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Exercise Sheet 6

Applied Analysis

Discussion on Thursday 28-11-2013 at 16ct

Convention: We will use on this sheet the real versions (i.e. $\mathbb{K} = \mathbb{R}$) of the spaces ℓ^p, that is

$$\ell^p = \{x = (x_k)_{k\in\mathbb{N}} \text{ a sequence in } \mathbb{R} : \|x\|_p < \infty\}.$$

Exercise 1 (*Compactness revisited - examples in ℓ^p*) (3+3+5+5*)

Proof the following statements about compact sets in ℓ^p.

(a) The closed ball $\overline{B(0,1)} \subset \ell^p$ is not compact (with $p \in [1, \infty]$).

(b) The set

$$\{x = (x_k) \in \ell^\infty | x_k \in [0, 1]\}$$

is no compact subset of ℓ^∞.

(c) The set

$$\{x = (x_k) \in \ell^\infty | x_k \in [0, a_k]\}$$

with $a_k \to 0$ ($k \to \infty$) and $a_k > 0$ (for all $k \in \mathbb{N}$) is a compact subset of ℓ^∞.
Hint: Let a sequence (x^n) in the set be given. By choosing a subsequence (how?) if necessary, we may assume that (x^n_k) converges for every $k \in \mathbb{N}$. Conclude carefully that (x^n) converges.

(d) Essentially the same set with ℓ^1 instead of ℓ^∞ is not necessarily compact, i.e.

$$\left\{x = (x_k) \in \ell^1 \middle| x_k \in [0, a_k]\right\}$$

with $a = (a_k) \in \mathfrak{c}_0$ and $a_k > 0$ (for all $k \in \mathbb{N}$) is not necessarily compact.
Hint: You have to choose a suitable $a = (a_k)$. A good candidate is some $a \notin \ell^1$ but with $a \in \mathfrak{c}_0$ (you know at least one example from an older sheet!).

Exercise 2 (*Linear operators*) (3+3+4+5+5)

Solve the following problems about linear operators.

(a) Show that every linear operator $T : \mathbb{R} \to \mathbb{R}$ is given by $T(x) = a \cdot x$ for some $a \in \mathbb{R}$.

(b) Find a matrix $A \in \mathbb{R}^{2\times 2}$ such that

$$T : \mathbb{R}^2 \to \mathbb{R}^2, \quad T : x \to Ax$$

is the rotation by 180° (or equivalently π).

(c) Show that

$$T : x = (x_k)_{k\in\mathbb{N}} \mapsto (x_{k+1})_{k\in\mathbb{N}}, \quad T : \ell^p \to \ell^p$$

is a bounded linear operator.

(d) Given $x = (x_k) \in \ell^q$ (q is the Hölder conjugate to p, i.e. $p^{-1} + q^{-1} = 1$). Proof that

$$T : \ell^p \to \mathbb{R}, \quad y = (y_k) \mapsto \sum_{k=1}^{\infty} x_k \cdot y_k$$

is a bounded linear operator.

please turn over!

(e) Let us again suppose that $x = (x_k) \in \ell^q$ (again $p^{-1} + q^{-1} = 1$, with $p, q \in [1, \infty]$) is given. Show that

$$T\colon \ell^p \to \ell^1, \quad y = (y_k) \mapsto Ty = (x_k \cdot y_k)$$

defines a bounded linear operator.

Exercise 3 (*Multiple Choice*) (9)

Decide which of the following assertions are true. Try to give an argument for your answer.

(a) A strict contraction $f\colon M \to M$ in a compact metric space M has a fixed point.
☐ true ☐ false

(b) A linear map between Banach spaces is always bounded (try **not** to find a proof for this part).
☐ true ☐ false

(c) $\mathbb{R}^n$ is a n-dimensional real vector space.
☐ true ☐ false

(d) A linear map between finite dimensional Banach spaces is always bounded.
☐ true ☐ false

(e) ℓ^2 is finite dimensional.
☐ true ☐ false

(f) The set $\{x = (x_k) \in \ell^\infty : x_k > 1\}$ is open in ℓ^∞.
☐ true ☐ false

(g) The set $\{x = (x_k) \in \ell^\infty : x_k \leq 1\}$ is closed in ℓ^∞.
☐ true ☐ false

(h) The set $\{x = (x_k) \in \ell^\infty : \sup_k x_k < 1\}$ is open in ℓ^∞.
☐ true ☐ false

(i) The set $\{x = (x_k) \in \ell^\infty : \sup_k x_k \leq 1\}$ is closed in ℓ^∞.
☐ true ☐ false