

Integrability of Hamiltonian systems and transseries expansions

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Abstract This paper studies analytic Liouville-non-integrable and C^∞ -Liouville-integrable Hamiltonian systems with two degrees of freedom. We will show that considerably general Hamiltonians than the one studied in [2] have the property. We also show that a certain monodromy property of an ordinary differential equation obtained as a subsystem of a given Hamiltonian and the transseries expansion of a first integral play an important role in the analysis. In the former half we will show that the analytic Liouville-non-integrability holds for a rather wide class of Hamiltonians under a certain monodromy condition. For these analytic non integrable Hamiltonians we cannot construct nonanalytic first integrals concretely as in [2]. In the latter half, we show the nonanalytic integrability from the viewpoint of a transseries expansion of a first integral. We will construct a first integral in transseries formally under general situation. Then we show convergence of transseries or existence of the first integral which is asymptotically equal to a given formal transseries solution.

Keywords nonintegrability · Hamiltonian systems · transseries · summability

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1 Introduction

A Hamiltonian system in n degrees of freedom is said to be C^∞ -Liouville-integrable if there are n smooth first integrals in involution which are functionally independent on an open dense set. If the first integrals are analytic, then we say that it is analytic-Liouville-integrable. In [2], Gorni and Zampieri showed the existence of a Hamiltonian

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system with two degrees of freedom which is not analytic-Liouville-integrable, while it is C^∞ -Liouville-integrable. The geometrical motivation to study such an example comes from the integrability of a geodesic flows and the Taimanov's problem. (cf. [2]). We note that the proof of analytic-nonintegrability relies on the power series expansion of a first integral, and the C^∞ -integrability was proved by constructing concretely a smooth first integral. (cf. Remark after Corollary 1.) In this paper, we are interested in the analytical structures which yield nonintegrability in the framework of rather general Hamiltonians than those in previous works. In fact, we will show that the monodromy property of a certain subsystem of a given Hamiltonian plays an important role.

For these analytic-nonintegrable Hamiltonians, we cannot construct a nonanalytic first integral concretely, and instead we make use of a transseries in order to construct such an integral. An integral in a transseries expansion is constructed via the Lagrange-Charpit system of a certain vector field obtained by restricting a given Hamiltonian vector field to an invariant manifold. The construction of a first integral as a formal transseries is elementary, while the convergence part is complicated due to the degeneracy of a given Hamiltonian. We will study the convergence from two different points of view, transformation of transseries and Borel summability method for transseries.

This paper is organized as follows. In §2 we prove the analytic-Liouville- nonintegrability under the monodromy condition. (Theorem 1). In §3 we give the proof of Theorem 1. In §4 we prove the transseries expansion of the integrals in a formal sense. The convergence in a curved region and transformation of transseries are discussed in §5. In §6 we study the asymptotic property of transseries in terms of Borel summability method.

2 Analytic nonintegrability

Let $\sigma \geq 1$ be an integer and let $r(q_1, q_2, p_1, p_2)$ be an analytic function of $(q_1, q_2, p_1, p_2) \in \mathbb{R}^4$ in some neighborhood of the origin $0 \in \mathbb{R}^4$ such that

$$r \equiv r(q_1, q_2, p_1, p_2) = q_1^{2\sigma} + a(q_1^{2\sigma})q_2^2 + \tilde{r}(q_1, q_2, p_1, p_2)q_2^3, \quad (1)$$

where $\tilde{r}(q_1, q_2, p_1, p_2)$ is analytic at the origin and $a(t)$ ($t = q_1^{2\sigma}$) is a polynomial of t such that $a(0) > 0$. We are interested in the following analytic Hamiltonian in \mathbb{R}^4 with two degrees of freedom

$$\mathcal{H} = -q_2 p_2 \partial_{q_1} r + \left(r^2 + q_2 \partial_{q_2} r \right) p_1, \quad (2)$$

where $\partial_{q_1} = \frac{\partial}{\partial q_1}$ and $\partial_{q_2} = \frac{\partial}{\partial q_2}$. The associated Hamiltonian system is given by

$$\begin{cases} \dot{q}_1 = \partial \mathcal{H} / (\partial p_1) &= r^2 + q_2 \partial_{q_2} r + (2r \partial_{p_1} r + q_2 \partial_{q_2} \partial_{p_1} r) p_1 - q_2 p_2 \partial_{p_1} \partial_{q_1} r, \\ \dot{q}_2 = \partial \mathcal{H} / (\partial p_2) &= -q_2 \partial_{q_1} r - q_2 p_2 \partial_{q_1} \partial_{p_2} r + p_1 (2r \partial_{p_2} r + q_2 \partial_{q_2} \partial_{p_2} r), \\ \dot{p}_1 = -\partial \mathcal{H} / (\partial q_1) &= q_2 p_2 \partial_{q_1}^2 r - (2r \partial_{q_1} r + q_2 \partial_{q_1} \partial_{q_2} r) p_1, \\ \dot{p}_2 = -\partial \mathcal{H} / (\partial q_2) &= p_2 \partial_{q_1} r + q_2 p_2 \partial_{q_1} \partial_{q_2} r - \left(2r \partial_{q_2} r + \partial_{q_2} r + q_2 \partial_{q_2}^2 r \right) p_1. \end{cases} \quad (3)$$

We need a definition in order to state our theorem.

Definition 1 We say that a polynomial $a(t)$ satisfies the monodromy condition (M) if the following equation has no polynomial solution $U(t)$

$$2\sigma t^2 U' - 4\sigma U + (1 - 6\sigma)tU = (t + 1)a(t). \quad (4)$$

Then we have

Theorem 1 *Assume that $a(t)$ satisfy (M). Then the Hamiltonian system (3) is not analytic-Liouville-integrable in any neighborhood of the origin. More precisely, for any analytic first integral $u = u(q_1, q_2, p_1, p_2)$ of (3) in \mathbb{R}^4 , there exists a function ϕ of one-variable, being analytic at $0 \in \mathbb{R}$ such that $u = \phi \circ \mathcal{H}$.*

If $a(t) \equiv 1$, then we can easily see, from the direct computations or by Lemma 2 of [3] that (M) is satisfied. Hence we have

Corollary 1 *Suppose that $\sigma = 1$, $a(t) \equiv 1$ and $\tilde{r} \equiv 0$ in (1). Then the Hamiltonian system (3) is not analytic-Liouville-integrable in any neighborhood of the origin.*

Remark. (a) As to the fundamental properties of (M) we refer [3]. In this paper, we change the terminology for the sake of simplicity. We remark that (M) is a generic condition.

(b) Theorem 1 is a generalization of [4, Theorem 1], where the function r in (1) was supposed to be independent of p_1 and p_2 . Corollary 1 was proved in [2]. In this case, it is not difficult to see that (3) in Corollary 1 is C^∞ -Liouville-integrable, because it has a smooth first integral

$$u = \begin{cases} q_2 \exp\left(-\frac{1}{r}\right) & \text{if } (q_1, q_2) \neq (0, 0), \\ 0 & \text{if } (q_1, q_2) = (0, 0). \end{cases} \quad (5)$$

On the other hand, it is not known whether (3) in a general case has a nonanalytic first integral because one cannot construct the first integral of (3) concretely since r also depends on p_1 and p_2 . In §4 we will study the integrability from the viewpoint of transseries.

3 Proof of theorem

The proof of Theorem 1 is done by the argument in [3]. For the sake of completeness we give the proof.

Proof (of Theorem 2.1) By the suitable change of the variable q_2 one may assume that $a(0) = 1$. Let $u = u(q_1, q_2, p_1, p_2)$ be any analytic first integral of (3). We note that u is the first integral of the Hamiltonian system (3) if and only if u is a solution of the following first order equation

$$\begin{aligned} \{\mathcal{H}, u\} &\equiv \left(q_2 p_2 \partial_{q_1}^2 r - (2r \partial_{q_1} r + q_2 \partial_{q_1} \partial_{q_2} r) p_1 \right) \frac{\partial u}{\partial p_1} \\ &+ \left(p_2 \partial_{q_1} r + q_2 p_2 \partial_{q_1} \partial_{q_2} r - \left(2r \partial_{q_2} r + \partial_{q_2} r + q_2 \partial_{q_2}^2 r \right) p_1 \right) \frac{\partial u}{\partial p_2} \\ &+ \left(r^2 + q_2 \partial_{q_2} r + (2r \partial_{p_1} r + q_2 \partial_{q_2} \partial_{p_1} r) p_1 - q_2 p_2 \partial_{p_1} \partial_{q_1} r \right) \frac{\partial u}{\partial q_1} \\ &+ \left(-q_2 \partial_{q_1} r - q_2 p_2 \partial_{q_1} \partial_{p_2} r + p_1 (2r \partial_{p_2} r + q_2 \partial_{q_2} \partial_{p_2} r) \right) \frac{\partial u}{\partial q_2} = 0. \end{aligned} \quad (6)$$

We define

$$v \equiv v(q_1, p_1, p_2) := u(q_1, 0, p_1, p_2). \quad (7)$$

By setting $q_2 = 0$ in (6) and noting that $\partial_{q_2} r(q_1, 0) \equiv 0$ and $r(q_1, 0) = q_1^{2\sigma}$ by (1), we obtain

$$2\sigma p_2 \frac{\partial v}{\partial p_2} - 4\sigma q_1^{2\sigma} p_1 \frac{\partial v}{\partial p_1} + q_1^{2\sigma+1} \frac{\partial v}{\partial q_1} = 0. \quad (8)$$

We expand v into the power series of p_2 , $v = \sum_{j=0}^{\infty} v_j(q_1, p_1) p_2^j$. Then we see that $v_j(q_1, p_1)$ ($j = 0, 1, \dots$) satisfy

$$2\sigma j v_j - 4\sigma q_1^{2\sigma} p_1 \frac{\partial v_j}{\partial p_1} + q_1^{2\sigma+1} \frac{\partial v_j}{\partial q_1} = 0, \quad j = 0, 1, 2, \dots \quad (9)$$

We want to show that $v_j = 0$ if $j \neq 0$, and $v = \phi(p_1 q_1^{4\sigma})$ for some analytic function $\phi(t)$ of one variable. Indeed, by substituting the expansion of v_j $v_j(q_1, p_1) = \sum_{\nu=0}^{\infty} v_{j,\nu}(q_1) p_1^\nu$ into (9) we obtain

$$2\sigma j v_{j,\nu} - 4\sigma \nu q_1^{2\sigma} v_{j,\nu} + q_1^{2\sigma+1} \frac{\partial v_{j,\nu}}{\partial q_1} = 0, \quad j = 0, 1, 2, \dots \quad (10)$$

If we expand $v_{j,\nu}$ into the power series of q_1 , then we can easily see that $v_{j,\nu} \equiv 0$ for all $\nu = 0, 1, \dots$, if $j \neq 0$. Hence we have $v_j = 0$ if $j \neq 0$. It follows that $v = v_0(q_1, p_1)$. Moreover, by (9) v satisfies the equation

$$-4\sigma p_1 \frac{\partial v}{\partial p_1} + q_1 \frac{\partial v}{\partial q_1} = 0.$$

If we substitute the expansion of v into the equation, then, by simple computations, we easily see that $v = \phi(p_1 q_1^{4\sigma})$ for some analytic function $\phi(t)$ of one variable. This proves the assertion.

It follows from (2) that $v = v_0 = \phi(p_1 q_1^{4\sigma}) = \phi(\mathcal{H}|_{q_2=0})$. We define

$$g(q_1, q_2, p_1, p_2) := u(q_1, q_2, p_1, p_2) - \phi(\mathcal{H}). \quad (11)$$

By (7) and by recalling that \mathcal{H} is a first integral we see that g is an analytic solution of (6) such that $g(q_1, 0, p_1, p_2) \equiv 0$. In order to prove Theorem 1 we shall show $g(q_1, q_2, p_1, p_2) \equiv 0$ in some neighborhood of the origin. First we will show that

$$g(q_1, q_2, p_1, p_2) = \phi_1(p_1 q_1^{4\sigma}) p_2 q_2 + h_2(q_1, p_1, p_2) q_2^2 + \tilde{h}_3(q_1, q_2, p_1, p_2) q_2^3, \quad (12)$$

for some analytic function ϕ_1 of one variable and analytic functions h_2 and \tilde{h}_3 . Because g is analytic we have the expansion

$$g(q_1, q_2, p_1, p_2) = g_1(q_1, p_1, p_2) q_2 + h_2(q_1, p_1, p_2) q_2^2 + \tilde{h}_3(q_1, q_2, p_1, p_2) q_2^3. \quad (13)$$

We substitute (13) with $u = g$ into (6) and compare the coefficients of q_2 . By (1) we have

$$-4\sigma q_1^{2\sigma} p_1 \frac{\partial g_1}{\partial p_1} + 2\sigma \left(p_2 \frac{\partial g_1}{\partial p_2} - g_1 \right) + q_1^{2\sigma+1} \frac{\partial g_1}{\partial q_1} = 0. \quad (14)$$

By substituting the expansion $g_1(q_1, p_1, p_2) = \sum_{m=0}^{\infty} g_{1,m}(q_1, p_1) p_2^m$ into (14) and by comparing the coefficients of p_2^m we obtain

$$-4\sigma q_1^{2\sigma} p_1 \frac{\partial g_{1,m}}{\partial p_1} + 2\sigma(m-1)g_{1,m} + q_1^{2\sigma+1} \frac{\partial g_{1,m}}{\partial q_1} = 0. \quad (15)$$

If $m \neq 1$, then we obtain a similar equation as for v_j in (9). Hence we have $g_{1,m} = 0$ if $m \neq 1$. It follows that $g_1 = g_{1,1}p_2$, and $g_{1,1}$ satisfies the equation $-4\sigma p_1 \frac{\partial g_{1,1}}{\partial p_1} + q_1 \frac{\partial g_{1,1}}{\partial q_1} = 0$. By the same argument as in the above, we see that $g_1 = g_{1,1}p_2 = \phi_1(p_1 q_1^{4\sigma}) p_2$ for some analytic function ϕ_1 of one variable. This proves the assertion.

Let us now suppose that

$$g(q_1, q_2, p_1, p_2) = \phi_{n-1}(p_1 q_1^{4\sigma}) p_2^{n-1} q_2^{n-1} + h_n(q_1, p_1, p_2) q_2^n + \tilde{h}_{n+1}(q_1, q_2, p_1, p_2) q_2^{n+1}, \quad (16)$$

for some $n \geq 2$, some analytic function ϕ_{n-1} of one variable and analytic functions $h_n(q_1, p_1, p_2)$ and $\tilde{h}_{n+1}(q_1, q_2, p_1, p_2)$. Then we substitute (16) into (6) with $u = g$ and we compare the coefficients of q_2^n . By (1) we have

$$2p_2^n \sigma (2\sigma - 1) q_1^{6\sigma-2} \phi'_{n-1} - 4a(q_1^{2\sigma})(q_1^{2\sigma} + 1)(n-1)p_1 p_2^{n-2} \phi_{n-1} - 4\sigma q_1^{4\sigma-1} p_1 \frac{\partial h_n}{\partial p_1} + 2\sigma q_1^{2\sigma-1} \left(p_2 \frac{\partial h_n}{\partial p_2} - n h_n \right) + q_1^{4\sigma} \frac{\partial h_n}{\partial q_1} = 0. \quad (17)$$

By substituting the expansion $h_n(q_1, p_1, p_2) = \sum_{m=0}^{\infty} h_{n,m}(q_1, p_1) p_2^m$ into (17) and by comparing the coefficients of p_2^{n-2} we obtain

$$-4\sigma q_1^{4\sigma-1} p_1 \frac{\partial h_{n,n-2}}{\partial p_1} - 4a(q_1^{2\sigma})(q_1^{2\sigma} + 1)(n-1)p_1 \phi_{n-1} - 4\sigma q_1^{2\sigma-1} h_{n,n-2} + q_1^{4\sigma} \frac{\partial h_{n,n-2}}{\partial q_1} = 0. \quad (18)$$

We will show that

$$h_{n,n-2} = 0, \quad \phi_{n-1} = 0. \quad (19)$$

If we can prove $\phi_{n-1} = 0$, then it follows from (18) that $v := h_{n,n-2}$ satisfies a similar equation as (9). Hence, by the same argument as for (9) we have $h_{n,n-2} = 0$. In order to show $\phi_{n-1} = 0$ we insert the expansions

$$\phi_{n-1}(p_1 q_1^{4\sigma}) = \sum_{k=0}^{\infty} \phi_{n-1,k} p_1^k q_1^{4\sigma k}, \quad h_{n,n-2}(q_1, p_1) = \sum_{k=0}^{\infty} h_{n,n-2,k}(q_1) p_1^k \quad (20)$$

into (18) and we compare the coefficients of p_1^k . Then we obtain, for $k \geq 0$

$$-4\sigma q_1^{4\sigma-1} k h_{n,n-2,k} - 4\sigma q_1^{2\sigma-1} h_{n,n-2,k} + q_1^{4\sigma} \frac{\partial h_{n,n-2,k}}{\partial q_1} = 4a(q_1^{2\sigma})(q_1^{2\sigma} + 1)(n-1) \phi_{n-1,k-1} q_1^{4\sigma(k-1)}, \quad (21)$$

where we set $\phi_{n-1,-1} = 0$. If we set $q_1 = 0$ and $k = 1$ in (21), then, by $a(0) = 1$, we obtain $0 = 4(n-1)\phi_{n-1,0}$, which implies $\phi_{n-1,0} = 0$.

Suppose that $\phi_{n-1,k-1} \neq 0$ for some $k \geq 2$. We divide both sides of (21) by $q_1^{2\sigma-1}$. Then the right-hand side of (21) is divisible by q_1^N , $N = 4\sigma(k-1) + 1 - 2\sigma \geq 2\sigma + 1$. Because the operator $-4\sigma k q_1^{2\sigma} + q_1^{2\sigma+1}(d/dq_1)$ in the left-hand side of the equation increases the power of q_1 , it follows that $h_{n,n-2,k}$ is divisible by q_1^N . We set $h_{n,n-2,k}(q_1) = q_1^N W(q_1)$. Then we have $q_1(d/dq_1)h_{n,n-2,k} = q_1^N(N + q_1(d/dq_1))W$. It follows from (21) that W satisfies

$$\begin{aligned} (N - 4\sigma k)q_1^{2\sigma}W - 4\sigma W + q_1^{2\sigma+1}\frac{dW}{dq_1} \\ = 4a(q_1^{2\sigma})(n-1)\phi_{n-1,k-1}(q_1^{2\sigma} + 1). \end{aligned} \quad (22)$$

We set $W = \sum_{j=0}^{2\sigma-1} q_1^j W_j(q_1^{2\sigma})$. Because the right-hand side of (22) is a function of $q_1^{2\sigma}$, W_j ($1 \leq j < 2\sigma$) satisfy

$$q_1^{2\sigma}(N - 4\sigma k + j)W_j - 4\sigma W_j + q_1^{2\sigma+1}\frac{dW_j}{dq_1} = 0. \quad (23)$$

By a similar argument as for (9) we have $W_j = 0$ for $1 \leq j < 2\sigma$. Hence we have $W(q_1) = W_0(q_1^{2\sigma}) =: V(t)$ ($t = q_1^{2\sigma}$). Because $q_1(d/dq_1)V = 2\sigma t(d/dt)V$, it follows from (22) that

$$(1 - 6\sigma)tV - 4\sigma V + 2\sigma t^2\frac{dV}{dt} = 4a(t)(n-1)(t+1)\phi_{n-1,k-1}.$$

If we set $U := V/(2(n-1)\phi_{n-1,k-1})$, then U is an analytic solution of (4). This contradicts to the assumption of the theorem, because we assume that (M) is not verified. Hence we have $\phi_{n-1,k-1} = 0$. Because k is arbitrary we have $\phi_{n-1} = 0$.

Next we set $\phi_{n-1} = 0$ in (17) and consider the coefficients of p_2^m ($m \neq n$). Then we see that $h_{n,m}$ satisfies a similar equation as for (9). Hence we have $h_{n,m} = 0$ if $n \neq m$, and $h_{n,n} = \phi_n(p_1 q_1^{4\sigma})$ for some analytic function ϕ_n of one variable. It follows that $h_n(q_1, p_1, p_2) = h_{n,n}(q_1, p_1)p_2^n = \phi_n(p_1 q_1^{4\sigma})p_2^n$. Hence we have (16) with n replaced by $n+1$. By induction we obtain (16) for an arbitrary integer $n \geq 2$.

It follows from (16) with n replaced by $n+2$ that, for every $n \geq 0$ we have $\partial_{q_2}^n g(q_1, 0, p_1, p_2) \equiv 0$, where (q_1, p_1, p_2) is in some neighborhood of the origin which may depend on n . On the other hand $\partial_{q_2}^n g(q_1, 0, p_1, p_2)$ is analytic in some neighborhood of the origin independent of n . By analytic continuation, we have $\partial_{q_2}^n g(q_1, 0, p_1, p_2) \equiv 0$ in some neighborhood of the origin independent of n . By the partial Taylor expansion $g = \sum_n \partial_{q_2}^n g(q_1, 0, p_1, p_2) q_2^n / n!$, we have $g = 0$.

4 Transseries expansion of first integral

In this section, we shall construct a first integral of (3) as a transseries. (cf. [1]). In order to introduce such a series we consider the terms in (6) which preserve the order of q_2

$$\mathcal{L}u := q_1^{2\sigma-1} \left(2\sigma p_2 \frac{\partial u}{\partial p_2} - q_2 \frac{\partial u}{\partial q_2} \right) + q_1^{2\sigma} \left(q_1 \frac{\partial u}{\partial q_1} - 4\sigma p_1 \frac{\partial u}{\partial p_1} \right). \quad (24)$$

We note that we can write (6) in the form

$$\mathcal{L}u + Ru = 0, \quad Ru := \{\mathcal{H}, u\} - \mathcal{L}u. \quad (25)$$

In order to construct an inverse of \mathcal{L} we consider the Lagrange-Charpit system corresponding to \mathcal{L}

$$\frac{dq_1}{q_1^{4\sigma}} = \frac{dq_2}{-2\sigma q_1^{2\sigma-1} q_2} = \frac{dp_1}{-4\sigma q_1^{4\sigma-1} p_1} = \frac{dp_2}{2\sigma q_1^{2\sigma-1} p_2}. \quad (26)$$

We integrate (26) by taking q_1 as an independent variable. By simple computations we can easily see that the solution of (26) is given by

$$q_2 = q_2^0 \exp\left(q_1^{-2\sigma}\right), \quad p_2 = p_2^0 \exp\left(-q_1^{-2\sigma}\right), \quad p_1 = p_1^0 q_1^{-4\sigma}, \quad (27)$$

where q_2^0 , p_2^0 and p_1^0 are certain constants.

We note that the solution of the homogeneous equation $\mathcal{L}v = 0$ is given by

$$v = \phi(p_1 q_1^{4\sigma}, p_2 \exp\left(q_1^{-2\sigma}\right), q_2 \exp\left(-q_1^{-2\sigma}\right)), \quad (28)$$

with $\phi(p_1^0, p_2^0, q_2^0)$ being an arbitrary function of p_1^0 , p_2^0 and q_2^0 . We then construct a solution u of (25) in the form

$$u = \sum_{j=0}^{\infty} u_j(q_1, p_1 q_1^{4\sigma}, p_2 \exp\left(q_1^{-2\sigma}\right)) \left(\exp\left(-q_1^{-2\sigma}\right) q_2\right)^j, \quad (29)$$

where $u_0(q_1, p_1 q_1^{4\sigma}, p_2 \exp\left(q_1^{-2\sigma}\right)) \equiv u_0(p_1 q_1^{4\sigma}, p_2 \exp\left(q_1^{-2\sigma}\right))$. We call (29) the transseries solution of (25). Then we have

Proposition 1 *Let $u_0(p_1^0, p_2^0)$ be a given analytic function of p_1^0 and p_2^0 such that $\partial u_0 / \partial p_2^0 \neq 0$. Then (3) is formally Liouville-integrable in the sense that (29) is a formal integral of (3) which is functionally independent of \mathcal{H} .*

Proof. We note that R in (25) has analytic coefficients and R raises the power of q_2 at least by one. On the other hand we have

$$\mathcal{L} \left(u_j \left(\exp\left(-q_1^{-2\sigma}\right) q_2 \right)^j \right) = (\mathcal{L}u_j) \left(\exp\left(-q_1^{-2\sigma}\right) q_2 \right)^j. \quad (30)$$

Hence, if we substitute (29) into (25) and compare the coefficients of q_2^j of both sides, then we have

$$\mathcal{L}u_j = (\text{linear functions of } u_k \text{ and their derivatives } (k < j), \quad j = 1, 2, \dots \quad (31)$$

We note that the right-hand side is a known quantity if we determine u_j recursively.

We will solve $\mathcal{L}v = f$, where

$$v = v\left(q_1, p_1 q_1^{4\sigma}, p_2 \exp\left(q_1^{-2\sigma}\right)\right).$$

By making the change of variables $(q_1, p_1, p_2) \mapsto (q_1, p_1^0, p_2^0)$ given by (27), the equation $\mathcal{L}v = f(q_1, p_1, p_2)$ is written in the form

$$q_1^{4\sigma} (\partial v / \partial q_1) = g(q_1, p_1^0, p_2^0), \quad (32)$$

where

$$g \equiv g(q_1, p_1^0, p_2^0) = f(q_1, p_1^0 q_1^{-4\sigma}, p_2^0 \exp\left(-q_1^{-2\sigma}\right)).$$

Hence the solution of (32) is given by

$$v = \int_{a_0}^{q_1} s^{-4\sigma} g(s, p_1^0, p_2^0) ds, \quad (33)$$

where a_0 is an arbitrary complex constant. If we go back to the original variables q_1 , p_1 and p_2 , then we obtain a solution of $\mathcal{L}v = f$. Therefore we have a solution u of (25) given by (29).

Finally, we will show that u converges, then u is an integral of (3) functionally independent of \mathcal{H} . Hence our Hamiltonian system is formally Liouville-integrable. Indeed, if this is not the case, then we have $u = \phi(\mathcal{H})$ for some smooth function ϕ of one variable. If we set $q_2 = 0$, then we obtain

$$u_0 \left(p_1 q_1^{4\sigma}, p_2 \exp \left(q_1^{-2\sigma} \right) \right) = \phi(\mathcal{H})|_{q_2=0} = \phi(p_1 q_1^{4\sigma}).$$

This is a contradiction to the assumption that $\partial u_0 / \partial p_2^0 \neq 0$. This ends the proof.

5 Convergence of transseries

In this section we consider the Hamiltonian corresponding to $r = q_1^{2\sigma} + a(q_1^{2\sigma})q_2^2$ in (2)

$$\mathcal{H} = -2\sigma q_1^{2\sigma-1} q_2 p_2 (1 + q_2^2 a') + p_1 \left((q_1^{2\sigma} + a q_2^2)^2 + 2a q_2^2 \right), \quad (34)$$

where we assume $a(0) = 1$ for the sake of simplicity. We study the convergence of transseries solutions (29), where we set $a_0 = 0$ in (33). Note that $u_j(0, p_1^0, p_2^0)$ identically vanishes for any $j \geq 1$.

Clearly, the integral u is the solution of (25), where \mathcal{L} and R are given, respectively, by (24) and

$$\begin{aligned} Ru &= (\tilde{\alpha} p_1 + \tilde{\beta} p_2) \frac{\partial u}{\partial p_1} + (\tilde{\gamma} p_2 + \tilde{\delta} p_1) \frac{\partial u}{\partial p_2} \\ &+ E(q_1, q_2) \frac{\partial u}{\partial q_1} - 2\sigma q_1^{2\sigma-1} a' q_2^3 \frac{\partial u}{\partial q_2}, \end{aligned} \quad (35)$$

where

$$E \equiv E(q_1, q_2) := a q_2^2 (a q_2^2 + 2q_1^{2\sigma} + 2), \quad (36)$$

$$\tilde{\alpha} := -4\sigma q_1^{2\sigma-1} q_2^2 (a + a' + a' q_1^{2\sigma} + a a' q_2^2), \quad (37)$$

$$\tilde{\beta} := 2\sigma(2\sigma - 1) q_2 q_1^{2\sigma-2} (1 + a' q_2^2) + 4\sigma^2 q_1^{4\sigma-2} a'' q_2^3, \quad (38)$$

$$\tilde{\gamma} := 6\sigma q_1^{2\sigma-1} a' q_2^2, \quad \tilde{\delta} := -4q_2 a (q_1^{2\sigma} + a q_2^2 + 1). \quad (39)$$

Let ε_0 be a small positive constant. Then we define

$$S_0 := \{q_1 \in \mathbb{C}; |q_1| < \varepsilon_0\} \cap \left\{ q_1 \in \mathbb{C}; \Re q_1^{2\sigma} < 0 \right\}, \quad (40)$$

where $\Re q_1^{2\sigma}$ denotes the real part of $q_1^{2\sigma}$. Then we have

Theorem 2 *Let $u_0(p_1^0, p_2^0)$ be an analytic function of p_1^0 and p_2^0 such that $\partial u_0 / \partial p_2^0 \neq 0$ in some neighborhood of the origin. Then there exist a $\delta > 0$, an $\varepsilon_0 > 0$, neighborhoods V_1, V_2 of the origin in \mathbb{C} such that u in (29) converges in the domain*

$$\{(q_1, q_2, p_1, p_2); q_1 \in S_0, p_1 \in V_1, p_2 \in V_2, |\exp(-q_1^{-2\sigma}) q_2| < \delta\}.$$

Proof Let \mathcal{L} be given by (24). The Lagrange-Charpit system corresponding to $\mathcal{L} + R$ is given by

$$\frac{dq_1}{q_1^{4\sigma} + E} = \frac{dq_2}{-2\sigma q_1^{2\sigma-1} q_2 T} = \frac{dp_1}{(\tilde{\alpha} - 4\sigma q_1^{4\sigma-1})p_1 + \tilde{\beta}p_2} = \frac{dp_2}{(2\sigma q_1^{2\sigma-1} + \tilde{\gamma})p_2 + \tilde{\delta}p_1}, \quad (41)$$

where we set

$$T \equiv T(q_1, q_2) := 1 + a' q_2^2. \quad (42)$$

We integrate (41) by taking $q_1 \in S_0$ as an independent variable. We want to show that the solutions are perturbations of the solutions (27) in S_0 . Namely, we will prove

$$q_2 = q_2^0 \exp(q_1^{-2\sigma}) \exp(q_1 Q_2(q_1, q_2^0)), \quad (43)$$

$$p_1 = q_1^{-4\sigma} \left(p_1^0 (1 + P_1(q_1, q_2^0)) + p_2^0 \tilde{P}_1(q_1, q_2^0) \right), \quad (44)$$

$$p_2 = \exp(-q_1^{-2\sigma}) \left(p_1^0 P_2(q_1, q_2^0) + p_2^0 (1 + \tilde{P}_2(q_1, q_2^0)) \right), \quad (45)$$

for some functions $Q_2 \equiv Q_2(q_1, q_2^0)$, $P_j \equiv P_j(q_1, q_2^0)$ and $\tilde{P}_j \equiv \tilde{P}_j(q_1, q_2^0)$ ($j = 1, 2$), which are holomorphic and bounded when $q_1 \in S_0$ and q_2^0 in some neighborhood of $q_2^0 = 0$. Here p_1^0 and p_2^0 are arbitrary constants.

In order to verify these properties we first consider the following equation

$$\frac{dq_2}{dq_1} = \frac{-2\sigma q_2 q_1^{2\sigma-1} (1 + a' q_2^2)}{q_1^{4\sigma} + E} = \frac{-2\sigma q_2}{q_1^{2\sigma+1}} \left(1 + \frac{a' q_2^2 - q_1^{-4\sigma} E}{1 + q_1^{-4\sigma} E} \right). \quad (46)$$

Clearly, $q_2 = 0$ is a solution of (46). We assume $q_2 \neq 0$. We note that $v := q_2^0 \exp(q_1^{-2\sigma})$ satisfies the equation $dv/dq_1 = -2\sigma v q_1^{-2\sigma-1}$. We set $U := q_1 Q_2$, and we substitute (43) into (46). Then we have

$$\frac{dU}{dq_1} = -\frac{2\sigma(a' q_2^2 - q_1^{-4\sigma} E)}{q_1^{2\sigma+1}(1 + q_1^{-4\sigma} E)} =: f(q_1, U), \quad (47)$$

where $q_2 = q_2^0 \exp(q_1^{-2\sigma}) e^U$. Because $\Re q_1^{2\sigma} < 0$ in S_0 , we see that $f(q_1, U)$ is holomorphic when $(q_1, U) \in S_0 \times \Omega$, and continuous up to its closure, where Ω is a neighborhood of the origin. Moreover, its maximal norm when $(q_1, U) \in S_0 \times \Omega$ can be made arbitrarily small if we shrink S_0 sufficiently small.

We will solve (47) in S_0 . If we replace U with $U + c$ for a constant c we see from (43) that q_2^0 is replaced by $e^c q_2^0$. Hence we may assume that U vanishes at $q_1 = 0$. We will look for the solution U as the solution of the following equation

$$U = \int_0^{q_1} f(s, U) ds, \quad (48)$$

where the integral is taken along the straight line in S_0 which connects 0 and q_1 . We note that we can make $|f(q_1, U)|$ arbitrarily small if we take ε_0 in S_0 sufficiently small and U is bounded. We can easily show that the right-hand side operator of (48) is a contraction mapping on a small ball in the set of functions holomorphic in S_0 and continuous on its closure, and $U(0) = 0$. Hence we have a holomorphic solution U in S_0 of (48). If we set $Q_2 := q_1^{-1} U$ we obtain the desired solution. The analyticity of U with respect to q_2^0 is easy to verify because of the definition of f .

Next we study the equations for p_1 and p_2 . It follows from (41) that

$$\frac{dp_1}{dq_1} = \alpha p_1 + \beta p_2, \quad \frac{dp_2}{dq_1} = \gamma p_2 + \delta p_1, \quad (49)$$

where

$$\alpha = \frac{\tilde{\alpha} - 4\sigma q_1^{4\sigma-1}}{q_1^{4\sigma} + E}, \quad \beta = \frac{\tilde{\beta}}{q_1^{4\sigma} + E}, \quad \gamma = \frac{2\sigma q_1^{2\sigma-1} + \tilde{\gamma}}{q_1^{4\sigma} + E}, \quad \delta = \frac{\tilde{\delta}}{q_1^{4\sigma} + E}. \quad (50)$$

We will construct the solution of (49) in the following form

$$p_\nu = \sum_{j=0}^{\infty} p_\nu^{(j)}, \quad \nu = 1, 2, \quad (51)$$

where

$$\frac{dp_1^{(0)}}{dq_1} = \alpha p_1^{(0)}, \quad \frac{dp_2^{(0)}}{dq_1} = \gamma p_2^{(0)}, \quad (52)$$

and $p_\nu^{(j)}$ ($\nu = 1, 2; j = 1, 2, \dots$) are determined by

$$\frac{dp_1^{(j)}}{dq_1} = \alpha p_1^{(j)} + \beta p_2^{(j-1)}, \quad \frac{dp_2^{(j)}}{dq_1} = \gamma p_2^{(j)} + \delta p_1^{(j-1)}. \quad (53)$$

First we solve (52). By the definition of γ in (50) we have the expression $\gamma = 2\sigma q_1^{-2\sigma-1} + \gamma_0$, where γ_0 is a bounded holomorphic function in S_0 . By the change of an unknown function similar in the argument for (46) the solution $p_2^{(0)}$ of (52) has the following expression

$$p_2^{(0)} = p_2^0 \exp(-q_1^{-2\sigma})(1 + \tilde{P}_2^{(0)}(q_1, q_2^0)) \quad (54)$$

for some $\tilde{P}_2^{(0)}$ which is bounded and holomorphic in $q_1 \in S_0$ and q_2^0 in some neighborhood of the origin, where p_2^0 is an arbitrary constant. Similarly, noting that $\alpha = -4\sigma q_1^{-1} + \alpha_0$ for some bounded holomorphic function α_0 in S_0 we see that the solution $p_1^{(0)}$ of (52) has the following expression

$$p_1^{(0)} = p_1^0 q_1^{-4\sigma}(1 + \tilde{P}_1^{(0)}(q_1, q_2^0)) \quad (55)$$

for some $\tilde{P}_1^{(0)}$ which is bounded and holomorphic in $q_1 \in S_0$ and q_2^0 in some neighborhood of the origin, where p_1^0 is an arbitrary constant.

Now we assume that $p_\nu^{(k)}$'s ($k = 0, 1, \dots, j-1, \nu = 1, 2$) are determined. We define $p_\nu^{(j)}$ by the following relations

$$p_1^{(j)} = \int_0^{q_1} \exp\left(\int_s^{q_1} \alpha(t) dt\right) \beta(s) p_2^{(j-1)}(s) ds, \quad (56)$$

$$p_2^{(j)} = \int_0^{q_1} \exp\left(\int_s^{q_1} \gamma(t) dt\right) \delta(s) p_1^{(j-1)}(s) ds. \quad (57)$$

Let $\mu > 4\sigma - 1$ be a given integer. Then we will show the following expressions for $p_\nu^{(j)}$

$$p_1^{(j)} = \begin{cases} p_1^0 q_1^{-4\sigma + (\mu - 2\sigma)k} P_1^{(j)}(q_1, q_2^0) & (j = 2k, k = 0, 1, 2, \dots) \\ p_2^0 q_1^{-2\sigma - 1 + (\mu - 2\sigma)k} P_1^{(j)}(q_1, q_2^0) & (j = 2k + 1, k = 0, 1, 2, \dots) \end{cases} \quad (58)$$

$$p_2^{(j)} = \begin{cases} p_2^0 \exp(-q_1^{-2\sigma}) q_1^{(\mu-2\sigma)k} P_2^{(j)}(q_1, q_2^0) & (j = 2k, k = 0, 1, \dots) \\ p_1^0 \exp(-q_1^{-2\sigma}) q_1^{-2\sigma+1+(\mu-2\sigma)(k+1)} P_2^{(j)}(q_1, q_2^0) & (j = 2k+1, k = 0, 1, \dots) \end{cases} \quad (59)$$

where $P_1^{(j)}$ and $P_2^{(j)}$ are bounded holomorphic functions of $q_1 \in S_0$ and q_2^0 in some neighborhood of the origin such that, for some constants $C_0 > 0$ and $C_1 > 0$ and the neighborhood of the origin U we have

$$\sup_{q_1 \in S_0, q_2^0 \in U} |P_1^{(j)}(q_1, q_2^0)| \leq C_0 C_1^j, \quad \sup_{q_1 \in S_0, q_2^0 \in U} |P_2^{(j)}(q_1, q_2^0)| \leq C_0 C_1^j, \quad j = 0, 1, \dots \quad (60)$$

We have already proved (58) and (59) for $j = 0$. Let $j = 1$. Since we have $\alpha(t) = -4\sigma t^{-1} + \alpha_0$ for some bounded holomorphic function α_0 in S_0 , we have

$$\exp\left(\int_s^{q_1} \alpha dt\right) = s^{4\sigma} q_1^{-4\sigma} (1 + O(s, q_1)),$$

where the term $O(s, q_1)$ represents a function holomorphic and bounded in S_0 . By the definition of $\beta(s)$ and (43) we have

$$\beta(s) = q_2^0 2\sigma(2\sigma - 1) \exp(s^{-2\sigma}) s^{-2\sigma-2} (1 + O(s)), \quad (61)$$

where the term $O(s)$ represents the function holomorphic and bounded in S_0 . Hence we have

$$\begin{aligned} p_1^{(1)}(q_1) &= \int_0^{q_1} s^{4\sigma} q_1^{-4\sigma} q_2^0 2\sigma(2\sigma - 1) s^{-2\sigma-2} p_2^0 (1 + O(s, q_1)) ds \\ &= q_1^{-4\sigma} p_2^0 q_2^0 2\sigma(2\sigma - 1) \int_0^{q_1} s^{2\sigma-2} (1 + O(s, q_1)) ds. \end{aligned} \quad (62)$$

The right-hand side of (62) can be written in the form $p_2^0 q_1^{-2\sigma-1} P_2^{(1)}(q_1, q_2^0)$ for some bounded holomorphic function $P_2^{(1)}(q_1, q_2^0)$ in $q_1 \in S_0$ and q_2^0 in some neighborhood of the origin. This proves (58) for $j = 1$.

In order to estimate $p_2^{(1)}$ we first note that $\gamma = 2\sigma q_1^{-2\sigma-1} (1 + O(q_1))$. It follows that

$$\exp\left(\int_s^{q_1} \gamma\right) = \exp\left(-q_1^{-2\sigma} + s^{-2\sigma} + O(q_1, s)\right). \quad (63)$$

On the other hand we have

$$\delta(s) = \frac{-4aq_2(s^{2\sigma} + aq_2^2 + 1)}{s^{4\sigma} + E} = -4aq_2 s^{-2\sigma} (1 + O(s)), \quad q \in S_0. \quad (64)$$

By (43) we have $aq_2 s^{-2\sigma-\mu} = O(s)$ in S_0 , it follows that $\delta(s) = s^\mu O(s)$, where $\mu > 4\sigma - 1$ is a given integer. In terms of the estimate of $p_1^{(0)}$ and (57) we have

$$p_2^{(1)} = p_1^0 \int_0^{q_1} \exp\left(-q_1^{-2\sigma} + s^{-2\sigma}\right) s^{\mu-4\sigma} O(q_1, s) ds = p_1^0 \exp(-q_1^{-2\sigma}) q_1^{\mu+1-4\sigma} O(q_1), \quad (65)$$

where $O(q_1)$ is a bounded holomorphic function in S_0 . This proves (58) and (59) for $j = 0, 1$.

Let us now assume that (58) and (59) hold up to some $k \geq 0$. We first show (58) for $j = 2(k+1)$. By (56) with $j = 2k+2$, (59) with $j = 2k+1$ and the same argument as in the estimate of $p_1^{(1)}$ we obtain (58) for $j = 2(k+1)$. Next we show (59) for $j = 2(k+1)$. By (57) with $j = 2(k+1)$, (58) with $j = 2k+1$ and the same argument as in the estimate of $p_2^{(1)}$ we obtain (59) with $j = 2(k+1)$. Next we show (58) with $j = 2(k+1) + 1$. By (56) with $j = 2k+3$, (59) with $j = 2(k+1)$ and the same argument as for the estimate of $p_1^{(1)}$ we obtain (58) with $j = 2k+3$. To show (59) with $j = 2k+3$ we use (57) with $j = 2k+3$, (58) with $j = 2k+2$ and the same argument as for the estimate of $p_2^{(1)}$. Hence we have proved (58) and (59). In order to show the estimate (60) we note that $p_\nu^{(j)}$ in (56) and (57) are determined by the integral of the previous ones multiplied by some bounded holomorphic functions in S_0 . Hence we can easily show the desired estimates. It follows from (58) and (59) that the series (51) converges in $q_1 \in S_0$ and q_2^0 in some neighborhood of the origin. Moreover we have the representations (44) and (45).

Now we construct the solution u of $\{\mathcal{H}, u\} = 0$ in the form (29) where q_2^0 , p_1^0 and p_2^0 are given by (43), (44) and (45). For the sake of simplicity we set $\tilde{\mathcal{L}} := \mathcal{L} + R$. We regard q_2^0 as the function of q_1 and q_2 through the relation (43) in view of the implicit function theorem. In terms of (41) and the implicit function theorem we can easily see that $\tilde{\mathcal{L}}q_2^0 = 0$.

Next we prove that $\tilde{\mathcal{L}}p_\nu^0 = 0$ ($\nu = 1, 2$). Indeed, by (44) and (45) we can determine p_ν^0 as a linear function of p_1 and p_2 , which is also holomorphic in $q_1 \in S_0$. By applying $\tilde{\mathcal{L}}$ to both sides of (44), we see that the left-hand side is equal to $(q_1^{4\sigma} + E)(\alpha p_1 + \beta p_2)$. On the other hand the right-hand side can be written in

$$q_1^{-4\sigma}((\tilde{\mathcal{L}}p_1^0)(1 + P_1) + (\tilde{\mathcal{L}}p_2^0)\tilde{P}_1) + \dots \quad (66)$$

where the dots denotes the terms which does not contain the differentiations of p_ν^0 . Because $\tilde{\mathcal{L}}q_2^0 = 0$, if the differentiations of $\tilde{\mathcal{L}}$ are applied to q_2^0 , then it vanishes. Hence, in the dotted terms one may consider the differentiations with respect to q_1 . In view of the definition of (41) one can show that the dotted term is then equal to the left-hand side. Therefore we obtain the following

$$q_1^{-4\sigma}((\tilde{\mathcal{L}}p_1^0)(1 + P_1) + (\tilde{\mathcal{L}}p_2^0)\tilde{P}_1) = 0. \quad (67)$$

We similarly obtain

$$\exp(-q_1^{-2\sigma})((\tilde{\mathcal{L}}p_1^0)P_2 + (\tilde{\mathcal{L}}p_2^0)(1 + \tilde{P}_2)) = 0. \quad (68)$$

It follows that $\tilde{\mathcal{L}}p_\nu^0 = 0$ ($\nu = 1, 2$).

By (41) we have $\tilde{\mathcal{L}}(u_j(q_1, p_1^0, p_2^0)(q_2^0)^j) = (\tilde{\mathcal{L}}u_j)(q_2^0)^j$ and

$$\tilde{\mathcal{L}}u_j = (q_1^{4\sigma} + E)\partial_1 u_j, \quad (69)$$

$$E = (q_2^0)^2 \exp(2q_1 Q_2 + 2q_1^{-2\sigma})a \left((q_2^0)^2 \exp(2q_1 Q_2 + 2q_1^{-2\sigma})a + 2q_1^{2\sigma} + 2 \right),$$

where the differentiation ∂_1 means the one with respect to the first variable by considering q_1 , p_1^0 , p_2^0 and q_2^0 as the independent variables. Hence we can write $\tilde{\mathcal{L}} = 0$ in the following form

$$(q_1^{4\sigma} + E)\partial_1 u = 0. \quad (70)$$

Because E is an exponentially small when $q_1 \rightarrow 0$ ($q_1 \in S_0$), it follows that $q_1^{4\sigma} + E$ does not vanish identically in S_0 . Hence $\tilde{\mathcal{L}}u = 0$ is equivalent to $\partial_1 u = 0$.

In order to determine u we study the transformation among two coordinates of transseries. First we set

$$\tilde{q}_2^0 = q_2 \exp(-q_1^{-2\sigma}), \quad \tilde{p}_1^0 = p_1 q_1^{4\sigma}, \quad \tilde{p}_2^0 = p_2 \exp(q_1^{-2\sigma}). \quad (71)$$

Then it follows from (43), (44) and (45) that

$$\tilde{q}_2^0 = q_2^0 \exp(q_1 Q_2), \quad (72)$$

$$\tilde{p}_1^0 = p_1^0 (1 + P_1) + p_2^0 \tilde{P}_1, \quad (73)$$

$$\tilde{p}_2^0 = p_1^0 P_2 + p_2^0 (1 + \tilde{P}_2). \quad (74)$$

By (72) and the implicit function theorem we can express \tilde{q}_2^0 as a holomorphic function of q_2^0 in some neighborhood of the origin, $q_2^0 = 0$ when $q_1 \in S_0$. Conversely, q_2^0 can be expressed as the holomorphic function of \tilde{q}_2^0 in some neighborhood of the origin, $\tilde{q}_2^0 = 0$ when $q_1 \in S_0$. We also recall that $Q_2, P_\nu, \tilde{P}_\nu$ are holomorphic functions of q_2^0 in some neighborhood of the origin, $q_2^0 = 0$.

We replace p_1^0, p_2^0 and q_2^0 in (29) with $\tilde{p}_1^0, \tilde{p}_2^0$ and \tilde{q}_2^0 , respectively and we rewrite (29) in the following form

$$u(q_1, \tilde{q}_2^0, \tilde{p}_1^0, \tilde{p}_2^0) = \sum_{j=0}^{\infty} \tilde{u}_j(q_1, \tilde{p}_1^0, \tilde{p}_2^0) (\tilde{q}_2^0)^j, \quad (75)$$

where \tilde{u}_0 is independent of q_1 and analytic in some neighborhood of the origin, $\tilde{p}_1^0 = 0, \tilde{p}_2^0 = 0$. We recall that $\tilde{u}_j(0, \tilde{p}_1^0, \tilde{p}_2^0) \equiv 0$ for every $j \geq 1$. We substitute the relations (72), (73) and (74) into (29). Then we see that the power series (29) is expressed as the power series of q_2^0, p_2^0 and p_1^0

$$u(q_1, \tilde{q}_2^0, \tilde{p}_1^0, \tilde{p}_2^0) = u_0(p_1^0, p_2^0) + u_1(p_1^0, p_2^0) q_2^0 + u_2(p_1^0, p_2^0) (q_2^0)^2 + \dots \quad (76)$$

where we used the relations $\partial_1 u_j = \partial_{q_1} u_j = 0$ which follows from the fact that $\partial_1 u = 0$.

We first determine u_0 . In view of the constructions of P_ν and \tilde{P}_ν ($\nu = 1, 2$) we can easily see that these functions vanish when $q_2^0 = 0$. Moreover, in view of the construction we can easily see that if we expand P_ν and \tilde{P}_ν ($\nu = 1, 2$) into the power series of q_2^0 the coefficients vanish when $q_1 \rightarrow 0$. We substitute the relations (72) - (74) into (75) and we look for terms in $\tilde{u}_0(\tilde{p}_1^0, \tilde{p}_2^0)$ which does not contain q_2^0 . We can easily see that the term is given by $\tilde{u}_0(p_1^0, p_2^0)$. Hence we have $u_0(p_1^0, p_2^0) = \tilde{u}_0(p_1^0, p_2^0)$. This shows that $u_0(p_1^0, p_2^0)$ is analytic at $p_1^0 = 0, p_2^0 = 0$. Next we will show that $u_1(p_1^0, p_2^0) = 0$. For this purpose we calculate the limit of the coefficients of q_2^0 in (75) when $q_1 \rightarrow 0$, where we substitute (72) - (74) into (75). In view of the definitions of P_ν and \tilde{P}_ν ($\nu = 1, 2$) we can easily see that $\lim_{q_1 \rightarrow 0} P_\nu = 0$ and $\lim_{q_1 \rightarrow 0} \tilde{P}_\nu = 0$. Hence we have $\lim_{q_1 \rightarrow 0} \tilde{p}_1^0 = p_1^0$ and $\lim_{q_1 \rightarrow 0} \tilde{p}_2^0 = p_2^0$. Therefore, if we neglect terms which vanish when $q_1 \rightarrow 0$, then we have

$$u_1(p_1^0, p_2^0) = \lim_{q_1 \rightarrow 0} \tilde{u}_1(q_1, \tilde{p}_1^0, \tilde{p}_2^0) = \tilde{u}_1(0, p_1^0, p_2^0) \equiv 0. \quad (77)$$

Similarly we can show that $u_j(p_1^0, p_2^0) \equiv 0$ for all $j \geq 1$. Therefore the right-hand side of (76) converges. If we return to the original variables the series (29) converges. This ends the proof of the theorem.

Remark. The proof of Theorem 2 also show that if we are given an analytic solution $u = u_0(p_1^0, p_2^0)$, then we obtain the solution (29). Indeed, by (72) - (74) we rewrite $u_0(p_1^0, p_2^0)$ as an analytic function of q_1, \tilde{p}_1^0 and \tilde{p}_2^0 in the following form

$$\tilde{u}_0(q_1, \tilde{p}_1^0, \tilde{p}_2^0) + \tilde{u}_1(q_1, \tilde{p}_1^0, \tilde{p}_2^0)q_2^0 + \cdots \quad (78)$$

If we consider terms which does not contain \tilde{q}_2^0 , then we obtain that $\tilde{u}_0(q_1, \tilde{p}_1^0, \tilde{p}_2^0) = u_0(\tilde{p}_1^0, \tilde{p}_2^0)$. Similarly we can show that $\tilde{u}_j|_{q_1=0} = 0$ for all $j \geq 1$. Hence we obtain the solution (29).

Remark. So far we studied the solvability of $\mathcal{L}u + Ru = 0$ in a neighborhood of the point $(q_1, q_2, p_1, p_2) = (0, 0, 0, 0)$. We will briefly mention the solvability of $\mathcal{L}u + Ru = 0$ at other points. As to the solvability at the points (q_1, q_2, p_1, p_2) such that $q_1 \neq 0, q_2 = 0$, we see that the term $q_1^{4\sigma} \partial / \partial q_1$ appears in $\mathcal{L}u + Ru = 0$, which implies that $\mathcal{L}u + Ru = 0$ is a nonsingular vector field. Hence, by the standard existence theorem for the noncharacteristic Cauchy problem, $\mathcal{L}u + Ru = 0$ is solvable under a suitable initial condition on u .

By a similar argument, we see that at the point such that $q_2 \neq 0, q_1 = 0, p_1 \neq 0$, either the coefficient of $\partial u / \partial p_2$ or that of $\partial u / \partial q_1$ does not vanish because $a(0) \neq 0$. Hence the vector field $\mathcal{L} + R$ is nonsingular. At the point such that $q_2 \neq 0, q_1 = 0, p_1 = 0$ we easily see that if $2 + aq_2^2 \neq 0$, then the coefficient of $\partial u / \partial q_1$ does not vanish. Hence $\mathcal{L} + R$ is nonsingular.

Finally, in a neighborhood of the point (q_1, q_2, p_1, p_2) such that $q_1 = 0, q_2 = \infty$, $\mathcal{L}u + R$ is nonsingular because $a^2 \partial / \partial q_1$ does not vanish. In every case we can construct a solution by solving a noncharacteristic Cauchy problem. Summing up the above, if the condition $(q_1, q_2) \neq (0, 0)$ holds, then $\mathcal{L}u + Ru = 0$ is solvable under an appropriate initial condition except for some special cases.

6 Asymptotic properties of transseries

In Theorem 2 we proved the solvability of $\mathcal{L}u + Ru = 0$ by using a transseries solution near $q_2 = 0$ in some narrow region near the origin. We want to give a meaning to formal transseries solutions constructed in §4 by Borel-Laplace resummation method in a larger region. More precisely we will prove

Theorem 3 *Suppose that $a \equiv 1$ in the equation $(\mathcal{L} + R)v = 0$. Moreover, assume that $v = \sum_{j=0}^{\infty} u_j(q_1, p_1^0, p_2^0)(q_2^0)^j$ be the solution of the equation $(\mathcal{L} + R)v = 0$ constructed in §4 for a polynomial $u_0 \equiv u_0(p_1^0, p_2^0)$ such that $\partial u_0 / \partial p_2^0 \neq 0$. Then, for every $T_0 > 0$ there exist $\nu_0 > 0, \varepsilon_0 > 0$ and $\varepsilon_1 > 0$ such that, for every $\nu \geq \nu_0$ the equation $(\mathcal{L} + R)u = 0$ has a solution*

$$u = \sum_{j=0}^{\nu} u_j(q_1, p_1^0, p_2^0)(q_2^0)^j + U(q_1, p_1, p_2, q_2) \quad (79)$$

such that $U(q_1, p_1, p_2, q_2)$ is holomorphic in the domain

$$\{(q_1, q_2, p_1, p_2); |\arg q_1^{2\sigma} - \pi| < \pi/2 + \varepsilon_0, |q_2| < \varepsilon_1 |q_1^{2\sigma}|, |p_1| < T_0 |q_1^{4\sigma} \exp(-q_1^{-2\sigma})|, |p_2| < T_0 |q_1^{4\sigma} \exp(-q_1^{-2\sigma})|\}. \quad (80)$$

In order to prove Theorem 3 we prepare some notation. In terms of the condition $a \equiv 1$ R is given by (35) with

$$E \equiv E(q_1, q_2) = q_2^2(q_2^2 + 2q_1^{2\sigma} + 2), \quad \tilde{\alpha} = -4\sigma q_1^{2\sigma-1} q_2^2, \quad (81)$$

$$\tilde{\beta} = 2\sigma(2\sigma - 1)q_2 q_1^{2\sigma-2}, \quad \tilde{\gamma} = 0, \quad \tilde{\delta} = -4q_2(q_1^{2\sigma} + q_2^2 + 1). \quad (82)$$

Now we make the change of variables

$$t = q_1^{2\sigma}, \quad st = q_2. \quad (83)$$

The Jacobian is nonsingular if $q_1 \neq 0$. We can easily see that

$$q_1 \frac{\partial}{\partial q_1} = 2\sigma \left(t \frac{\partial}{\partial t} - s \frac{\partial}{\partial s} \right), \quad q_2 \frac{\partial}{\partial q_2} = s \frac{\partial}{\partial s}.$$

Therefore we have

$$q_1^{-2\sigma+1} \mathcal{L} = 2\sigma \left(p_2 \frac{\partial}{\partial p_2} - s \frac{\partial}{\partial s} \right) + 2\sigma t \left(t \frac{\partial}{\partial t} - s \frac{\partial}{\partial s} - 2p_1 \frac{\partial}{\partial p_1} \right). \quad (84)$$

In order to calculate $q_1^{-2\sigma+1} R$ we note

$$q_1^{-2\sigma+1} E \frac{\partial}{\partial q_1} = 2\sigma s^2 t (s^2 t^2 + 2t + 2) \left(t \frac{\partial}{\partial t} - s \frac{\partial}{\partial s} \right). \quad (85)$$

Similarly we have $q_1^{-2\sigma+1} \tilde{\alpha} = -4\sigma s^2 t^2$, $q_1^{-2\sigma+1} \tilde{\beta} = 2\sigma(2\sigma - 1)st^{1-1/(2\sigma)}$ and

$$q_1^{-2\sigma+1} \tilde{\delta} = -4s(t^{1+1/(2\sigma)} + s^2 t^{2+1/(2\sigma)} + t^{1/(2\sigma)}). \quad (86)$$

Hence we have

$$\begin{aligned} q_1^{-2\sigma+1} R &= \left(-4p_1 \sigma s^2 t^2 + p_2 2\sigma(2\sigma - 1)st^{1-1/(2\sigma)} \right) \frac{\partial}{\partial p_1} \\ &- 4p_1 s \left(t^{1+1/(2\sigma)} + s^2 t^{2+1/(2\sigma)} + t^{1/(2\sigma)} \right) \frac{\partial}{\partial p_2} \\ &+ 2\sigma s^2 t \left(s^2 t^2 + 2t + 2 \right) \left(t \frac{\partial}{\partial t} - s \frac{\partial}{\partial s} \right). \end{aligned} \quad (87)$$

Therefore, by (84) and (87) the equation $q_1^{-2\sigma+1} (\mathcal{L}v + Rv) = 0$ is written in the following form

$$\begin{aligned} &2\sigma \left(p_2 \frac{\partial v}{\partial p_2} - s \frac{\partial v}{\partial s} \right) + 2\sigma t \left(t \frac{\partial v}{\partial t} - s \frac{\partial v}{\partial s} - 2p_1 \frac{\partial v}{\partial p_1} \right) \\ &+ \left(-4p_1 \sigma s^2 t^2 + 2\sigma(2\sigma - 1)p_2 t^{1-1/(2\sigma)} s \right) \frac{\partial v}{\partial p_1} \\ &- 4p_1 t^{1/(2\sigma)} (t + s^2 t^2 + 1) s \frac{\partial v}{\partial p_2} \\ &+ 2\sigma s^2 t \left(s^2 t^2 + 2t + 2 \right) \left(t \frac{\partial v}{\partial t} - s \frac{\partial v}{\partial s} \right) = 0. \end{aligned} \quad (88)$$

For the sake of simplicity we write the right-hand side operators of (84) and (87) by \mathcal{L} and R , respectively. Then (88) can be written in $\mathcal{L}v + Rv = 0$.

Now we will construct a transseries solution v of (88) by the same method as in §4. Indeed, we replace the variables q_1 and q_2 with t and s , respectively. The Lagrange-Charpit system corresponding to \mathcal{L} in the new coordinates is given by

$$\frac{dt}{t^2} = \frac{ds}{-s(1+t)} = \frac{dp_1}{-2tp_1} = \frac{dp_2}{p_2}. \quad (89)$$

Hence the transseries solution is given by

$$v = \sum_{j=0}^{\infty} v_j(t, p_1 t^2, p_2 e^{1/t}) (ste^{-1/t})^j, \quad (90)$$

where $v_0(t, p_1^0, p_2^0)$, $(p_1^0 = p_1 t^2, p_2^0 = p_2 e^{1/t})$ is a given polynomial of p_1^0 and p_2^0 with degree $N_0 \geq 1$ which is independent of t and s . We note that $\mathcal{L}v_0 = 0$. The v_j 's are given by the formula like (33) which is obtained by inserting (90) into the equation $\mathcal{L}v + Rv = 0$ and by equating the coefficients of the powers of s . In order to determine v_j , ($j \geq 1$) we impose the initial conditions on v_j

$$v_j(a_0, p_1^0, p_2^0) \equiv 0, \quad j = 1, 2, \dots \quad (91)$$

For $\nu \geq 1$ we define

$$w = \sum_{j=0}^{\nu} v_j(t, p_1 t^2, p_2 e^{1/t}) (ste^{-1/t})^j, \quad (92)$$

and

$$f_\nu(t, s, p_1, p_2) := \mathcal{L}w + Rw. \quad (93)$$

By definition $f_\nu(t, s, p_1, p_2)$ is a polynomial of s , p_1 and p_2 and is holomorphic in t in $\mathbb{C} \setminus [0, \infty)$ and satisfies $f_\nu(t, s, p_1, p_2) = O(s^{\nu+1})$, as $s \rightarrow 0$.

We define the path $\gamma(\pi)$ as follows; Let $0 < \theta < \pi/2$ be a number and $\rho_0 > 0$ be a small number. We start from the origin and go to the point $\rho_0 e^{i\theta}$ along the straight line. Then we go to the point $\rho_0 e^{2\pi i - i\theta}$ along the circle with center at the origin and radius ρ_0 counterclockwise. Finally we return to the origin along the straight line. Let $K > 0$ be a given constant, and define

$$\Sigma := \{(t, p_1, p_2); |p_1| < K|t^2 e^{-1/t}|, |p_2| < K|t^2 e^{-1/t}|, \theta \leq \arg t \leq 2\pi - \theta\}. \quad (94)$$

Then we have

Lemma 1 *There exists $K > 0$ such that the restriction of f_ν to Σ , $f_\nu(t, s, p_1, p_2)|_\Sigma$ is a holomorphic function of t in $\mathbb{C} \setminus [0, \infty)$, and it is continuous in t on the path $\gamma(\pi)$ such that $\lim_{t \rightarrow 0, t \in \gamma(\pi)} t^{-2} f_\nu(t, s, p_1, p_2)|_\Sigma = 0$.*

Proof We note that \mathcal{L} preserves the order of s . It follows that $\mathcal{L}w$ cancels with the corresponding terms in Rw . Because Rw is the polynomial of s with degree $\nu + 4$, it is sufficient to show that restrictions to Σ of terms of Rw are continuous in t on the path $\gamma(\pi)$, and it has the desired estimate. In view of the definition of R in (87) we have

$$Rw = \sum_j (Rv_j)(ste^{-1/t})^j + 2\sigma s^2 (s^2 t^2 + 2t + 2) \sum_j jv_j (ste^{-1/t})^j. \quad (95)$$

By definition $v_0 = v_0(p_1 t^2, p_2 e^{1/t})$ is a polynomial of p_1^0 and p_2^0 . Hence v_0 is holomorphic in t in the domain $\mathbb{C} \setminus \{0\}$. If $(t, p_1, p_2) \in \Sigma$, then we easily see that $\lim_{t \rightarrow 0, t \in \gamma(\pi)} v_0|_{\Sigma} = 0$. Let $g_1(t)$ be the terms with degree 1 in s of Rw . Because Rw is divisible by s , $g_1(t)$ consists of terms with degree 1 in s of Rv_0 . It follows that

$$g_1(t) = 2\sigma(2\sigma - 1)t^{1-1/(2\sigma)} s p_2 \frac{\partial v_0}{\partial p_1} - 4t^{1/(2\sigma)}(t+1) s p_1 \frac{\partial v_0}{\partial p_2}. \quad (96)$$

We can easily see that $g_1(t)$ is holomorphic in $\mathbb{C} \setminus [0, \infty)$. We want to show that there exists an $\varepsilon > 0$ such that

$$\lim_{t \rightarrow 0, t \in \gamma(\pi)} t^{-2-\varepsilon} g_1(t)|_{\Sigma} = 0. \quad (97)$$

Indeed, we recall that v_0 is a function of $p_1^0 = p_1 t^2$ and $p_2^0 = p_2 e^{1/t}$. By (94) we have that for every $i_k, j_k = 1, 2$ ($k = 1, 2, \dots, N_0$) with N_0 being the degree of v_0 the term

$$t^{-2} \left(\prod_{k=1}^{N_0} p_{i_k} \frac{\partial}{\partial p_{j_k}} \right) v_0|_{\Sigma} \quad (98)$$

is bounded when $t \rightarrow 0, t \in \gamma(\pi)$. Hence (97) follows from (96) and (98).

We compare the powers of s in (88) with degree 1. By (89) and the construction of a transseries solution v_1 satisfies

$$2\sigma t^2 \partial_t v_1(t, p_1^0, p_2^0) = t^{-1} e^{1/t} g_1(t), \quad (99)$$

where $p_1^0 = p_1 t^2$ and $p_2^0 = p_2 e^{1/t}$. By integrating (99) we obtain

$$v_1 = v_1(t, p_1^0, p_2^0) = \frac{1}{2\sigma} \int_{a_0}^t z^{-3} e^{1/z} g_1(z) dz, \quad (100)$$

where $a_0 \neq 0$. In case t is close to the origin, then we first integrate from a_0 to some $a_1 \neq 0$ on the straight line which connects the origin and t , then we integrate from a_1 to t along the straight line.

Next we will show that

$$\lim_{t \rightarrow 0, t \in \gamma(\pi)} e^{-1/t} v_1|_{\Sigma} = 0. \quad (101)$$

Because $e^{-1/t}$ tends to zero when $t \rightarrow 0, t \in \gamma(\pi)$, the integral from a_0 to a_1 in the integral (100) has no contribution when we calculate the limit of $v_1|_{\Sigma}$ as $t \rightarrow 0, t \in \gamma(\pi)$. Hence we consider

$$e^{-1/t} \int_{a_1}^t z^{-3} e^{1/z} g_1(z) dz = \int_{a_1}^{(1+\delta)t} z^{-3} e^{1/z-1/t} g_1(z) dz + \int_{(1+\delta)t}^t z^{-3} e^{1/z-1/t} g_1(z) dz, \quad (102)$$

where $\delta > 0$ is a small positive constant.

We consider the first term in the right-hand side of (102). By setting $z = r e^{i\mu}$, $t = \rho e^{i\mu}$, ($\rho < r$) we have

$$\Re\left(\frac{1}{z} - \frac{1}{t}\right) = -\left(\frac{1}{\rho} - \frac{1}{r}\right) \cos \mu \leq -\left(1 - \frac{1}{1+\delta}\right) \frac{1}{\rho} \cos \mu = -\frac{\delta}{1+\delta} \frac{1}{\rho} \cos \mu,$$

where $\cos \mu > 0$. It follows that we have $|e^{1/z-1/t}z^{-3}| \leq K_1|t|$ for some $K_1 > 0$ independent of t when $t \rightarrow 0, t \in \gamma(\pi)$. Hence we have

$$\left| \int_{a_1}^{(1+\delta)t} z^{-3} e^{1/z-1/t} g_1(z) dz \right| \leq \left| \int_{|a_1|}^{(1+\delta)\rho} K_1 |t| |g_1(z)| |dz| \right|. \quad (103)$$

We restrict p_1 and p_2 to Σ . We note that $|p_1 z^2| \leq |p_1| |a_1|^2 < K|t^2 e^{-1/t}| |a_1|^2$ tends to 0. Similarly, $|p_2 e^{1/z}| < K|t^2 \exp(1/z - 1/t)| \leq K|t|^2$ tends to 0. Hence the restriction to Σ of the right-hand side tends to zero.

Next we will estimate the second term of the right-hand side of (102). Noting that $|e^{1/z-1/t}| \leq 1$ we will consider the integral

$$\begin{aligned} & \left| \int_{(1+\delta)\rho}^{\rho} |z^{-3} g_1(z)| |dz| \right| = \left| \int_{(1+\delta)\rho}^{\rho} |z^{-1+\varepsilon}| |z^{-2-\varepsilon} g_1(z)| |dz| \right| \\ & \leq \max_{|t| \leq |z| \leq (1+\delta)|t|} |z^{-2-\varepsilon} g_1(z)| \left| \int_{(1+\delta)\rho}^{\rho} |z^{-1+\varepsilon}| |dz| \right|, \end{aligned} \quad (104)$$

where $\varepsilon > 0$. By (94) we see that the restriction of the right-hand side of (104) to Σ tends to zero. This proves (101).

Let $g_2(t) \equiv g_2(t, p_1^0, p_2^0)$ be the terms with degree 2 in s of Rw . Clearly, g_2 appears from $R(v_0 + v_1 s t e^{-1/t})$. We define

$$v_2 = \frac{1}{2\sigma} \int_{a_0}^t z^{-4} e^{2/z} g_2(z) dz. \quad (105)$$

Then we will show that

$$\lim_{t \rightarrow 0, t \in \gamma(\pi)} t^2 e^{-2/t} v_2|_{\Sigma} = 0. \quad (106)$$

The terms with degree 2 appearing from Rv_0 is given by $-4p_1 \sigma s^2 t^2 \partial_{p_1} v_0 + 4\sigma s^2 (t+1)t^2 \partial_t v_0$. We consider the second term. We can easily see that the term with the largest growth in v_0 is given by $p_2 e^{1/t}$. Hence, in order to show (106) we may study the integral $p_2 \int_{a_1}^t z^{-4} e^{3/z} dz$. Because $|p_2| \leq |t^2 e^{-1/t}|$ on Σ , the restriction of the integral to Σ satisfies (106).

We now consider remaining terms \tilde{g}_2 in g_2 . We will show that there exists an $\varepsilon > 0$ such that

$$\lim_{t \rightarrow 0, t \in \gamma(\pi)} t^{-2-\varepsilon} \tilde{g}_2(t)|_{\Sigma} = 0. \quad (107)$$

Clearly, if we can show this, then we have (107) by the same argument as for v_1 . By (98) the term $-4p_1 \sigma s^2 t^2 \partial_{p_1} v_0$ satisfies the estimate like (107). Next we consider terms with degree 2 in s appearing from $R(v_1 s t e^{-1/t})$

$$\left(2\sigma(2\sigma-1)st^{1-1/2\sigma} p_2 \partial_{p_1} v_1 - 4st^{1/2\sigma} (t+1)p_1 \partial_{p_2} v_1 \right) (s t e^{-1/t}). \quad (108)$$

In view of (96) and (100) we see that differentiations in (108) are estimated by

$$(p_{i_1} \partial_{p_{j_1}})(p_{i_2} \partial_{p_{j_2}})v_0|_{\Sigma}, \quad i_1, i_2, j_1, j_2 = 1, 2. \quad (109)$$

By (98) we see that t^{-2} times of the terms in (109) are bounded. Hence we have (107).

We will show that

$$\lim_{t \rightarrow 0, t \in \gamma(\pi)} t^3 e^{-3/t} v_3|_{\Sigma} = 0, \quad (110)$$

where

$$v_3 = \frac{1}{2\sigma} \int_{a_0}^t z^{-5} e^{3/z} g_3(z) dz, \quad (111)$$

where $g_3(t)$ consists of terms with degree 3 in s of Rw . Clearly, $g_3(t)$ is equal to terms with degree 3 in s of $R(v_0 + (tse^{-1/t})v_1 + (tse^{-1/t})^2v_2)$.

First we note that if the terms in g_3 satisfies an estimate like $\tilde{g}_2|_{\Sigma}$ in (107), then the similar argument as for v_1 or v_2 shows that (110) holds. The terms with s^3 in Rv_0 is given by $-4p_1 s^3 t^{2+1/(2\sigma)} (\partial v_0 / \partial p_2)$. The restriction of the term to Σ has the similar estimate like $g_2(t)|_{\Sigma}$ in (107). Next we will consider $R((tse^{-1/t})v_1)$. The terms with s^3 appear from $(Rv_1)(tse^{-1/t})$ and $-4\sigma s^3(t+1)te^{-1/t}v_1$. As to $(Rv_1)(tse^{-1/t})$ we need to estimate

$$-4\sigma s^3 t^3 e^{-1/t} p_1 \partial_{p_1} v_1 + 4\sigma s^3 (t+1) e^{-1/t} t^3 \partial_t v_1.$$

The first term can be estimated by a similar method as in the above. Hence we consider the second term. Because $2\sigma t^2 \partial_t v_1 = t^{-1} e^{1/t} g_1(t)$, we see that the second term is equal to $2s^3(t+1)g_1(t)$. The term has a similar estimate like (107).

Next we consider $-4\sigma s^3(t+1)te^{-1/t}v_1$. In view of (111) we consider the integral

$$\begin{aligned} \int_{a_0}^t z^{-4} e^{2/z} v_1(z) dz &= \int_{a_0}^t \left(-z^{-3} e^{2/z} - \frac{1}{2} (z^{-2} e^{2/z})' \right) v_1(z) dz \\ &= -\frac{1}{2} \int_{a_0}^t (z^{-2} e^{2/z})' v_1(z) dz - \int_{a_0}^t z^{-3} e^{2/z} v_1(z) dz. \end{aligned} \quad (112)$$

In order to estimate the first term of the right-hand side we consider

$$z^{-2} e^{2/z} v_1(z) \Big|_{a_0}^t - \int_{a_0}^t z^{-2} e^{2/z} v_1'(z) dz. \quad (113)$$

By (101) $t^3 e^{-3/t}$ times the first term of the right-hand side of (113) restricted to Σ tends to 0 when $t \rightarrow 0, t \in \gamma(\pi)$. In order to estimate the second term we consider, by $2\sigma v_1'(z) = z^{-3} e^{1/z} g_1(z)$,

$$\frac{1}{2\sigma} \int_{a_0}^t z^{-5} e^{3/z} g_1(z) dz. \quad (114)$$

Multiplying the term with $t^3 e^{-3/t}$ leads to

$$\frac{1}{2\sigma} \int_{a_0}^t (t/z)^3 z^{-2} e^{3/z-3/t} g_1(z) dz. \quad (115)$$

Because one may assume $|t/z| \leq 1$, we are lead to the same estimate like (102). Hence $t^3 e^{-3/t}$ times the first term of the right-hand side of (112) restricted to Σ tends to 0 when $t \rightarrow 0, t \in \gamma(\pi)$. We note that in the second term of the right-hand side of (112) the negative power in the integrand is improved compared to the left-hand side term by partial integration. By (101) $t^3 e^{-3/t}$ times of the restriction to Σ of the second term

of the right-hand side of (112) tends to zero because $|z/t| \leq 1$, $|\exp(3/z - 3/t)| \leq 1$. This completes the estimate of $R((tse^{-1/t})v_1)$.

Next we will consider s^3 terms in $R((tse^{-1/t})^2v_2)$. Clearly we may consider $(tse^{-1/t})^2Rv_2$. The terms with degree 3 in s are given by

$$2\sigma(2\sigma - 1)e^{-2/t}t^{3-1/(2\sigma)}s^3p_2\frac{\partial v_2}{\partial p_1} - 4t^{2+1/(2\sigma)}e^{-2/t}(t+1)s^3p_1\frac{\partial v_2}{\partial p_2}. \quad (116)$$

Inserting the formula of v_2 we are lead to the estimate of v_0 . It is estimated by (98). Hence we see that $R((tse^{-1/t})^2v_2)$ have the estimate like (107). This proves (110).

We proceed by induction and show that for $\nu = 2, 3, \dots$,

$$\lim_{t \rightarrow 0, t \in \gamma(\pi)} t^\nu e^{-\nu/t} v_\nu|_\Sigma = 0. \quad (117)$$

Indeed, the estimates of differentiations with respect to p_j 's are reduced to (98). The differentiations with respect to t are estimated via partial integration and the argument as for v_1, v_2 and v_3 . Moreover, if we denote by $g_{\nu+1}$ the terms with degree ν in s in $\mathcal{L}w + Rv$, $w = \sum_{j=0}^\nu v_j(ste^{-1/t})^j$, then

$$\lim_{t \rightarrow 0, t \in \gamma(\pi)} t^{-2-\varepsilon} g_{\nu+1}(t)|_\Sigma = 0. \quad (118)$$

As to the terms with degree $> \nu + 1$ in s of $\mathcal{L}w + Rv$, $w = \sum_{j=0}^\nu v_j(ste^{-1/t})^j$ we can estimate the terms by the same argument as for the estimate of g_j 's. This proves the desired estimate in Lemma 1. This ends the proof.

We use the partial Borel transform with respect to t

$$g(\zeta) \equiv (\mathcal{B}f)(\zeta) = \frac{1}{2\pi i} \int_{\gamma(\pi)} t f(t) \exp(\zeta/t) dt^{-1}, \quad (119)$$

where ζ is the dual variable of t with respect to the Borel transform.

We assume that $f(t)$ in (119) is analytic in the sectorial domain bounded by $\gamma(\pi)$ and continuous up to the boundary. The inverse Borel transform of $g(\zeta)$ is given by the Laplace integral

$$(\mathcal{B}^{-1}g)(t) = t^{-1} \int_0^{\infty(\tau)} g(\zeta) \exp(-\zeta/t) d\zeta (= f(t)). \quad (120)$$

The integration in (120) is taken along the ray which starts from the origin and goes to infinity in the direction τ which is sufficiently close to π .

We introduce a function space. Let $\varepsilon_0 > 0$ be a given small constant and N_0 be a positive integer. We define

$$\Gamma_0 := \{\zeta \in \mathbb{C}; |\zeta| < \varepsilon_0 \text{ or } |\arg \zeta - \pi| < \varepsilon_0\}. \quad (121)$$

Let Ω_0 and Ω be defined by

$$\Omega_0 = \left\{ (\zeta, p_1, p_2) \in \mathbb{C}^3; \zeta \in \Gamma_0, |p_1| < \varepsilon_0, |p_2| < \varepsilon_0 \right\}, \quad (122)$$

$$\Omega := \{s \in \mathbb{C}; |s| < \varepsilon_0\} \times \Omega_0. \quad (123)$$

We denote by $H(\Omega)$ the set of functions w which are polynomials of p_1 and p_2 with degree at most N_0 , holomorphic in Ω and continuous up to the boundary such that there exist constants $c > 0$ and $K > 0$ for which

$$|w(s, \zeta, p_1, p_2)| \leq K |\exp(c|\zeta|)|, \quad \forall (s, \zeta, p_1, p_2) \in \Omega. \quad (124)$$

We define the norm $\|w\|$ of w as the infimum of K satisfying (124). The space $H(\Omega)$ is a Banach space with the norm $\|\cdot\|$.

We recall the following formula

$$\mathcal{B}(t^2 \partial_t f)(\zeta) = \zeta(\mathcal{B}f)(\zeta) - \mathcal{B}(tf)(\zeta). \quad (125)$$

The formula is proved by the direct computations of the Borel transform.

Let λ be a positive number. We denote by $\alpha_\lambda \equiv \alpha_\lambda(\zeta, D)$ the Borel transform of the multiplication operator $f(t) \mapsto t^\lambda f(t)$. Namely, α_λ is given by $\mathcal{B}(t^\lambda f) = \alpha_\lambda \mathcal{B}(f)$.

Lemma 2 *There exists a constant $K_0 > 0$ such that the Borel transform of the operator $f \mapsto t^\lambda f$ ($\lambda > 0$) has the estimate*

$$\|\mathcal{B}(t^\lambda f)\| \leq K_0 \|\mathcal{B}(f)\|, \quad (126)$$

for every f holomorphic in $\gamma(\pi)$ and continuous up to the boundary such that $\mathcal{B}(f) \in H(\Omega)$. Moreover, the constant $K_0 > 0$ can be chosen arbitrarily small if we take c in (124) sufficiently large.

Proof For the sake of simplicity we write $H := H(\Omega)$. We recall that the Borel transform of t^λ is equal to $\zeta^\lambda / \Gamma(1 + \lambda)$. Since $\hat{f} := \mathcal{B}(f) \in H$ we have the formula

$$\mathcal{B}(t^\lambda f)(\zeta) = \frac{1}{\Gamma(1 + \lambda)} \frac{d}{d\zeta} \int_0^\zeta (\zeta - \eta)^\lambda \hat{f}(\eta) d\eta = \frac{1}{\Gamma(\lambda)} \int_0^\zeta (\zeta - \eta)^{\lambda-1} \hat{f}(\eta) d\eta, \quad (127)$$

where the integral is taken along the straight line in the sector Γ_0 which connects the origin and $\zeta \in \Gamma_0$.

We first consider the case $0 < \lambda < 1$. Let $\theta = \arg \zeta$ and $\varepsilon > 0$ be a small number chosen later. Then the $\Gamma(\lambda)$ -times of the right-hand side of (127) can be written in the following form

$$\int_0^{\zeta - \varepsilon e^{i\theta}} (\zeta - \eta)^{\lambda-1} \hat{f}(\eta) d\eta + \int_{\zeta - \varepsilon e^{i\theta}}^\zeta (\zeta - \eta)^{\lambda-1} \hat{f}(\eta) d\eta \equiv I_1 + I_2. \quad (128)$$

We estimate I_2 . By the definition of the integral we have

$$\begin{aligned} |I_2| &\leq \int_{|\zeta|-\varepsilon}^{|\zeta|} |\zeta - \eta|^{\lambda-1} |\hat{f}(\eta)| d|\eta| \leq \|\hat{f}\| \int_{|\zeta|-\varepsilon}^{|\zeta|} (|\zeta| - |\eta|)^{\lambda-1} e^{c|\eta|} d|\eta| \\ &\leq \|\hat{f}\| e^{c|\zeta|} \int_{|\zeta|-\varepsilon}^{|\zeta|} (|\zeta| - |\eta|)^{\lambda-1} d|\eta| = \|\hat{f}\| e^{c|\zeta|} \int_0^\varepsilon s^{\lambda-1} ds = \|\hat{f}\| e^{c|\zeta|} \varepsilon^\lambda / \lambda. \end{aligned} \quad (129)$$

As to I_1 we can similarly estimate

$$|I_1| \leq \int_0^{|\zeta|-\varepsilon} |\zeta - \eta|^{\lambda-1} |\hat{f}(\eta)| d|\eta| \leq \|\hat{f}\| \int_0^{|\zeta|-\varepsilon} (|\zeta| - |\eta|)^{\lambda-1} e^{c|\eta|} d|\eta|. \quad (130)$$

Because $|\zeta| - |\eta| \geq \varepsilon$ and $0 < \lambda < 1$, we have $(|\zeta| - |\eta|)^{\lambda-1} \leq \varepsilon^{\lambda-1}$. It follows that the right-hand side of (130) can be estimated by

$$\|\hat{f}\| \varepsilon^{\lambda-1} \int_0^{|\zeta|-\varepsilon} e^{c|\eta|} d|\eta| \leq \frac{\|\hat{f}\| \varepsilon^\lambda}{c\varepsilon} e^{c|\zeta| - c\varepsilon}. \quad (131)$$

Therefore we have

$$|\mathcal{B}(t^\lambda f)(\zeta)| \leq \|\hat{f}\| e^{c|\zeta|} \varepsilon^\lambda (\lambda^{-1} + (c\varepsilon e^{c\varepsilon})^{-1}) / \Gamma(\lambda). \quad (132)$$

We take $\varepsilon > 0$ sufficiently small, then $c > 0$ sufficiently large. We can make the constant $K_0 = \varepsilon^\lambda (\lambda^{-1} + (c\varepsilon e^{c\varepsilon})^{-1}) / \Gamma(\lambda)$ arbitrarily small. This proves (126).

Next we consider the case $\lambda \geq 1$. We see that $\Gamma(\lambda) \mathcal{B}(t^\lambda f)(\zeta)$ is given by the integral in the right-hand side of (127). By similar calculations as in the case $0 < \lambda < 1$, we see that it is estimated by

$$\|\hat{f}\| \int_0^{|\zeta|} (|\zeta| - |\eta|)^{\lambda-1} e^{c|\eta|} d|\eta|. \quad (133)$$

If $\lambda = 1$, then the integral (133) is bounded by $\|\hat{f}\| e^{c|\zeta|} / c$. In the case $\lambda > 1$, by the partial integration it is equal to

$$\begin{aligned} & \|\hat{f}\| \left(\left[\frac{e^{cs}}{c} (|\zeta| - s)^{\lambda-1} \right]_{s=0}^{s=|\zeta|} + \frac{\lambda-1}{c} \int_0^{|\zeta|} (|\zeta| - |\eta|)^{\lambda-2} e^{c|\eta|} d|\eta| \right) \\ & \leq \|\hat{f}\| \left(-\frac{1}{c} |\zeta|^{\lambda-1} + \frac{\lambda-1}{c} \int_0^{|\zeta|} (|\zeta| - |\eta|)^{\lambda-2} e^{c|\eta|} d|\eta| \right) \\ & \leq \|\hat{f}\| \frac{\lambda-1}{c} \int_0^{|\zeta|} (|\zeta| - |\eta|)^{\lambda-2} e^{c|\eta|} d|\eta|. \end{aligned} \quad (134)$$

This implies that the estimate of our integral is reduced to that of the same integral with λ replaced by $\lambda - 1$. Hence, by the inductive argument we have the same estimate. This proves the lemma.

Proof (of Theorem 3) Let ν be a positive integer chosen later and define w by (92). Let u the solution of (88). If we set $u = w + v$, $v = O(s^{\nu+1})$, then v satisfies the equation

$$\mathcal{L}v + Rv = -\mathcal{L}w - Rw \equiv f, \quad (135)$$

where $f = O(s^{\nu+1})$. Let $\gamma(\pi)$ be the path given in Lemma 1. We use the coordinate (t, s) . The function f is holomorphic in $\mathbb{C} \setminus [0, \infty)$, continuous on $\gamma(\pi)$, and satisfies $f = O(s^{\nu+1})$ by Lemma 1. In order to construct the solution in $\Sigma \times \{|s| < \varepsilon_1\}$ we make the change of the variables $p_1 = \tilde{p}_1 t^2 e^{-1/t}$, $p_2 = \tilde{p}_2 t^2 e^{-1/t}$. Clearly, for $(t, p_1, p_2) \in \Sigma$ we have $|\tilde{p}_j| < K$ ($j = 1, 2$) and vice versa. Then we can easily verify that $p_i \frac{\partial}{\partial p_j} = \tilde{p}_i \frac{\partial}{\partial \tilde{p}_j}$ for $j = 1, 2$. It follows that in the equation (88) we may replace p_j with \tilde{p}_j . For the sake of simplicity we denote the variable \tilde{p}_j with p_j in (88). We note that we may think that p_j moves in some neighborhood of the origin.

We set $g := s^{-\nu-1} f$ and $V := s^{-\nu-1} v$. By taking the partial Borel transform with respect to t , we have

$$(\hat{\mathcal{L}} - 2\sigma(1 + \alpha_1)(\nu + 1))\hat{V} + \hat{R}\hat{V} = \hat{g}, \quad (136)$$

where $\hat{g} = \mathcal{B}(g)$, $\hat{V} := \mathcal{B}(V)$ and

$$\begin{aligned}\hat{\mathcal{L}}\hat{V} &:= 2\sigma \left(p_2 \frac{\partial}{\partial p_2} - (1 + \alpha_1) s \frac{\partial}{\partial s} + \zeta - \alpha_1 - 2\alpha_1 p_1 \frac{\partial}{\partial p_1} \right) \hat{V}, \\ \hat{R}\hat{V} &:= \left(-4p_1 \sigma s^2 \alpha_2 + 2\sigma(2\sigma - 1) p_2 s \alpha_{1-1/(2\sigma)} \right) \frac{\partial \hat{V}}{\partial p_1} \\ &\quad - 4p_1 s \left(\alpha_{1+1/(2\sigma)} + s^2 \alpha_{2+1/(2\sigma)} + \alpha_{1/(2\sigma)} \right) \frac{\partial \hat{V}}{\partial p_2} \\ &\quad + 2\sigma s^2 \left(s^2 \alpha_2 + 2\alpha_1 + 2 \right) \left(\zeta - \alpha_1 \left(1 + s \frac{\partial}{\partial s} \right) \right) \hat{V}.\end{aligned}$$

Here we used the fact that the Borel transform of $s^2 t \left(s^2 t^2 + 2t + 2 \right) \left(t \frac{\partial u}{\partial t} - s \frac{\partial u}{\partial s} \right)$ is given by

$$s^2 \left(s^2 \alpha_2 + 2\alpha_1 + 2 \right) \mathcal{B} \left(t^2 \frac{\partial u}{\partial t} - s t \frac{\partial u}{\partial s} \right) = s^2 \left(s^2 \alpha_2 + 2\alpha_1 + 2 \right) \left(\zeta - \alpha_1 \left(1 + s \frac{\partial}{\partial s} \right) \right) \hat{u}.$$

We expand \hat{g} in the power series of s , $\hat{g} = \sum_{k=0}^{\infty} \hat{g}_k(\zeta, p_1, p_2) s^k$ and we look for the solution \hat{V} of (136) in the form

$$\hat{V} = \sum_{k=0}^{\infty} \hat{V}_k(\zeta, p_1, p_2) s^k. \quad (137)$$

By inserting these expansions into (136) we have the recurrence formula

$$\begin{aligned}\left(\hat{\mathcal{L}}_0 - 2\sigma(\nu + 1 + k)(1 + \alpha_1) \right) \hat{V}_k &= \hat{g}_k \\ - 4\sigma \alpha_2 p_1 \frac{\partial}{\partial p_1} \hat{V}_{k-2} + 2\sigma(2\sigma - 1) \alpha_{1-1/(2\sigma)} p_2 \frac{\partial}{\partial p_1} \hat{V}_{k-1} \\ - 4(\alpha_{1+1/(2\sigma)} + \alpha_{1/(2\sigma)}) p_1 \frac{\partial}{\partial p_2} \hat{V}_{k-1} - 4\alpha_{2+1/(2\sigma)} p_1 \frac{\partial}{\partial p_2} \hat{V}_{k-3} \\ + 2\sigma \alpha_2 (\zeta - \alpha_1(k - 3)) \hat{V}_{k-4} + 4\sigma(\alpha_1 + 1)(\zeta - \alpha_1(k - 1)) \hat{V}_{k-2},\end{aligned} \quad (138)$$

where $k = 0, 1, \dots$ and

$$\hat{\mathcal{L}}_0 = \hat{\mathcal{L}} + 2\sigma(1 + \alpha_1) s \frac{\partial}{\partial s}. \quad (139)$$

In the following $H(\Omega_0)$ denotes the set of holomorphic functions $w(\zeta, p_1, p_2)$ being holomorphic in Ω_0 and polynomials of p_1 and p_2 with degree at most N_0 . We equip $H(\Omega_0)$ with the norm similar to $H(\Omega)$. We shall show that there exists ν such that $\hat{\mathcal{L}}_0 - 2\sigma(\nu + 1 + k)(1 + \alpha_1)$ on $H(\Omega_0)$ is invertible when $k = 0, 1, 2, \dots$. Because a function in $H(\Omega_0)$ is a polynomial of p_1 and p_2 with degree at most N_0 , the operators $p_j(\partial/\partial p_j)$ in $\hat{\mathcal{L}}_0$ are bounded continuous operators on $H(\Omega_0)$. Because the operators α_λ is a bounded continuous operator whose norm can be made arbitrarily small by Lemma 2, it follows that $\hat{\mathcal{L}}_0 - 2\sigma(\nu + 1 + k)(1 + \alpha_1)$ is invertible for $\zeta \in \Gamma_0$ if we take ν sufficiently large. Moreover, by the Neumann series we can easily show that there exists $K_1 > 0$ such that, for $k = 0, 1, 2, \dots$ and all $\zeta \in \Gamma_0$

$$\|(\hat{\mathcal{L}}_0 - 2\sigma(\nu + 1 + k)(1 + \alpha_1))^{-1}\| \leq \frac{K_1}{1 + k + |\zeta|}. \quad (140)$$

It follows that if we operate $(\hat{\mathcal{L}}_0 - 2\sigma(\nu + 1 + k)(1 + \alpha_1))^{-1}$ to (138), then the right-hand side terms in (138) are bounded by some constants which can be taken arbitrarily small if ε_0 is sufficiently small for all $\zeta \in I_0$ and $k = 0, 1, 2, \dots$. It follows that the V_k 's can be determined recursively from (138) in $H(\Omega_0)$. Moreover, there exist $K_2 > 0$ and $C_0 > 0$ such that we have the estimate

$$\|V_k\| \leq K_2 C_0^k, \quad k = 0, 1, 2, \dots \quad (141)$$

This proves that the series (137) converges in $H(\Omega)$. If we make the partial Laplace transform with respect to ζ in (136), then we have the solution U as in Theorem 3. This ends the proof.

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