 178–199.