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Exercise Sheet 2

Applied Analysis

Discussion on Thursday 31-10-2013 at 16ct

Exercise 1 (*Countability*) (1+1+2+2+2+5)

Let $A, B \subset X$ be (at most) countable sets. Show that the following sets M are (at most) countable. So one has to construct a surjective function $f\colon \mathbb{N} \to M$.

(a) $M = \mathbb{N}$
(b) $M = \{1, 2, 3, 4, 5\}$
(c) $M = \mathbb{Z}$
(d) $M \subset A$ with $M \neq \emptyset$
(e) $M = A \cup B$
(f) $M = A \times B$

Exercise 2 (*A non countable set*) (5)

Show that the power set

$$\mathcal{P}(\mathbb{N}) = \{A : A \subset \mathbb{N}\}$$

of $\mathbb{N}$ (that is, the set of all subsets of $\mathbb{N}$) is not countable.
Hint: Suppose that there is a surjective function $f\colon \mathbb{N} \to \mathcal{P}(\mathbb{N})$. This will lead to a contradiction. One possibility is to look at the set

$$A := \{n \in \mathbb{N} : n \notin f(n)\}\,.$$

Exercise 3 (*Polish spaces*) (5+5*+2+5)

Of importance to probability theory are the **Polish metric spaces**. They can be defined as metric spaces (M, d), which are complete (i.e. every Cauchy sequence converges) and separable (i.e. there is a countable dense subset of M).

(a) Show that $\mathbb{R}^n$ together with the Euclidean metric d_2 defines a Polish metric space $(\mathbb{R}^n, d_2)$.
(b*) Show that every closed subset $F \subset \mathbb{R}^n$ with the Euclidean metric d_2 defines a Polish metric space (F, d_2). *Remark*:* This part is more difficult than the rest.
(c) Let d be the discrete metric (i.e. $d(x, y) = 1$ for $x \neq y$ and $d(x, x) = 0$ for all x) on $\mathbb{R}$. Prove that $(\mathbb{R}, d)$ is no Polish metric space.
(d) Construct a metric space (M, d), which is separable but not a Polish metric space.

Exercise 4 (*Multiple Choice*) (10)

Decide which of the following claims are true. Try to find an argument for your guess.

(a) $\mathbb{Q}$ is a countable set.
☐ true ☐ wrong
(b) $\mathbb{R}$ is a countable set.
☐ true ☐ wrong
(c) A finite subset A of a metric space (M, d) is open.
☐ true ☐ wrong
(d) A finite subset A of a metric space (M, d) is compact and closed.
☐ true ☐ wrong

please turn over!

(e) The compact subsets of $\mathbb{R}$ with the Euclidean metric d_2 are precisely the bounded and closed subsets.

□ true □ wrong

(f) Every bounded and closed subset in a metric space is compact.

□ true □ wrong

(g) If $A \subset M$ is a subset of a complete metric space (M, d), then $(A, d|_{A\times A})$ is complete.

□ true □ wrong

(h) Let (M, d) be a metric space and $F \subset M$ be a non empty closed subset of M. Then there exists a continuous function $f\colon M \to F$ such that $f(x) = x$ for all $x \in F$.

□ true □ wrong

(i) Every Lipschitz continuous function is continuous.

□ true □ wrong

(j) Let (M, d) and (N, d') be metric spaces. If $f\colon M \to N$ is a Lipschitz continuous function and $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in M, then $(f(x_n))_{n\in\mathbb{N}}$ is Cauchy too.

□ true □ wrong