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Exercise Sheet 9

Applied Analysis

Discussion on Thursday 9-1-2013 at 16ct

We make some definitions:

(i) A **probability space** (Ω, Σ, μ) is a measure space such that $\mu(\Omega) = 1$.

(ii) A **random variable** $X\colon \Omega \to \mathbb{R}$ on a probability space is a $\Sigma/\mathcal{B}(\mathbb{R})$-measurable map.

(iii) A **Rademacher random variable** $X\colon \Omega \to \mathbb{R}$ on a probability space (Ω, Σ, μ) is a random variable with $X(\Omega) = \{1, -1\}$ and $\mu(\{X = 1\}) = \mu(\{X = -1\}) = 0.5$.

(iv) Finitely many random variables $X_1, ..., X_n$ on a probability space (Ω, Σ, μ) are called **(stochastically) independent**, iff for all $A_1, ..., A_n \in \mathcal{B}(\mathbb{R})$ we get

$$\mu(\{X_1 \in A_1, ..., X_n \in A_n\}) = \prod_{i=1}^{n} \mu(\{X_i \in A_i\}).$$

In the following one can use (without a proof)

$$\mathcal{B}(\mathbb{R}) = \sigma\left(\{(-\infty, a] \subset \mathbb{R} : a \in \mathbb{R}\}\right).$$

Exercise 1 (*Measures and cumulative distribution functions*) (10+5)

(a) Let us suppose that $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$ is a probability space. We define the cumulative distribution function $F_\mu\colon \mathbb{R} \to [0, 1]$ by

$$F_\mu(x) := \mu((-\infty, x]).$$

i. Show $\lim_{x \to -\infty} F_\mu(x) = 0$ and $\lim_{x \to \infty} F_\mu(x) = 1$.
ii. Show that F_μ is monotonically increasing (i.e. $F_\mu(x) \geq F_\mu(y)$ if $x \geq y$)
iii. Show that F_μ is right-continuous (i.e. if (x_n) is a sequence in $[x, \infty)$ converging to x, then we get $F_\mu(x_n) \to F_\mu(x)$ $(n \to \infty)$).
iv. Show that F_μ determines μ.

(b) Let (Ω, Σ, μ) be a fixed probability space and $X\colon \Omega \to \mathbb{R}$ be a random variable. In the lecture we defined the push forward μ_X of μ under X. By $F_X\colon \mathbb{R} \to [0, 1]$ we mean the cumulative distribution function of X defined by $F_X := F_{\mu_X}$ (see the part (a) of this exercise). Find a probability space (Ω, Σ, μ) and two Rademacher random variables X, Y such that $X \neq Y$ but $F_X = F_Y$.
Remark: This is very easy, but important to know.

Exercise 2 (*Dynkin's theorem and the independence of random variables*) (20)

Let us suppose that (Ω, Σ, μ) is a probability space. Suppose moreover that X_1, X_2 are two random variables. Then the following properties are equivalent:

(a) X_1 and X_2 are independent, i.e. for all $A, B \in \mathcal{B}(\mathbb{R})$ we have

$$\mu\left(\{X_1 \in A, X_2 \in B\}\right) = \mu\left(\{X_1 \in A\}\right) \cdot \mu\left(\{X_2 \in B\}\right).$$

(b) For all $x, y \in \mathbb{R}$ we get

$$\mu\left(\{X_1 \in (-\infty, x], X_2 \in (-\infty, y]\}\right) = \mu\left(\{X_1 \in (-\infty, x]\}\right) \cdot \mu\left(\{X_2 \in (-\infty, y]\}\right).$$

please turn over!

Show this equivalence.
Part (b) can be paraphrased as

$F_{(X_1,X_2)}$ is the product of F_{X_1} and F_{X_2}, i.e. $F_{(X_1,X_2)}(x,y) = F_{X_1}(x)F_{X_2}(y)$ for all $x, y \in \mathbb{R}$.

But we have not jet defined the cumulative distribution function of a pair of random variables. This function $F_{(X_1,X_2)} \colon \mathbb{R}^2 \to [0,1]$ is simply given by

$$F_{(X_1,X_2)}(x,y) := \mu\left(\{\omega : X_1(\omega) \in (-\infty, x] \text{ and } X_2(\omega) \in (-\infty, y]\}\right).$$

Hint: For "(b)⇒(a)" show that the good sets

$$\mathcal{G} := \{A \in \mathcal{B}(\mathbb{R}) | \text{ part (a) holds for all } B = (-\infty, y] \text{ with } y \in \mathbb{R}\}$$

define a Dynkin system. Then use Dynkin's theorem and the fact

$$\mathcal{B}(\mathbb{R}) = \sigma\left(\{(-\infty, a] \subset \mathbb{R} : a \in \mathbb{R}\}\right)$$

from above. The same procedure on

$$\mathcal{G}' := \{B \in \mathcal{B}(\mathbb{R}) | \text{ part (a) holds for all } A \in \mathcal{B}(\mathbb{R})\}.$$

gives the claim. See also exercise 3 on sheet 8 for a very similar proof.