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winter semester 2013/2014

Exercise Sheet 3

Applied Analysis

Discussion on Thursday 7-11-2013 at 16ct

Exercise 1 (*Some special series*) (5+5)

(a) Prove that the following series converges for $|q| < 1$

$$\sum_{k=0}^{\infty} q^k$$

and show the equality

$$\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}.$$

Hint: Use the well-known formula for the geometric sum.

(b) Show that the series

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

does not converge. *Hint:* Prove the following inequality

$$\sum_{k=2}^{2^n} \frac{1}{k} \geq \sum_{j=1}^{n} \sum_{i=2^{j-1}+1}^{2^j} \frac{1}{2^j} = \frac{n}{2}.$$

Exercise 2 (*Three norms in* $\mathbb{R}^2$) (2+3+3+2)

On $\mathbb{R}^2$ we can define three different norms by

$$\|(x,y)\|_1 = |x| + |y|, \ \|(x,y)\|_\infty = \max\{|x|,|y|\}, \ \|(x,y)\|_2 = \sqrt{x^2+y^2}$$

for $(x,y) \in \mathbb{R}^2$.

(a) Draw a picture of the unit balls for the norms $\|\cdot\|_2$, $\|\cdot\|_1$, and $\|\cdot\|_\infty$.

(b) Find constants $c_1, c_2, c_3 > 0$ such that

$$\|v\|_1 \leq c_1\|v\|_2 \leq c_2\|v\|_\infty \leq c_3\|v\|_1$$

holds for all $v \in \mathbb{R}^2$.

(c) Show that for a sequence $((x_n,y_n))_{n\in\mathbb{N}}$ in $\mathbb{R}^2$ and $(x,y) \in \mathbb{R}^2$ the following statements are equivalent:

i. $((x_n,y_n))_{n\in\mathbb{N}}$ converges in one of the three norms to (x,y).
ii. $((x_n,y_n))_{n\in\mathbb{N}}$ converges in all of the three norms to (x,y).
iii. $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are convergent sequences to x respectively y in $\mathbb{R}$ with the usual Euclidean metric.

Exercise 3 (*Accumulation points*) (5+5*)

(a) Calculate all accumulation points (an accumulation point is by definition the limit point of a subsequence) of the sequence $(x_n)_{n\in\mathbb{N}}$ with

$$x_n = (-1)^n + \frac{1}{n}.$$

please turn over!

(b*) Find a sequence $(x_n)_{n\in\mathbb{N}}$ with uncountably many accumulation points.

Exercise 4 (*Existence of minimum and maximum*) (5)

Show that the function $f\colon [0,1]^2 \to \mathbb{R}$ defined by

$$f(x,y) = xy + x^4 - 3x^2 y$$

attains its infimum and supremum on $[0,1]^2$.

Exercise 5 (*Multiple Choice*) (10)

Decide which of the following claims are true. Try to find an argument for your guess.

(a) If (M,d) is a metric space, then M is open and closed.
□ true □ false

(b) $\{(x,y) \in \mathbb{R}^2 : x^3 + 2x - yx + 7y^8 \leq 0\}$ is open in $\mathbb{R}^2$ with the usual Euclidean metric.
□ true □ false

(c) $\{(x,y) \in \mathbb{R}^2 : x^3 + 2x - yx + 7y^8 > 4\}$ is open in $\mathbb{R}^2$ with the usual Euclidean metric.
□ true □ false

(d) $\{(x,y) \in \mathbb{R}^2 : x^4 + 8y^6 + 4 \leq 0\}$ is open in $\mathbb{R}^2$ with the usual Euclidean metric.
□ true □ false

(e) Let $(\mathbb{R}, d)$ be a metric space. Then $\|x\| := d(0,x)$ defines a norm.
□ true □ false

(f) If $(V, \|\cdot\|)$ is a normed space, then $d(x,y) := \|x-y\|$ (for $x, y \in V$) is a metric on V.
□ true □ false

(g) The closed and bounded subsets of $\mathbb{R}^n$ (with the usual Euclidean norm) are precisely the compact ones.
□ true □ false

(h) Every function $f\colon [0,1] \to \mathbb{R}$ defined on the compact set $[0,1]$ has a minimum and a maximum.
□ true □ false

(i) Suppose we have a convergent sequence $(x_n)_{n\in\mathbb{N}}$ with

$$x_{n+1} = (2 - x_n)\, x_n + 1.$$

Then the limit point of $(x_n)_{n\in\mathbb{N}}$ is a solution of $x = (2-x)x + 1$.
□ true □ false

(j) Suppose we have a sequence $(x_n)_{n\in\mathbb{N}}$ with

$$x_{n+1} = (2 - x_n)\, x_n + 1.$$

Then the accumulation points of $(x_n)_{n\in\mathbb{N}}$ are solutions of $x = (2-x)x + 1$.
□ true □ false