ulm university universität uulm

Manfred Sauter
Manuel Bernhard
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Exercise Sheet 8

Applied Analysis

Discussion on Thursday 12-12-2013 at 16ct

Exercise 1 (*Riemann integral*) (4+4)

(a) Let $f\colon [-1,1] \to \mathbb{R}$ be a Riemann integrable function with $f(-x) = -f(x)$ for all $x \in [-1,1]$. Show directly with the definition that

$$R - \int_{-1}^{1} f(x)\,dx = 0.$$

(b) Calculate the following Riemann integrals

i. $R - \int_0^1 x^2 - x\,dx$ ii. $R - \int_0^1 xe^{x^2}\,dx$

iii. $R - \int_{-1}^{2} \operatorname{sgn}(x)\,dx$ iv. $R - \int_{-1}^{1} x^3 e^{x^2}\,dx$

Here we denote by sgn the function defined by

$$\operatorname{sgn}(x) = \begin{cases} -1 & \text{, for } x < 0 \\ 0 & \text{, for } x = 0 \\ 1 & \text{, for } x > 0. \end{cases}$$

Exercise 2 (σ*-algebras and measurable functions*) (3+5+5+3)

Work on the following problems. For the all the parts the solution is enough (you do not have to prove your claims).

(a) List all possible σ-algebras on $\Omega = \{1,2,3\}$.

(b) Let $f\colon \{1,2,3,4,5\} \to \{1,2,3,4\}$ be a function given by $f(1) = f(2) = 2$, $f(3) = 3$ and $f(4) = f(5) = 4$. We put the σ-algebra $\Sigma = \sigma(\{\{2\},\{3,4\}\})$ on the codomain $\{1,2,3,4\}$.

  i. Write down all elements of the σ-algebra $\sigma(f)$.
  ii. What are the $\sigma(f)/\Sigma$-measurable functions $g\colon \{1,2,3,4,5\} \to \{1,2,3,4\}$?

(c) Let us suppose that a function $f\colon \mathbb{R} \to \mathbb{R}$ is given. We equip the codomain with the Borel-σ-algebra $\mathcal{B}(\mathbb{R})$. Give a nice description (without a proof) of the σ-algebra $\sigma(f)$ in the following situations:

  i. $f(x) = \operatorname{sgn}(x)$
  ii. $f(x) = x^3$
  iii. $f(x) = |x|$.

(d) In the situation of (c) iii. is the function $g\colon \mathbb{R} \to \mathbb{R}$ given by

  i. $g(x) = x^2$ respectively
  ii. $g(x) = x^3$

$\sigma(f)/\mathcal{B}(\mathbb{R})$-measurable?

Exercise 3 (*The principle of good sets*) (5+5)

Use the principle of good sets twice to show that $A \times B \in \mathcal{B}(\mathbb{R}^2)$, if $A, B \in \mathcal{B}(\mathbb{R})$. We divide the proof into two steps:

please turn over!

(a) Show with the principle of good sets, that $A \times I \in \mathcal{B}(\mathbb{R}^2)$ for all open intervals I in $\mathbb{R}$ and all $A \in \mathcal{B}(\mathbb{R})$. The good sets are in this situation

$$\mathcal{G} = \{A \in \mathcal{B}(\mathbb{R}) : A \times I \in \mathcal{B}(\mathbb{R}^2) \text{ for all open intervals } I\}.$$

Hint: To show that $\mathcal{G}$ is a σ-algebra, you have to proof $A \in \mathcal{G} \Rightarrow A^c \in \mathcal{G}$. And here it might be helpful that

$$(A \times I)^c \cap (\mathbb{R} \times I) = (A^c \times I).$$

(b) Now use the principle of good sets again to show the claim. This time the good sets are

$$\mathcal{G}' = \{B \in \mathcal{B}(\mathbb{R}) : A \times B \in \mathcal{B}(\mathbb{R}^2) \text{ for all } A \in \mathcal{B}(\mathbb{R})\}.$$

Hint: We further remark (you will need this in the proof) that

$$(A \times B)^c \cap (A \times \mathbb{R}) = A \times B^c.$$

Exercise 4 (*Multiple Choice*) (6)

Which of the following statements are true?

(a) Every continuous function $f\colon [a,b] \to \mathbb{R}$ is Riemann integrable.
□ true □ false

(b) Every bounded function $f\colon [a,b] \to \mathbb{R}$ is Riemann integrable.
□ true □ false

(c) $\Sigma = \{A \subset \mathbb{N} : A \text{ is finite or } A^c \text{ is finite}\}$ is a σ-algebra on $\mathbb{N}$.
□ true □ false

(d) The union of two σ-algebras is a σ-algebra again.
□ true □ false

(e) $\{\emptyset, \{1\}, \{2\}, \{1,2\}, \{3,4\}, \{2,3,4\}, \{1,3,4\}, \{1,2,3,4\}\}$ is a σ-algebra on $\Omega = \{1,2,3,4\}$.
□ true □ false

(f) If Σ is a σ-algebra and elements $A_i \in \Sigma$ for all $i \in I$ (for some index set I) are given, then $\bigcup_{i \in I} A_i$ is an element of Σ too.
□ true □ false