Computation of the Stokes multipliers of Okubo's confluent hypergeometric system

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Abstract

The Stokes multipliers of Okubo's confluent hypergeometric system can, in general, not be expressed in closed form using known special functions. Instead they may themselves be regarded as highly interesting new functions of the system's parameters. In this article we study their dependence on the eigenvalues of the leading term. Doing so, we obtain several interesting representations in terms of power series in several variables. As an application we show that the Stokes multipliers may be obtained with help of the sum of a formal solution of a system of difference equations whose dimension is smaller than that of the hypergeometric one.

Introduction

The so-called *hypergeometric system* and its confluent form have recently been investigated in great detail – for a discussion of existing results, and for a representation of its solutions in terms of a single (scalar) function, compare an article of *B. and Röscheisen* [6], or the PhD thesis of *C. Röscheisen* [16]. In a very recent paper of the author's [2], it has been made clear that all the entries in the Stokes multipliers of (0.1) can also be expressed explicitly in terms of *one* (scalar) *Stokes function* v that depends on the parameters of the system. By choice, this function is equal to the entry v_{21} in the $(2,1)$-position of *one* Stokes multiplier, and *the entries in all the other multipliers may be expressed using the same function, evaluated for suitably permuted parameter values.*

In this publication we shall continue the study of the Stokes function v. To do so, we denote the confluent hypergeometric system as

$$z\,x' \;=\; A(z)\,x\,, \qquad A(z) \;=\; z\,\Lambda \;+\; A_1\,, \quad \Lambda \;=\; \operatorname{diag}[\lambda_1,\ldots,\lambda_n] \qquad (0.1)$$

and assume for the moment that the values $\lambda_1,\ldots,\lambda_n$ are mutually distinct. In the theory of formal and proper invariants, presented in work by *Balser, Jurkat, and Lutz* [3, 4, 15], the diagonal elements of A_1 have been shown to be so-called *formal invariants*, and hence are of a special nature. Therefore, we shall always split $A_1 = \Lambda' + A$, with

$$\Lambda' \;=\; \mathrm{diag}\,[\lambda'_1, \dots, \lambda'_n]\,, \qquad A \;=\; \begin{bmatrix} 0 & a_{12} & \dots & a_{1n} \\ a_{21} & 0 & \dots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & 0 \end{bmatrix} \tag{0.2}$$

By choice of v, the values λ_1, λ_2 play a special role in our investigations. To simplify some formulas and/or proofs, we shall not aim at covering the most general situation, but instead shall make the following assumptions:

- Throughout this article, we assume that the matrix Λ is such that

  $$\lambda_1 \;=\; 0\,, \qquad \lambda_2 \;=\; 1\,, \qquad \lambda_k \;\not\in \{0, 1\} \quad (k \ge 3) \tag{0.3}$$

  without assuming that the values $\lambda_3, \dots, \lambda_n$ are mutually distinct.

- Concerning the values $\lambda'_1, \dots, \lambda'_n$, we restrict ourselves to the situation of

  $$\lambda'_1 \;=\; 0 \tag{0.4}$$

- Assuming that (0.3), (0.4) hold, let α, β be so that

  $$\alpha + \beta \;=\; \lambda'_2\,, \quad \alpha\,\beta \;=\; -a_{12}\,a_{21} \tag{0.5}$$

  In fact, these numbers are the (not necessarily distinct) eigenvalues of the 2×2 matrix

  $$A_2 \;:=\; \begin{bmatrix} 0 & a_{12} \\ a_{21} & \lambda'_2 \end{bmatrix}$$

  which for $n = 2$ is equal to A_1. With α and β as in (0.5), we assume that neither α nor β is equal to zero or a negative integer. In other words, we assume that[1]

  $$p(j) \;:=\; (j + \alpha)\,(j + \beta) \;=\; j\,(j + \lambda'_2) \;-\; a_{12}\,a_{21} \;\ne\; 0 \quad \forall\, j \in \mathbb{N}_0 \tag{0.6}$$

  Note that this assumption implies that neither a_{12} nor a_{21} are allowed to vanish!

Remark 0.1 *Observe that some of these assumptions may be made to hold by means of prenormalizing transformations: An* exponential shift $x = \mathrm{e}^{\lambda_1 z}\tilde{x}$ *together with a change of variable* $z = (\lambda_2 - \lambda_1)\,\zeta$, *provided that* $\lambda_2 \ne \lambda_1$, *leads to a new system for which* (0.3) *is satisfied. Then, the transformation* $x = z^{\lambda'_1}\,\tilde{x}$ *may be used to make* (0.4) *hold. It is well known that neither one of these transformations changes the Stokes multipiers, while the effect on the parameters of the system* (0.1) *is easily found to be as follows:*

- Given a general system (0.1), the normalizing transformations described above lead to a new system with the same matrix A, but with new values $\tilde{\lambda}_1, \dots, \tilde{\lambda}_n$ and $\tilde{\lambda}'_1, \dots, \tilde{\lambda}'_n$ given by $\tilde{\lambda}_1 = 0$, $\tilde{\lambda}_2 = 1$, $\tilde{\lambda}'_1 = 0$, and

  $$\tilde{\lambda}_k \;=\; (\lambda_k - \lambda_1)/(\lambda_2 - \lambda_1) \qquad (3 \le k \le n)\,,$$
  $$\tilde{\lambda}'_k \;=\; \lambda'_k - \lambda'_1 \qquad (2 \le k \le n)\,.$$

[1] Observe that in this article $\mathbb{N} = \{1, 2, \dots\}$ denotes the set of natural numbers, while $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

In accordance with this, one can easily extend the results derived in this article to a confluent hypergeometric system with general matrices Λ *and* Λ'*.*

Concerning the last one of the assumptions made above, note that it has been shown in [16] *that the Stokes multipliers of* (0.1) *are entire functions of the entries in the matrix* A*. Therefore, while assumption* (0.6) *certainly is restrictive, most of the results that shall be obtained in this article carry over to cases for which* (0.6) *is violated. We shall not go into details about this, however.*

In this article we shall, under the assumptions listed above, analyze the dependence of $v = v_{21}$ on the variables $\lambda_3, \ldots, \lambda_n$. Since it shall turn out later that the inverses of the λ_k are the more natural variables, we shall throughout write (using superscript τ to denote the transposed of a vector or matrix)

$$v \;=\; v(w)\,, \qquad w \;=\; (w_3, \ldots, w_n)^\tau\,, \quad w_k \;=\; \lambda_k^{-1} \tag{0.7}$$

In detail, we shall show that $v(w)$ may be expanded into a power series in the variables w_k which converges for w with $\|w\|_\infty = \sup\{|w_\nu| \;:\; 3 \le \nu \le n\} < 1$. In fact, we shall instead of $v(w)$ analyze another function denoted as $\gamma(w)$, which shall be defined in Theorem 2.1. For the elementary relation between $v(w)$ and $\gamma(w)$, compare Theorem 2.4. As an application of our investigations, we shall show that the Stokes' function can be explicitly expressed in terms of a solution of an $(n-1)$-dimensional system of difference equations. This solution, in turn, can be computed as the 1-sum of a formal solution – for details compare Theorem 4.3 and the discussion in the final section.

In the case of dimension $n = 2$ the set of variables $w_3, \ldots, w_n$ is empty, and the (constant) entries v and γ have been computed in a paper of *Balser, Jurkat, and Lutz* [4] – also compare the book of *W. B. Jurkat* [15]: In this situation we have, with α, β as in (0.5),

$$v \;=\; 2\pi i\, e^{-i\pi\lambda_2'}\; \gamma\,, \qquad \gamma \;=\; \frac{a_{21}}{\Gamma(1+\alpha)\,\Gamma(1+\beta)} \tag{0.8}$$

In this dimension, observe that assumption (0.6) is violated if, and only if, $v = \gamma = 0$, in which case the system (0.1) is said to be *reducible*. As shall follow from Theorem 3.1 for higher dimensions of $n \ge 3$, the numbers v and γ are equal to the constant terms in the power series expansion of the functions $v(w)$ and $\gamma(w)$, resp. Accordingly, assumption (0.6) is equivalent to the constant term of $v(w)$ and $\gamma(w)$ being non-zero!

1 A formal power series solution

In what follows, we shall always assume that $n \ge 3$, although our results, when properly interpreted, stay correct even for $n = 2$.

Under the assumptions (0.3), (0.4), it is well understood that the system (0.1) has a formal (vector) solution that is a power series in inverse powers of z. We here shall pay special attention to the dependence of this solution on the vector $w = (w_3, \ldots, w_n)^\tau$, and thus we state this result as follows:

Lemma 1.1 *Suppose that* (0.3), (0.4) *hold. Then* (0.1) *has exactly one formal solution of the form*

$$\hat{x}(z;w) \;=\; \sum_{j=0}^{\infty} z^{-j}\, x_j(w)\,, \qquad x_0(w) \;=\; e_1\,,$$

with e_1 denoting the first unit vector in the canonical basis of $\mathbb{C}^n$. For $j \geq 1$, we choose to write $x_j(w) = (x_{1,j}(w), \ldots, x_{n,j}(w))^{\tau}$. The entries $x_{\nu,j}(w)$ can be recursively computed from the identities

$$-j\, x_{1,j}(w) \;=\; \sum_{k=2}^{n} a_{1k}\, x_{k,j}(w) \tag{1.1}$$

$$-x_{2,j+1}(w) \;=\; (j+\lambda_2')\, x_{2,j}(w) \;+ \sum_{\substack{1\le k\le n\\ k\neq 2}} a_{2k}\, x_{k,j}(w) \tag{1.2}$$

$$-x_{\nu,j+1}(w) \;=\; w_\nu\, \Big((j+\lambda_\nu')\, x_{\nu,j}(w) \;+ \sum_{\substack{1\le k\le n\\ k\neq \nu}} a_{\nu k}\, x_{k,j}(w)\Big) \tag{1.3}$$

which hold for every $j \geq 0$ and $3 \leq \nu \leq n$. In particular, for $j = 0$ the first identity is trivially satisfied, while we conclude from (1.2), (1.3) *that*

$$x_{2,1}(w) \;=\; -a_{21}\,, \qquad x_{\nu,1}(w) \;=\; -w_\nu\, a_{\nu 1} \qquad (3 \le \nu \le n) \tag{1.4}$$

For $j \geq 1$, each $x_{\nu,j}(w)$ is a polynomial in the variables w_k ($k \geq 3$) of total degree j whenever $\nu \neq 2$, resp. $j - 1$ for $\nu = 2$.

Proof: Follows immediately by inserting the power series $\hat{x}(z;w)$ into (0.1) and comparing coefficients. □

For $j \geq 1$, note that we may use (1.1) to eliminate $x_{1,j}(w)$ from (1.2), (1.3), and doing so, we obtain a system of $n-1$ linear difference equations for the remaining entries $x_{\nu,j}(w)$, $\nu \geq 2$.

2 The asymptotic behaviour of the coefficients

The following result on the asymptotic behaviour of the entries $x_{\nu,j}(w)$ for $j \to \infty$ has been obtained earlier by *R. Schäfke* [17, 18] and, independently, by *Balser, Jurkat, and Lutz* [5]. However, here we pay special attention to the holomorphy with respect to w of the main term in the asymptotics.

Theorem 2.1 *Suppose that* (0.3), (0.4) *and* (0.6) *hold. Then for $\|w\|_\infty < 1$ the limit*

$$\gamma(w) \;:=\; \lim_{j\to\infty} \frac{(-1)^j\, \Gamma(j)}{\Gamma(j+\alpha)\, \Gamma(j+\beta)}\;\; x_{2,j}(w)$$

exists, with convergence being uniform on compact subsets of the unit polydisc, and the function $\gamma(w)$ is holomorphic for $\|w\|_\infty < 1$ in $\mathbb{C}^{n-2}$. Moreover, we have

$$\frac{(-1)^j\, \Gamma(j)}{\Gamma(j+\alpha)\, \Gamma(j+\beta)}\; x_{\nu,j}(w) \;=\; O(1/j) \quad (j \to \infty) \qquad\qquad (\nu = 3, \ldots, n)$$

and for $\|w\|_\infty \leq c$, with arbitrary $c < 1$, the O-constant may be taken independent of w.

Proof: As was said above, the existence of the limit $\gamma(w)$ has been shown before, and analyzing the proofs in the articles mentioned, it is even possible to obtain its analyticity. However, for convenience of the reader, we shall supply the necessary estimates here: We set

$$\gamma_{\nu,j}(w) \;:=\; \frac{(-1)^j\,\Gamma(j)}{\Gamma(j+\alpha)\,\Gamma(j+\beta)}\; x_{\nu,j}(w) \qquad (j \ge 1\,,\; 2 \le \nu \le n)$$

and, with $p(j)$ as in (0.6), use the abbreviations

$$r_{\nu k}(j) \;=\; \frac{p_{\nu k}(j)}{p(j)}\,, \qquad p_{\nu k}(j) \;=\; \begin{cases} (j+\lambda'_\nu)\,j \;-\; a_{\nu 1}\,a_{1\nu} & (\nu = k) \\[2ex] j\,a_{\nu k} \;-\; a_{\nu 1}\,a_{1k} & (\nu \ne k) \end{cases} \tag{2.1}$$

Observe that $r_{22}(j) \equiv 1$ for all j, and that by assumption $p(j) \ne 0$ for all $j \ge 0$. From (1.2), (1.3), after using (1.1) to eliminate $x_{1,j}(w)$, we conclude that

$$\left.\begin{aligned} \gamma_{2,j+1}(w) \;&=\; \gamma_{2,j}(w) \;+\; \sum_{k=3}^{n} r_{2k}(j)\;\gamma_{k,j}(w) \\ \gamma_{\nu,j+1}(w) \;&=\; w_\nu \sum_{k=2}^{n} r_{\nu k}(j)\,\gamma_{k,j}(w) \qquad (3 \le \nu \le n) \end{aligned}\right\} \quad (j \ge 1) \tag{2.2}$$

Let $c \in (0,1)$ be given and restrict to $\|w\|_\infty \le c$. We may estimate for $j \ge 1$

$$|r_{\nu k}(j)| \;\le\; \begin{cases} 1 \;+\; r/j & (\nu = k) \\[2ex] r/j & (\nu \ne k) \end{cases} \tag{2.3}$$

with sufficiently large $r > 0$. Defining $a_{j+1} = a_j + (n-2)\,r\,b_j/j$ and $b_{j+1} = c\,[b_j\,(1 + r\,(n-2)/j) + r\,a_j/j]$, beginning with a_1, b_1 sufficiently large and independent of w, we may estimate (2.2) to find by induction with respect to j that (a_j) and (b_j) are majorant sequences for $(\gamma_{2,1}(w))$ and $(\gamma_{\nu,1}(w))$, respectively. Since a_1, b_1 are independent of w, we conclude the same for a_j, b_j, $j \ge 2$. For arbitrary j_0, to be selected later, the above recursion for a_j implies that

$$a_j \;=\; a_{j_0} \;+\; (n-2)\,r \sum_{\ell=j_0}^{j-1} b_\ell/\ell \qquad (j \ge j_0) \tag{2.4}$$

For some $j \ge j_0$, let $b \ge 0$ be so large that $b_\ell \le b/\ell$ for every $\ell \le j$. Such a number b certainly exists, but we aim at showing that we can, in fact, find one that is independent of j. To do so, we estimate the recursions to find

$$a_j \;\le\; a_{j_0} \;+\; (n-2)\,r \sum_{\ell=j_0}^{j-1} b/\ell^2\,, \quad b_{j+1} \;\le\; c\,[b\,(1 + r\,(n-2)/j) + r\,a_j]/j\,.$$

With d so that $\sum_{\ell=j_0}^{\infty} \ell^{-2} \le d/j_0$, this implies

$$b_{j+1} \;\le\; c\,[b\,(1 + r\,(n-2)/j) + r\,a_{j_0} + r^2(n-2)\,d\,b/j_0\,]/j\,,$$

and the right hand side is at most $b/(j+1)$ if, and only if, we have for every $j \geq j_0$

$$c\,[1 + r\,(n-2)/j + r\,a_{j_0}/b + r^2(n-2)\,d/j_0\,]\,(1+1/j) \;\leq\; 1\,.$$

Using the fact that $c < 1$, we see that this is correct for j_0 and b sufficiently large, and thus we have shown existence of $b > 0$ such that $b_j \leq b/j$ for every $j \geq 1$. This then implies, in view of (2.4), that the sequence (a_j) is bounded.

Hence, summing up, we have shown that $a, b > 0$ exist such that

$$|\gamma_{2,j}(w)| \;\leq\; a_j \;\leq\; a\,, \qquad |\gamma_{\nu,j}(w)| \;\leq\; b_j \;\leq\; b/j \qquad (3 \leq \nu \leq n) \tag{2.5}$$

for every $j \geq 1$ and $\|w\|_\infty \leq c$. From (2.2) we find, observing $\gamma_{2,1}(w) = \gamma$ with γ as in (0.8) (also see Remark 2.3 below), that

$$\gamma_{2,j}(w) \;=\; \gamma \;+\; \sum_{k=3}^{n}\sum_{\ell=1}^{j-1} r_{2k}(\ell)\,\gamma_{k,\ell}(w) \qquad (j \geq 1) \tag{2.6}$$

Owing to (2.3) and (2.5) we see that the series $\sum_{\ell=1}^{\infty} r_{2k}(\ell)\,\gamma_{k,\ell}(w)$, for $k \geq 3$, all are absolutely convergent, and convergence is uniform for $\|w\|_\infty \leq c$. Therefore, for these w's the limit $\gamma(w)$ exists and is holomorphic at interior points. Hence the proof is completed. □

Remark 2.2 *For fixed w, the identities* (2.2) *may be viewed as a linear system of difference equations with rational coefficients. There is a well-developped theory of formal solutions and their multi-summability of even non-linear systems of difference equations that could be applied to prove* Theorem 2.1. *For the very simple situation studied here, this theory is not needed, but it can prove very useful for more complicated systems. We refer the reader to papers of* Braaksma *and others* [7–11] *for a presentation of this very beautiful theory that deserves to be better known among the specialists for difference equations – also compare the last section of this article for some more details.*

Remark 2.3 *With γ as in* (0.8) *we conclude from* (1.4) *and the definition of $\gamma_{\nu,j}(w)$ that*

$$\gamma_{2,1}(w) \;= \gamma\,, \quad \gamma_{\nu,1}(w) \;= w_\nu\,\gamma^{(\nu)}\,, \qquad \gamma^{(\nu)} \;:=\; \frac{a_{\nu 1}}{\Gamma(1+\alpha)\,\Gamma(1+\beta)} \tag{2.7}$$

Furthermore, we obtain from (2.6) *the following representation for the limit $\gamma(w)$ that shall be very important in our investigations:*

$$\gamma(w) \;=\; \gamma \;+\; \sum_{k=3}^{n}\;\sum_{j=1}^{\infty} r_{2k}(j)\,\gamma_{k,j}(w) \qquad (\|w\|_\infty < 1) \tag{2.8}$$

The $\gamma_{\nu,j}(w)$ are polynomials in w and, roughly speaking, the functions

$$\gamma_\nu(t;w) \;:=\; \sum_{j=1}^{\infty}(-t)^j\,\gamma_{\nu,j}(w)$$

are a formal Borel-like transformation in the sense of [1, Section 5.5], *corresponding to the moment sequence*

$$m(j) \;:=\; \frac{\Gamma(j+\alpha)\,\Gamma(j+\beta)}{\Gamma(j)}\;,$$

of the formal solution $\hat{x}(z;w)$. This shall not be needed here, however.

The Stokes function $v(w)$ and the limit $\gamma(w)$, whose existence and analyticity has been shown above, satisfy the following elementary relation which agrees with (0.8) in dimension $n = 2$:

Theorem 2.4 *Under the assumptions* (0.3), (0.4), *and* (0.6) *we have*

$$v(w) \;=\; 2\pi i\ \mathrm{e}^{-i\pi\lambda_2'}\ \gamma(w) \qquad (\|w\|_\infty < 1) \tag{2.9}$$

Proof: The proof follows directly from [5, Proposition 3 or Corollary 1]; also compare the PhD thesis of *R. Schäfke* [17]. □

Due to this elementary relation, we may from now on restrict our investigations to the function $\gamma(w)$. In fact, it also follows that it is not necessary to assume that the numbers $w_3, \ldots, w_n$ are mutually distinct. It may even be shown that $v(w)$, and then by means of (2.9) the function $\gamma(w)$ as well, can be analytically continued, in every variable w_ν, along every path that avoids the point $w_\nu = 1$. So the Stokes function is resurgent in the sense of *Ecalle's* [12,13]. Note, however, that $w = 1$ is, in general, a branch point!

3 Power series expansions

It follows from Lemma 1.1 that the functions $\gamma_{\nu,j}(w)$, which differ from the coefficients $x_{\nu,j}(w)$ by constants only, are polynomials in $w_3, \ldots, w_n$. Therefore, the series (2.8) can be used to find the power series expansion of $\gamma(w)$: Let $p = (p_3, \ldots, p_n) \in \mathbb{N}_0^{n-2}$ be a multi-index, and set as usual

$$w^p \;=\; w_3^{p_3} \cdot \ldots \cdot w_n^{p_n}\,.$$

In particular, with $e^{(\nu)}$ denoting the multi-index with entries $\delta_{j,\nu-2}$, $3 \le \nu \le n$, we have

$$w^{e^{(\nu)}} \;=\; w_\nu\,, \qquad 3 \le \nu \le n\,.$$

With $|p| := p_2 + \ldots + p_n$ denoting the length of p, we use (2.2) to see by means of induction with respect to j that the polynomials $\gamma_{\nu,j}(w)$ may be written in the form

$$\left.\begin{aligned} \gamma_{2,j}(w) \;&=\; \sum_{0\le|p|\le j-1} \gamma_{2,j,p}\ w^p \\ \gamma_{\nu,j}(w) \;&=\; w_\nu \sum_{0\le|p|\le j-1} \gamma_{\nu,j,p}\ w^p \qquad (3 \le \nu \le n) \end{aligned}\right\} \tag{3.1}$$

Due to (2.7) we find that $\gamma_{2,1,0} = \gamma$ and $\gamma_{\nu,1,0} = \gamma^{(\nu)}$ for $\nu = 3, \ldots, n$. Inserting (3.1) into (2.2) and comparing coefficients, we find that the other coefficients may be recursively computed from the identity

$$\gamma_{\nu,j+1,p} \;=\; r_{\nu 2}(j)\, \gamma_{2,j,p} \;+\; \sum_{k=3}^{n} r_{\nu k}(j)\, \gamma_{k,j,p-e^{(k)}} \qquad (2 \le \nu \le n) \tag{3.2}$$

observing that $r_{22}(j) \equiv 1$, and for simplicity of notation setting $\gamma_{2,j,p} = 0$ when $|p| \ge j$, and $\gamma_{k,j,p-e^{(\nu)}} = 0$ if $p_\nu = 0$. In particular, we conclude for $p = 0$ and $2 \le \nu \le n$ that

$$\gamma_{\nu,j,0} \;=\; r_{\nu 2}(j-1)\ \gamma \qquad (j \ge 1)\,. \tag{3.3}$$

Check that this agrees with the above formulas in case of $j = 1$. For $\nu = 2$ we have that $\gamma_{2,j,0} \equiv \gamma$ for every $j \geq 1$. For general p we obtain from (3.2), setting $\nu = 2$, that

$$\gamma_{2,j,p} \;=\; \sum_{\ell=|p|}^{j-1} \sum_{k=3}^{n} r_{2k}(\ell)\, \gamma_{k,\ell,p-e^{(k)}} \qquad (j \geq |p|+1)\,.$$

This equation may be used to get a recursion formula for the remaining $\gamma_{\nu,j,p}$:

$$\begin{aligned}\gamma_{\nu,j+1,p} \;=\; & r_{\nu 2}(j) \sum_{\ell=|p|}^{j-1} \sum_{k=3}^{n} r_{2k}(\ell)\, \gamma_{k,\ell,p-e^{(k)}} \\ & + \sum_{k=3}^{n} r_{\nu k}(j)\, \gamma_{k,j,p-e^{(k)}} \qquad (3 \leq \nu \leq n\,, \quad j \geq |p|) \end{aligned} \tag{3.4}$$

Observe that this identity allows recursive computation of $\gamma_{\nu,j,p}$, not only with respect to j, but also the length of p. For example, we obtain for multi-indices of length 1 that

$$\gamma_{\nu,j+1,e^{(\mu)}} \;=\; \gamma\, \Big(r_{\nu\mu}(j)\, r_{\mu 2}(j-1) \;+\; r_{\nu 2}(j) \sum_{\ell=1}^{j-1} r_{2\mu}(\ell)\, r_{\mu 2}(\ell-1) \Big)$$

In term of these entries $\gamma_{\nu,j,p}$ we obtain the power series expansion of $\gamma(w)$ as follows:

Theorem 3.1 *Suppose that* (0.3), (0.4) *and* (0.6) *hold. For w with $\|w\|_\infty < 1$ we have $\gamma(w) = \sum_{|p|\geq 0} \gamma_p\, w^p$, with coefficients γ_p given by*

$$\gamma_p \;=\; \lim_{j\to\infty} \gamma_{2,j,p} \;=\; \sum_{j=|p|}^{\infty} \sum_{\nu=3}^{n} r_{2\nu}(j)\, \gamma_{\nu,j,p-e^{(\nu)}} \tag{3.5}$$

In particular we obtain $\gamma_0 = \gamma$, with the number γ as in (0.8). *Furthermore, for the coefficients of length* 1 *we have the representation*

$$\gamma_{e^{(\nu)}} \;=\; \gamma \sum_{j=0}^{\infty} \frac{(j+1)\, a_{2\nu} - a_{21}\, a_{1\nu}}{(j+1+\alpha)\,(j+1+\beta)}\; \frac{j\, a_{\nu 2} - a_{\nu 1}\, a_{12}}{(j+\alpha)\,(j+\beta)} \tag{3.6}$$

for $\nu = 3, \ldots, n$.

Proof: According to the proof of Theorem 2.1, the series (2.8) converges compactly on the unit polydisc of $\mathbb{C}^{n-2}$, and therefore the proof follows from the Cauchy formula for (partial) derivatives of holomorphic functions in several variables [14]. □

Remark 3.2 *Based on the above formula for the coefficients γ_p, we can find a slightly different representation for $\gamma(w)$: Define*

$$\gamma^{(\nu)}(w) \;=\; \sum_{|p|\geq 0} w^p \sum_{j=|p|+1}^{\infty} r_{2\nu}(j)\, \gamma_{\nu,j,p} \qquad (3 \leq \nu \leq n)\,.$$

Then we obtain from the formula (3.5)*, through an interchange of summation and a change of summation index* $p - e^{(\nu)} \leftrightarrow p$*, that*

$$\gamma(w) \;=\; \gamma \;+\; \sum_{\nu=3}^{m} w_\nu\, \gamma^{(\nu)}(w)\,.$$

4 More on the coefficients of the power series

A different representation for the coefficients γ_p may be obtained ty means of a generalization of (2.8). In order to achieve this, we introduce the following numbers:

- With p as above, we define for $3 \le k \le n$ and $j \in \mathbb{N}_0$: $\varrho_k(p;j) = r_{2k}(j)$ for $p = 0$, with $r_{\nu k}(j)$ as in (2.1), and for $|p| \ge 1$:

$$\left.\begin{aligned} \varrho_k(p;j) \;&=\; \sum_{\nu=3}^{n} \Big[\, \varrho_\nu(p - e^{(\nu)}; j+1)\, r_{\nu k}(j) \\ &\qquad + r_{2k}(j) \sum_{\ell=j+1}^{\infty} \varrho_\nu(p - e^{(\nu)}; \ell+1)\, r_{\nu 2}(\ell) \Big] \end{aligned}\right\} \qquad (4.1)$$

  with the interpretation that $\varrho_\nu(p - e^{(\nu)}; j+1) = \varrho_\nu(p - e^{(\nu)}; \ell+1) = 0$ for multi-indices p with $p_\nu = 0$. Observe that by induction with respect to $|p|$ one can easily see that $\varrho_k(p;j) = O(1/j)$ as $j \to \infty$, and that therefore the series in (4.1) is absolutely convergent.

Remark 4.1 *It shall be convenient to use formula* (4.1) *with* $k = 2$ to define *numbers* $\varrho_2(p;j)$ *for* $|p| \ge 1$*, and in view of* $r_{22}(j) \equiv 1$ *this equation simplifies to*

$$\begin{aligned} \varrho_2(p;j) \;&=\; \sum_{\nu=3}^{n} \sum_{\ell=j}^{\infty} \varrho_\nu(p - e^{(\nu)}; \ell+1)\, r_{\nu 2}(\ell) \\ &=\; \varrho_2(p; j+1) \;+ \sum_{\nu=3}^{n} \varrho_\nu(p - e^{(\nu)}; j+1)\, r_{\nu 2}(j) \quad (j \ge 0\,,\; |p| \ge 1)\,. \end{aligned}$$

In addition we set $\varrho_2(0;j) = r_{22}(j)$ $(\equiv 1)$*. With this definition we may rewrite* (4.1) *as*

$$\varrho_k(p;j) \;=\; \varrho_2(p;j+1)\, r_{2k}(j) \;+\; \sum_{\nu=3}^{n} \varrho_\nu(p - e^{(\nu)}; j+1)\, r_{\nu k}(j) \qquad (4.2)$$

which then holds for $k = 2, \ldots, n$*,* $j \ge 1$*, and all multi-indices* p*.*

In order to understand the meaning of the numbers $\varrho_k(p;j)$, we (formally) define functions

$$y_k(j;w) \;:=\; \sum_{|p| \ge 0} w^p\, \varrho_k(p;j) \qquad (j \ge 0\,, \quad 2 \le k \le n) \qquad (4.3)$$

and prove the following result:

Lemma 4.2 *Suppose that* (0.3), (0.4) *and* (0.6) *hold.*

a) *For every* $\varepsilon > 0$ *there exists constants* C *and* $j_0 \geq 1$ *such that for every multi-index* p, *every* $j \geq j_0$, *and* $k = 3, \ldots, n$

$$|\varrho_k(p;j)| \;\leq\; C\,(1+\varepsilon)^{|p|}/j\,.$$

b) *The series* (4.3) *all converge absolutely for* $\|w\|_\infty < 1$, *and the functions so defined satisfy*

$$y_k(j;w) \;=\; y_2(j+1;w)\,r_{2k}(j) \;+\; \sum_{\nu=3}^{n} w_\nu\, y_\nu(j+1;w)\,r_{\nu k}(j) \tag{4.4}$$

for $j \geq 0$ *and* $2 \leq k \leq n$.

c) *For* $\|w\|_\infty < 1$ *and* $k = 3, \ldots, n$ *we have* $y_k(j;w) = O(1/j)$ *as* $j \to \infty$, *with a O-constant that is locally uniform in* w, *while*

$$\lim_{j\to\infty} \; y_2(j;w) \;=\; 1\,,$$

with convergence being locally uniform in w.

Proof: Let $\varepsilon > 0$ be given. For $|p| \geq 1$, assume that the first statement is correct for every multi-index of length $|p|-1$. This is so for $|p| = 1$ and arbitrary j_0, due to $\varrho_k(0;j) = r_{2k}(j) = O(1/j)$. Estimating (4.1), we then find for $j \geq j_0$ and every k

$$|\varrho_k(p;j)| \;\leq\; \frac{C\,(1+\varepsilon)^{|p|-1}}{j}\Big[1 + (n-2)\,r/j + (n-2)\,r^2 \sum_{\ell=j+1}^{\infty} \frac{1}{(\ell(\ell+1)}\Big].$$

For sufficiently large j_0, independent of p, the term in brackets is not larger than $1+\varepsilon$, hence by induction with respect to $|p|$ we obtain that a) is correct. Due to this estimate we see that convergence of the series (4.3) follows for $j \geq j_0$, and using (4.1) we obtain correctness of b) for such j. For smaller j, we may take the identity in b) as the definition for $y_k(j;w)$, implying that all these functions are holomorphic in the unit polydisc in $\mathbb{C}^{(n-2)}$. The coefficients of their power series expansion may then be verified to satisfy (4.1), so that b) follows *for all* $j \geq 0$. Statement c) then can be proven using a). □

As we shall make clear in the final section, the identities (4.4) can be equivalently written as a system of linear difference equations, and the functions $y_k(j;w)$ are the components of a vector solution that is uniquely characterized by the behaviour as $j \to \infty$.

In terms of the numbers $\varrho_k(p;j)$ we obtain the following formula for the coefficients γ_p:

Theorem 4.3 *Assume* (0.3), (0.4) *and* (0.6).

a) *For all* w *with* $\|w\|_\infty < 1$ *and every* $m \in \mathbb{N}_0$ *we have the following generalization of* (2.8):

$$\left.\begin{aligned} \gamma(w) \;&=\; \sum_{|p|\leq m} \gamma_p\, w^p \;+\; \varrho^{(m)}(w) \\ \varrho^{(m)}(w) \;&:=\; \sum_{|p|=m} w^p \sum_{k=3}^{n}\sum_{j=1}^{\infty} \varrho_k(p;j)\,\gamma_{k,j}(w) \end{aligned}\right\} \tag{4.5}$$

b) *The coefficients* γ_p *with* $|p| \geq 1$ *are given by the formula*

$$\gamma_p \;=\; \gamma\, \varrho_2(p;0) \;=\; \gamma \sum_{\nu=3}^{n} \sum_{j=0}^{\infty} \varrho_\nu(p - e^{(\nu)}; j+1)\, r_{\nu 2}(j) \tag{4.6}$$

the infinite series being absolutely convergent.

c) *For all* w *with* $\|w\|_\infty < 1$. *the function* $\gamma(w)$ *is given by*

$$\gamma(w) \;=\; \gamma\, y_2(0;w) \;=\; \gamma\, y_2(1;w) \;+\; \sum_{\nu=3}^{n} w_\nu\, y_\nu(1;w)\, \gamma^{(\nu)} \tag{4.7}$$

Proof: To prove a), we proceed by induction with respect to m: For $m = 0$, the statement is correct, owing to (2.8), hence we may assume correctness for some $m \geq 0$. Inserting (2.6) into the second line in recursion (2.2), we obtain for $3 \leq \nu \leq n$ and every $j \geq 1$:

$$\left.\begin{aligned} \gamma_{\nu,j+1}(w) \;&=\; w_\nu\, \gamma\, r_{\nu 2}(j) \;+ \\ & w_\nu \sum_{k=3}^{n} \Big[\, r_{\nu k}(j)\, \gamma_{k,j}(w) \;+\; r_{\nu 2}(j) \sum_{\ell=1}^{j-1} r_{2k}(\ell)\, \gamma_{k,\ell}(w) \Big] \end{aligned}\right\} \tag{4.8}$$

Splitting off the term for $j = 1$ from the infinite series in the definition of $\varrho^{(m)}(w)$ and observing (2.7), we can then use (4.8) to obtain

$$\begin{aligned} \varrho^{(m)}(w) \;=\; & \sum_{|p|=m} \sum_{\nu=3}^{n} w^{p+e^{(\nu)}} \Big[\gamma^{(\nu)}\, \varrho_\nu(p;1) \;+\; \gamma \sum_{j=1}^{\infty} \varrho_\nu(p; j+1)\, r_{\nu 2}(j) \Big] \\ & + \sum_{|p|=m} \sum_{\nu=3}^{n} w^{p+e^{(\nu)}} \sum_{j=1}^{\infty} \varrho_\nu(p; j+1) \sum_{k=3}^{n} r_{\nu k}(j)\, \gamma_{k,j}(w) \\ & + \sum_{|p|=m} \sum_{\nu=3}^{n} w^{p+e^{(\nu)}} \sum_{j=1}^{\infty} \varrho_\nu(p; j+1)\, r_{\nu 2}(j) \sum_{k=3}^{n} \sum_{\ell=1}^{j-1} r_{2k}(\ell)\, \gamma_{k,\ell}(w)\;. \end{aligned}$$

In the third line we may interchange summation with respect to j and ℓ, and afterwards rename the index ℓ by j and vice versa. Also, the double sums at the beginning of the lines are equivalent to one sum over all multi-indices $q := p + e^{(\nu)}$ of length $m+1$, but one has to be careful to observe that the same q can be written in more than one way in the form $p + e^{(\nu)}$. Doing all this, we obtain after interchanging the sums with respect to ν and k:

$$\begin{aligned} \varrho^{(m)}(w) = & \sum_{|q|=m+1} w^q \sum_{\nu=3}^{n} \Big[\gamma^{(\nu)}\, \varrho_\nu(q - e^{(\nu)}; 1) + \gamma \sum_{j=1}^{\infty} \varrho_\nu(q - e^{(\nu)}; j+1)\, r_{\nu 2}(j) \Big] \\ & + \sum_{|q|=m+1} w^q \sum_{k=3}^{n} \sum_{j=1}^{\infty} \gamma_{k,j}(w) \sum_{\nu=3}^{n} \Big[\, r_{\nu k}(j) \varrho_\nu(q - e^{(\nu)}; j+1) \\ & \qquad\qquad + r_{2k}(j) \sum_{\ell=j+1}^{\infty} \varrho_\nu(q - e^{(\nu)}; \ell+1)\, r_{\nu 2}(\ell) \Big]\,. \end{aligned}$$

Verifying that $\gamma^{(\nu)} = -\gamma\, a_{\nu 1}\, a_{12}/(\alpha\, \beta) = \gamma\, r_{\nu 2}(0)$, we can momentarily define γ_q in analogy with (4.6), and then read off that

$$\varrho^{(m)}(w) \;=\; \sum_{|q|=m+1} \gamma_q\, w^q \;+\; \varrho^{(m+1)}(w)\;.$$

Since the power series expansion of $\varrho^{(m+1)}$ contains only terms of total degree at least $m+2$, we conclude that the numbers γ_q are indeed equal to the coefficients of the expansion of $\gamma(w)$ of degree $m+1$, which completes the proof of a) as well as of b). The remaining statement then follows from the first two. □

Observe that according to (4.7) the function $y(0;w)$ is closely related to the Stokes function. In the final section we shall say more about the computation of the sequence $y(j;w)$ with help of 1-summability!

5 Another representation of the Stokes function

In this section we prove new identities for $\gamma(w)$, which lead to a more elementary representation of this function. To do so, we introduce a countable family of rational functions of j by the following recursive definition:

- Let $r_{\nu k}(j)$ and $p = (p_3, \ldots, p_n)$ be as in the previous section. Starting with $r_k(0;j) = r_{2k}(j)$, we define for $|p| \ge 1$, $3 \le k \le n$, and $j \in \mathbb{N}_0$

$$\left.\begin{aligned} r_k(p;j) \;=\; \sum_{\nu=3}^{n} \Big[& r_\nu(p-e^{(\nu)};j+1)\, r_{\nu k}(j) \\ & - r_{2k}(j) \sum_{\ell=1}^{j} r_\nu(p-e^{(\nu)};\ell+1)\, r_{\nu 2}(\ell) \Big] \end{aligned}\right\} \tag{5.1}$$

  with the interpretation that $r_\nu(p-e^{(\nu)};j+1) = r_\nu(p-e^{(\nu)};\ell+1) = 0$ for multi-indices p with $p_\nu = 0$. Note in particular that for $j=0$ the sum with repsect to ℓ is empty, so that we obtain

$$r_k(p;0) \;=\; \sum_{\nu=3}^{n} r_\nu(p-e^{(\nu)};1)\, r_{\nu k}(0) \qquad (3 \le k \le n)\,.$$

  Observe the strong similarity of (5.1) with (4.1); however, here there is only a finite sum instead of an infinite series involved! Therefore, the $r_k(p;j)$ are rational functions of j, hence much more elementary than the $\varrho_k(p;j)$.

- For $p \ne 0$ and $j \ge 1$ we define numbers $r_2(p;j)$ by

$$r_2(p;j) \;=\; -\sum_{\nu=3}^{n}\sum_{\ell=1}^{j-1} r_\nu(p-e^{(\nu)};\ell+1)\, r_{\nu 2}(\ell) \tag{5.2}$$

$$\;=\; r_2(p;j+1) \;+\sum_{\nu=3}^{n} r_\nu(p-e^{(\nu)};j+1)\, r_{\nu 2}(j) \tag{5.3}$$

Observe that this implies $r_2(p;1) = 0$, and that the definition of $r_2(p;j)$ coincides with (5.1) for $k = 2$. Note that we can use (5.3) (but not (5.2)) to define

$$r_2(p;0) \;=\; \sum_{\nu=3}^{n} r_\nu(p - e^{(\nu)};1)\, r_{\nu 2}(0)$$

For $p = 0$ we set $r_2(0;j) = r_{22}(j) \equiv 1$.

Remark 5.1 *By induction with respect to* $|p|$ *one can easily see that for every* p *and every* $k = 3,\ldots,n$ *we have* $r_k(p;j) = O(1/j)$ *as* $j \to \infty$*, while the sequence* $r_2(p;j)$ *is convergent. We may rewrite* (5.1) *as*

$$r_k(p;j) \;=\; r_2(p;j+1)\, r_{2k}(j) \;+\; \sum_{\nu=3}^{n} r_\nu(p - e^{(\nu)};j+1)\, r_{\nu k}(j) \qquad (5.4)$$

which then holds for $k = 2,\ldots,n$*, for* $j \geq 0$*, and all multi-indices* p*.*

Analogous to the previous section we define functions

$$\tilde{y}_k(j;w) \;:=\; \sum_{|p|\geq 0} w^p\, r_k(p;j) \qquad (j \geq 0\,, \quad 2 \leq k \leq n) \qquad (5.5)$$

Observe that in particular we have $\tilde{y}_k(1;w) \equiv 1$. Convergence of the series for w near the origin is shown in the next lemma:

Lemma 5.2 *Suppose that* (0.3)*,* (0.4) *and* (0.6) *hold. Then for all* $j \geq 0$ *we have:*

a) *The power series* (5.5) *converge for* $\|w\|_\infty < r_0$*, with some* r_0 *independent of* j*. Moreover, for these* w *we have*

$$y_k(j;w) \;=\; \tilde{y}_k(j;w)\, y_2(1;w) \qquad (2 \leq k \leq n) \qquad (5.6)$$

b) *The functions* $\tilde{y}_k(j;w)$ *can be continued meromorphically onto the unit polydisc of* $\mathbb{C}^{n-2}$*, with possible poles at places where* $y_1(1;w)$ *vanishes. For all other* w*, they satisfy*

$$\tilde{y}_k(j;w) \;=\; \tilde{y}_2(j+1;w)\, r_{2k}(j) \;+\; \sum_{\nu=3}^{n} w_\nu\, \tilde{y}_\nu(j+1;w)\, r_{\nu k}(j) \qquad (5.7)$$

for $2 \leq k \leq n$*.*

c) *For* $\|w\|_\infty < 1$ *with* $y_1(1;w) \neq 0$ *and* $k = 3,\ldots,n$ *we have* $\tilde{y}_k(j;w) = O(1/j)$ *as* $j \to \infty$*, with a* O*-constant that is locally uniform in* w*, while*

$$\lim_{j\to\infty} \tilde{y}_2(j;w) \;=\; 1/y_1(1;w)\,,$$

with convergence being locally uniform in w*.*

Proof: Since even in the definition of the $r_k(p;j)$ the case of $j = 0$ is different from the other ones, we first restrict ourselves to $j \geq 1$: For such j, similar estimates as in the proof of Lemma 4.2 a) show existence of c so that $|r_k(p;j)| \leq c^{|p|}/j$ for all p and $3 \leq k \leq n$, from which convergence of (5.5) follows for

$\|w\|_\infty < 1/c =: r_0$ and these j and k. For $k = 2$, we then obtain convergence using (5.2). Moreover, observe that (5.6) is equivalent to

$$\varrho_k(p;j) \;=\; \sum_{q \le p} r_k(p-q;j)\,\varrho_2(q;1) \qquad (2 \le k \le n\,,\ |p| \ge 0\,,\ j \ge 1) \tag{5.8}$$

with $q \le p$ meaning that $q_\nu \le p_\nu$ for $3 \le \nu \le n$. This identity is certainly correct for $p = 0$, hence let $p \ne 0$ be given and assume correctness for all multi-indices of length strictly less than $|p|$ (and all $j \ge 1$ and $2 \le k \le n$). Then we use the definition of $\varrho_2(p;j)$ together with the induction hypothesis to conclude (with $q < p$ meaning $q \le p$ and $q \ne p$)

$$\begin{aligned}
\varrho_2(p;j) \;&=\; \varrho_2(p;1) \;-\; \sum_{\nu=3}^{n}\sum_{\ell=1}^{j-1} \sum_{q \le p - e^{(\nu)}} r_k(p-q-e^{(\nu)};\ell+1)\,\varrho_2(q;1)\,r_{\nu 2}(\ell) \\
&=\; \varrho_2(p;1) \;-\; \sum_{q<p} \varrho_2(q;1) \sum_{\nu=3}^{n}\sum_{\ell=1}^{j-1} r_k(p-q-e^{(\nu)};\ell+1)\,r_{\nu 2}(\ell) \\
&=\; r_2(0;1)\,\varrho_2(p;1) \;+\; \sum_{q<p} r_2(p-q;j)\,\varrho_2(q;1)
\end{aligned}$$

(using the definition of $r_2(p-q;j)$ and the fact that $r_2(0;1) = 1$). This shows (5.6) for $k = 2$ and $j \ge 1$ (and the selected p). In the same manner one can prove correctness for $k = 3, \ldots, n$, using either (4.2) or (4.1). Thus, the proof of statement a) is completed, and the other two follow from a), using (5.4) and Lemma 4.2c). In order to cover the case $j = 0$, note that the definition of $r_k(p;0)$ immediately implies convergence of (5.5) even in this case, and then one can verify (5.7) for $j = 0$. From this and (4.4) we then obtain validity of (5.6) for this j. □

In terms of the entries introduced above, we show the following generalization of (2.8):

Proposition 5.3 *Assume* (0.3), (0.4) *and* (0.6). *For every integer* $m \ge 0$ *and* $\|w\|_\infty < 1$ *we have the following representation formula for* $\gamma(w)$:

$$\gamma(w)\Big(1 \;-\; \sum_{1 \le |p| \le m} \alpha_p\,w^p\Big) \;=\; \gamma \;+\; \sum_{1 \le |p| \le m} \beta_p\,w^p \;+\; r^{(m)}(w) \tag{5.9}$$

with entities of the form

$$\alpha_p \;=\; \sum_{\nu=3}^{n}\sum_{j=1}^{\infty} r_\nu(p-e^{(\nu)};j+1)\,r_{\nu 2}(j) \;=\; -\lim_{j\to\infty} r_2(p;j) \tag{5.10}$$

$$\beta_p \;=\; \sum_{\nu=3}^{n} \gamma^{(\nu)}\,r_\nu(p-e^{(\nu)};1) \;=\; \gamma\,r_2(p;0) \tag{5.11}$$

$$r^{(m)}(w) \;=\; \sum_{|p|=m} w^p \sum_{k=3}^{n}\sum_{j=1}^{\infty} r_k(p;j)\,\gamma_{k,j}(w) \tag{5.12}$$

Proof: The proof is very much analogous with that of Theorem 4.3: Assuming correctness of the statements for some $m \ge 0$ (which is so when $m = 0$), we use

(2.8) and (2.6) to obtain

$$\gamma_{2,j}(w) \;=\; \gamma(w) \;-\; \sum_{k=3}^{n} \sum_{\ell=j}^{\infty} r_{2k}(\ell)\, \gamma_{k,\ell}(w) \qquad (j \geq 1)$$

and insert this into the second line in recursion (2.2) to show for $3 \leq \nu \leq n$ and every $j \geq 1$:

$$\left.\begin{aligned} \gamma_{\nu,j+1}(w) \;&=\; w_\nu\, \gamma(w)\, r_{\nu 2}(j) \;+ \\ & w_\nu \sum_{k=3}^{n} \Big[\, r_{\nu k}(j)\, \gamma_{k,j}(w) \;-\; r_{\nu 2}(j) \sum_{\ell=j}^{\infty} r_{2k}(\ell)\, \gamma_{k,\ell}(w) \Big] \end{aligned}\right\} \tag{5.13}$$

Splitting off the term for $j = 1$ in the series for $r^{(m)}(w)$ we can then use (5.13) to prove

$$\begin{aligned} r^{(m)}(w) \;=\; & \sum_{|p|=m} \sum_{\nu=3}^{n} w^{p+e^{(\nu)}} \Big[\, \gamma^{(\nu)}\, r_\nu(p;1) \;+\; \gamma(w) \sum_{j=1}^{\infty} r_\nu(p;j+1)\, r_{\nu 2}(j) \Big] \\ & + \sum_{|p|=m} \sum_{\nu=3}^{n} w^{p+e^{(\nu)}} \sum_{j=1}^{\infty} r_\nu(p;j+1) \sum_{k=3}^{n} r_{\nu k}(j)\, \gamma_{k,j}(w) \\ & - \sum_{|p|=m} \sum_{\nu=3}^{n} w^{p+e^{(\nu)}} \sum_{j=1}^{\infty} r_\nu(p;j+1)\, r_{\nu 2}(j) \sum_{k=3}^{n} \sum_{\ell=j}^{\infty} r_{2k}(\ell)\, \gamma_{k,\ell}(w)\;. \end{aligned}$$

In the third term we again interchange summation with respect to j and ℓ, and afterwards rename the index ℓ by j and vice versa, to obtain

$$\begin{aligned} r^{(m)}(w) \;=\; & \sum_{|p|=m+1} w^p \sum_{\nu=3}^{n} \Big[\, \gamma^{(\nu)}\, r_\nu(p-e^{(\nu)};1) \\ & \qquad\qquad + \gamma(w) \sum_{j=1}^{\infty} r_\nu(p-e^{(\nu)};j+1)\, r_{\nu 2}(j) \Big] \\ & + \sum_{|p|=m+1} w^p \sum_{k=3}^{n} \sum_{j=1}^{\infty} \gamma_{k,j}(w) \sum_{\nu=3}^{n} \Big[\, r_{\nu k}(j) r_\nu(p-e^{(\nu)};j+1) \\ & \qquad\qquad - r_{2k}(j) \sum_{\ell=1}^{j} r_\nu(p-e^{(\nu)};\ell+1)\, r_{\nu 2}(\ell) \Big] \\ =\; & \sum_{|p|=m+1} w^p \Big[\alpha_p \;+\; \beta_p \;+\; \sum_{k=3}^{n} \sum_{j=1}^{\infty} r_k(p;j)\, \gamma_{k,j}(w) \Big] \end{aligned}$$

where for the last identity we use (5.1). Inserting into (5.9) and moving the term containing $\gamma(w)$ over to the left hand side, we complete the proof. □

Theorem 5.4 *Suppose that* (0.3), (0.4) *and* (0.6) *hold. Then the power series*

$$\alpha(w) \;:=\; \sum_{|p|\geq 1} \alpha_p\, w^p\,, \qquad \beta(w) \;:=\; \sum_{|p|\geq 1} \beta_p\, w^p$$

both converge for $\|w\|_\infty < r_0$*, with* r_0 *as in* Lemma 5.2 a)*, and for such* w *we have the identities*

$$\alpha(w) \;=\; 1 \;-\; \lim_{j\to\infty} \tilde{y}_2(j;w)\,, \qquad \beta(w) \;=\; \gamma\left(\tilde{y}_2(0;w) - 1\right) \tag{5.14}$$

$$\gamma(w) \;=\; \frac{\gamma + \beta(w)}{1-\alpha(w)} \;=\; \gamma\;\frac{\tilde{y}_2(0;w)}{\lim\limits_{j\to\infty} \tilde{y}_2(j;w)} \qquad (\|w\|_\infty < r_0) \tag{5.15}$$

Proof: From the estimate in the proof of Lemma 5.2 a) we conclude convergence of the two power series. Furthermore, note that $r^{(m)}(w)$ omits a power series containing only terms w^p with $|p| \ge m+1$, so that we conclude from Proposition 5.3

$$\gamma_p \;-\; \sum_{0<q\le p} \gamma_{p-q}\,\alpha_q \;=\; \beta_p \qquad (0 < |p| \le m)\,.$$

Since m is an arbitrary natural number, we obtain the first identity in (5.15), while the remaining one and (5.14) follow using the identities for α_p, β_p obtained in Proposition 5.3. □

6 A system of difference equations

In this section, we want to better understand the meaning of the sequence of functions $y_k(j;w)$ which have been introduced before, and which have been shown to satisfy the identity (4.4). In order to simplify this formula, we define for $j \ge 1$

$$y_1(j+1;w) \;:=\; -\,j^{-1}\big(y_2(j+1;w)\,a_{21} \;+\; \sum_{\nu=3}^{n} w_\nu\, y_\nu(j+1;w)\,a_{\nu 1}\big) \tag{6.1}$$

Observe that this definition becomes meaningless for $j = 0$, hence the function $y_1(1;w)$ remains undefined. With this new entry we then may reformulate (4.4), recalling the definition of $r_{\nu,k}(j)$ from (2.1), to obtain for $j \ge 1$

$$\begin{aligned} p(j)\,y_2(j;w) \;&=\; j\,\Big[y_1(j+1;w)\,a_{12} \;+\; y_2(j+1;w)\,(j+\lambda_2') \\ &\qquad\qquad + \sum_{\nu=3}^{n} w_\nu\, y_\nu(j+1;w)\,a_{\nu 2}\Big] \end{aligned} \tag{6.2}$$

$$\begin{aligned} p(j)\,y_k(j;w) \;&=\; j\,\Big[y_1(j+1;w)\,a_{1k} + y_2(j+1;w)\,a_{2k} \\ &\quad + w_k\,y_k(j+1;w)\,(j+\lambda_k') + \sum_{\substack{3\le\nu\le n\\ \nu\ne k}} w_\nu\, y_\nu(j+1;w)\,a_{\nu k}\Big] \end{aligned} \tag{6.3}$$

These identities may best be understood using a matrix-vector notation: For $\|w\|_\infty < 1$ and $j \ge 1$ we define

$$y(j;w) \;=\; [y_1(j;w), y_2(j;w),\, w_3\,y_3(j;w), \ldots, w_n\,y_n(j;w)]^\tau \tag{6.4}$$

ignoring the fact that $y_1(1;w)$ has not yet been defined. In terms of these vectors, equations (6.1) – (6.3) are equivalent with the simple matrix identity

$$p(j)\; y(j;w)^\tau\,\Lambda \;=\; j\; y(j+1;w)^\tau\,\big(j + A_1\big) \qquad (j \ge 1)$$

with Λ and A_1 as in (0.1). For fixed w and all sufficiently large j, this identity may be solved for $y(j+1;w)$, and then is a system of linear difference equations. Since the first diagonal entry of Λ vanishes, one may eliminate the first component of $y(j+1;w)$, such that the system is, in fact, of dimension $n-1$. Setting

$$y(j;w) \;=\; \frac{\Gamma(j+\alpha)\,\Gamma(j+\beta)}{\Gamma(j)\,\Gamma(j+\lambda_2')}\; x(z;w)\,, \qquad z = j + \lambda_2' \tag{6.5}$$

and then allowing z to vary freely in the complex plane, we can write this system in even simpler form as

$$z\, x(z;w)^\tau\,\Lambda \;=\; x(z+1;w)^\tau\,\big(z - \lambda_2' + A_1\big) \tag{6.6}$$

Solving for $x(z+1;w)$ wherever possible and fixing w, we obtain a system that is a very special case of the much more general ones treated in the articles [7–11]. Without going into any details, we briefly explain what can be concluded from the results in said papers:

a) The system (6.6) has a unique formal (vector) solution $\hat{x}(z;w)$ that is a power series in z^{-1} of Gevrey order 1 and has the second unit vector e_2 for its constant term. To show this is a bit tedious, and shall not be done here.

b) The formal Borel transform $\xi(t;w)$ of $\hat{x}(z+1;w)$ satisfies the system of Volterra-type integral equations

$$\xi(t;w)^\tau\,(\mathrm{e}^t\,\Lambda - I) = \Big(e_2^\tau \;+\; \int_0^t \xi(u;w)^\tau\,du\Big)\,(A_1 - \lambda_2') \tag{6.7}$$

The singularities of this system, aside from the ones of the form $2\,k\,\pi\,i$ with $k \in \mathbb{Z}$, are at all points of the form $\log w_k = \log|w_k| + i\arg w_k$, $k = 3,\ldots,n$. For $\|w\| < 1$, these point all have negative real parts. The (unique) solution of this equation is holomorphic in the largest star-shaped (with respect to the origin) region that does not contain any one of these singularities (except for the origin which, however, is removable) and is of exponential growth at most one as $t \to \infty$.

c) Using this information on $\xi(t;w)$, we conclude from the theory of k-summability [1] that the formal solution $\hat{x}(z+1;w)$ (and, equivalently, also $\hat{x}(z;w)$) is 1-summable in all directions d that avoid all the singular points. For $\|w\| < 1$ these include all d with $|d| < \pi/2$. The sum $x(z+1;w)$ is holomorphic for $z+1$ in $\mathbb{C} \setminus \{x + i\,y \;:\; x \le 0\}$, and is Gevrey-asymptotic of order 1 to the formal series $\hat{x}(z+1;w)$. Moreover, $x(z;w)$ is a solution of (6.6).

d) *Defining* $y(j;w)$ by (6.5), we find a vector whose components satisfy the identities (6.1) – (6.3). Because of $\|w\| < 1$ this vector is, up to a factor independent of j, the only solution that stays bounded as $j \to \infty$. Therefore we conclude that the components of $y(j;w)$ coincide with the functions $y_k(j;w)$ that we defined before.

e) For values w outside of the unit polydisc, we find that the formal solution $\hat{x}(z;w)$ remains 1-summable (at least) in direction $d=0$, as long as no w_ν is equal to a real number larger than 1. This shows that $x(z;w)$ as well as the $y_k(j;w)$, admit continuation with respect to w outside of the unit polydisc.

Roughly speaking, we conclude from above, with help of (4.7), that the Stokes function $\gamma(w)$ can be computed in terms of the sum of a formal solution of (6.6). While this from a theoretical point of view is a very satisfying result, it still is not so easy to directly use this for a computation of $\gamma(w)$. For such a practical approach, the results of Section 5 are more suitable, since the entries $r_k(p;j)$ are relatively simple rational functions of j, of which finitely many might be computed, say, with help of standard computer algebra software.

Observe that, due to (5.6), the functions $\tilde{y}_k(j;w)$ can also be linked with a solution $\tilde{x}(z;w)$ of the system of difference equations (6.6), differing from $x(z;w)$ by a constant factor. This shall not be investigated any further in this article.

References

[1] W. Balser, *Formal power series and linear systems of meromorphic ordinary differential equations*, Universitext, Springer-Verlag, New York, 2000. 6, 17

[2] ——, *Non-linear difference equations and Stokes matrices*, Adv. Dyn. Syst. Appl., 7 (2012), pp. 145–162. 1

[3] W. Balser, W. B. Jurkat, and D. A. Lutz, *A general theory of invariants for meromorphic differential equations. I. Formal invariants*, Funkcial. Ekvac., 22 (1979), pp. 197–221. 1

[4] ——, *A general theory of invariants for meromorphic differential equations. II. Proper invariants*, Funkcial. Ekvac., 22 (1979), pp. 257–283. 1, 3

[5] ——, *On the reduction of connection problems for differential equations with an irregular singular point to ones with only regular singularities. II*, SIAM J. Math. Anal., 19 (1988), pp. 398–443. 4, 7

[6] W. Balser and C. Röscheisen, *Solving the hypergeometric system of Okubo type in terms of a certain generalized hypergeometric function*, J. Differential Equations, 247 (2009), pp. 2485–2494. 1

[7] B. L. J. Braaksma, *Transseries for a class of nonlinear difference equations*, Journal of Difference Equations and Applications, 7 (2001), pp. 717–750. 6, 17

[8] B. L. J. Braaksma and B. F. Faber, *Multisummability for some classes of difference equations*, Ann. Inst. Fourier (Grenoble), 46 (1996), pp. 183–217. 6, 17

[9] B. L. J. Braaksma, B. F. Faber, and G. K. Immink, *Summation of formal solutions of a class of linear difference equations*, Pacific J. Math., 195 (2000), pp. 35–65. 6, 17

[10] B. L. J. Braaksma and R. Kuik, *Resurgence relations for classes of differential and difference equations*, Ann. Fac. Sci. Toulouse Math. (6), 13 (2004), pp. 479–492. 6, 17

[11] ——, *Asymptotics and singularities for a class of difference equations*, in Analyzable functions and applications, vol. 373 of Contemp. Math., Amer. Math. Soc., Providence, RI, 2005, pp. 113–135. 6, 17

[12] J. Ecalle, *Les fonctions résurgentes I–II*, Publ. Math. d'Orsay, Université Paris Sud, 1981. 7

[13] ——, *Les fonctions résurgentes III*, Publ. Math. d'Orsay, Université Paris Sud, 1985. 7

[14] M. Hervé, *Several complex variables. Local theory*, Published for the Tata Institute of Fundamental Research, Bombay, by Oxford University Press, London, 1963. 8

[15] W. B. Jurkat, *Meromorphe Differentialgleichungen*, vol. 637 of Lecture Notes in Math., Springer Verlag, Berlin, 1978. 1, 3

[16] C. Röscheisen, *On hypergeometric systems of differential equations*, PhD thesis, University of Ulm, Germany, 2009. 1, 3

[17] R. Schäfke, *Über das globale analytische Verhalten der Lösungen der über die Laplace–Transformation zusammenhängenden Differentialgleichungen* $t\,x'(t) = (A + tB)\,x$ *und* $(s - B)\,v' = (\varrho - A)\,v$, Dissertation, Essen, 1979. 4, 7

[18] ——, *Über das globale Verhalten der Normallösungen von* $x'(t) = (B + t^{-1}A)\,x(t)$, *und zweier Arten von assoziierten Funktionen*, Math. Nachr., 121 (1985), pp. 123–145. 4