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winter semester 2013/2014

Exercise Sheet 1

Applied Analysis

Discussion on Thursday 24-10-2013 at 16ct

Exercise 1 (*Open or closed in $\mathbb{R}^2$?*) (1+1+1+1+2)
Which of the following sets X are open respectively closed in $\mathbb{R}^2$ with the usual Euclidean metric

$$d_2\left((x_1, x_2), (y_1, y_2)\right) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}?$$

(a) $X = \emptyset$
(b) $X = (0,1) \times (0,1)$
(c) $X = (0,1) \times \{0\}$
(d) $X = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 \leq 0\right\}$
(e) $X = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < x_1 - x_2^3\right\}$

Exercise 2 (*Definition of metric spaces*) (1+1+1+1+1)
Which of the following tuples (M, d) define a metric space?

(a) $M = \mathbb{R}$ and $d(x, y) = 1$ for all $x, y \in \mathbb{R}$.
(b) $M = \mathbb{R}$ and $d(x, y) = x - y$ for all $x, y \in \mathbb{R}$.
(c) $M = \mathbb{R}$ and $d(x, y) = \left|x^2 - y^2\right|$ for all $x, y \in \mathbb{R}$.
(d) $M = \mathbb{R}$ and $d(x, y) = \left|x - y^2\right|$ for all $x, y \in \mathbb{R}$.
(e) $M = \mathbb{R}$ and $d(x, y) = \sqrt{|x - y|}$ for all $x, y \in \mathbb{R}$.

Exercise 3 (*A strange metric on $\mathbb{R}$*) (3+3+3)
Let us denote by $d\colon \mathbb{R} \times \mathbb{R} \to [0, \infty)$ the usual Euclidean metric in $\mathbb{R}$, i.e. $d(x, y) = |x - y|$ for all $x, y \in \mathbb{R}$. We construct a new metric $d_S\colon \mathbb{R} \times \mathbb{R} \to [0, \infty)$.

i. Let $S \subset \mathbb{R}^2$ be the circle of radius 1 and centre $(0, 1) \in \mathbb{R}^2$.

ii. We denote by $N = (0, 2) \in S$ the north pole of the circle.

iii. Draw a line $L_x \subset \mathbb{R}^2$ which contains N and $(x, 0) \in \mathbb{R}^2$.

iv. The line L_x intersects S in a point $\neq N$. Call this point $P(x) \in S \subset \mathbb{R}^2$.

We define (for $x, y \in \mathbb{R}$)

$$d_S(x, y) := d_2\left(P(x), P(y)\right).$$

(a) Draw a picture, which illustrates the construction of d_S.
(b) Are $(\mathbb{R}, d)$ and $(\mathbb{R}, d_S)$ complete?
(c) Show that a set is open in $(\mathbb{R}, d_S)$ if and only if it is open in $(\mathbb{R}, d)$.

Exercise 4 (*Two simple proofs*) (5+5)
Let (M, d) be a metric space, $x \in M$ and $(x_n)_{n\in\mathbb{N}}$ a sequence in M.

(a) Show the equivalence of the following assertions:

   i. $(x_n)_{n\in\mathbb{N}}$ converges.

   ii. Every subsequence $(x_{n_k})_{k\in\mathbb{N}}$ converges in M (here we do not assume that the limits of different subsequences are the same).

please turn over!

iii. Prove that the following properties are equivalent:
   i. $(x_n)_{n\in\mathbb{N}}$ converges to x.
   ii. For every subsequence $(x_{n_k})_{k\in\mathbb{N}}$ there is a convergent subsequence $\left(x_{n_{k_l}}\right)_{l\in\mathbb{N}}$ with limit x.

Exercise 5 (*Multiple Choice*) (10)

Given metric spaces (M, d) and (N, d'). On $\mathbb{R}$ we use the usual Euclidean metric. Which of the following statements are true? Try to find proofs or counterexamples of the claims.

(a) The sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ given by $x_n = n^{-1}$ is a Cauchy sequence.
☐ true ☐ wrong

(b) The sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ given by $x_n = (-1)^n$ is a Cauchy sequence.
☐ true ☐ wrong

(c) Every neighbourhood of a point $x \in M$ is open.
☐ true ☐ wrong

(d) A subset A of M is a neighbourhood of $x \in M$, if $x \in \mathring{A}$ (that is, x lies in the interior of A).
☐ true ☐ wrong

(e) The empty set and M itself are open and closed.
☐ true ☐ wrong

(f) A non-trivial (i.e. $X \neq M$ and $X \neq \emptyset$) subset $X \subset M$ of M is never both open and closed.
☐ true ☐ wrong

(g) A subset $X \subset M$, which can be covered by finitely many open balls, is compact.
☐ true ☐ wrong

(h) Let d be the discrete metric (i.e. $d(x, y) = 1$ for all $x \neq y \in M$ and $d(x, x) = 0$ for all $x \in M$). Then a subset $K \subset M$ of M is compact if and only if K contains finitely many points.
☐ true ☐ wrong

(i) If $f : M \to N$ is continuous and $K \subset N$ is compact, then $f^{-1}(K)$ is compact too.
☐ true ☐ wrong

(j) If $f : M \to N$ is continuous and $(x_n)_{n\in\mathbb{N}}$ a Cauchy sequence in M, then $(f(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in N.
☐ true ☐ wrong