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Suggested Solution to Exercise Sheet 10

Applied Analysis

Discussion on Thursday 9-1-2014 at 16ct

This is also the first mock exam. 100% corresponds to 110 points. In the final exam you are allowed to use a calculator and a double-sided handwritten A4 sheet. This is intended to be solved in 120 minutes.

Contents

Exercise 1 (*Three basic properties of metric spaces*) (3+4+5)

Let (M, d) be a metric space.

(a) Give a definition of the following properties of (M, d):

i. compactness,
ii. separability, and
iii. completeness.

(b) Which of the following implications are true? (no proof required)

i. (M, d) is compact $\Rightarrow$ (M, d) is complete.
ii. (M, d) is complete $\Rightarrow$ (M, d) is compact.
iii. (M, d) is compact $\Rightarrow$ (M, d) is separable.
iv. (M, d) is separable $\Rightarrow$ (M, d) is complete.

(c) Give a counterexample with explanation of one of the wrong implications in (b).

Solution of Exercise 1:

ad (a):

ad i.:

(M, d) is compact, iff for **every** given sequence $(x_n)_{n\in\mathbb{N}}$ in M we find a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ which converges in M.

ad ii.:

(M, d) is separable, iff we can find a dense and countable subset of M.

ad iii.:

(M, d) is complete, iff **every** Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in M converges in M.

ad (b):

ad i.:

true

ad ii.:

false

ad iii.:

true

ad iv.:

false

ad (c):

We have to find a counterexample for ii. **or** iv..

ad ii.:

$(\mathbb{R}, d_2)$ is complete (see lecture) but not compact (because it is not bounded).

ad iv.:

$(\mathbb{Q}, d_2)$ is separable (the dense countable subset is $\mathbb{Q}$) but not complete (see lecture).

Exercise 2 (*Compactness*) (4+4+5+5)

(a) Which of the following sets are compact? (no proof required)

i. $(\mathbb{Q}, d)$ where d is the discrete metric.

ii. $[0,1] \times \{1\}$ in $(\mathbb{R}^2, d_2)$. Here we denote by d_2 the euclidean metric.

iii. $(0,1]$ in $(\mathbb{R}, d_2)$.

iv. (M, d) a metric space where M is a finite set.

(b) Choose one of your claims in part (a) and prove them.

(c) Show that the closed unit ball in ℓ^∞ is not compact.

(d) Prove that the function $f\colon [0,1]^2 \to \mathbb{R}$ given by

$$f(x,y) = e^{x^2} + yx + ye^y - x$$

attains its infimum and supremum in $[0,1]^2$.

Solution of Exercise 2:

ad (a):

ad i.:

not compact

ad ii.:

compact

ad iii.:

not compact

ad iv.:

compact

ad (b):

We state again all proofs. But you have to give only one!

ad i.:

$\mathbb{Q}$ is not bounded, so it is not compact.

ad ii.:

$[0,1]$ and $\{1\}$ are both closed and bounded subset of $\mathbb{R}$. So $[0,1] \times \{1\}$ is a closed and bounded subset of $\mathbb{R}^2$. The claim follows because a closed and bounded subset of the Euclidean space $(\mathbb{R}^2, d_2)$ is compact.

ad iii.:

$(0,1]$ is not closed. Indeed the sequence (x_n) in $\mathbb{R}$ given by $x_n = \frac{1}{n}$ is convergent to 0 in $(\mathbb{R}, d_2)$, but $0 \notin (0,1]$. So $(0,1]$ is not closed and therefore not compact.

ad iv.:

Given an arbitrary sequence $(x_n)_{n\in\mathbb{N}}$ in M, then there exists a point $m \in M$ which is occurs infinitely often in this sequence. So we can choose a constant subsequence $(x_{n_k})_{k\in\mathbb{N}}$ with $x_{n_k} = m$ for every $k \in \mathbb{N}$. As every constant sequence in M converges in M our claim follows.

ad (c):

It is clear that $(e_n)_{n\in\mathbb{N}}$ is a sequence in the closed unit ball of ℓ^∞. We get immediately from the definition of the norm

$$\|e_n - e_m\|_\infty = 1$$

for $n \neq m$. So let us suppose that the closed unit ball is compact. In this case we could find a convergent subsequence $(e_{n_k})_{k\in\mathbb{N}}$. As every convergent subsequence is Cauchy, we get

$$\|e_{n_k} - e_{n_l}\|_\infty < 1$$

for k, l large enough. So for $k \neq l$ large enough we get the contradiction

$$1 = \|e_{n_k} - e_{n_l}\|_\infty < 1.$$

So our assumption that the closed unit ball of ℓ^∞ is compact gives us a contradiction. Hence the closed unit ball of ℓ^∞ is not compact.

ad (d):

f is as the composition/multiplication/addition of continuous functions continuous. Moreover $[0,1]^2$ is compact (because it is a bounded and closed subset of an Euclidean space). So we know from the lecture (every continuous function defined on a compact subset attains its supremum and infimum) that f attains its infimum and supremum in $[0,1]^2$.

Exercise 3 (*Banach's fixed point theorem*) (5+5+5)

(a) Formulate Banach's (classical) fixed point theorem.

(b) Use Banach's fixed point theorem to prove the existence of a unique solution $x^*, y^* \in [-1,1]$ of

$$\begin{aligned} 10x &= x^2 + y \\ 10y &= x^2 + y + 5 \end{aligned}$$

(c) Calculate the first two decimal digits of x^* and y^* by using the fixed point iteration starting with $x_0 = y_0 = 0$.

Solution of Exercise 3:

ad (a):

The classical version of Banach's fixed point theorem:

Every strict contraction $f\colon M \to M$ (i.e. $d\left(f(x), f(y)\right) \leq Ld(x,y)$ holds for all $x, y \in M$ and some fixed $L < 1$) on a complete metric space (M,d) has a unique fixed point.

ad (b):

Let us use the compact (because it is a bounded and closed subset of a Euclidean space) subset $[-1,1]^2$ of $\mathbb{R}^2$ with the metric

$$d_\infty(a,b) = \max\{|a_1 - b_1|, |a_2 - b_2|\}$$

for $a = (a_1, a_2) \in \mathbb{R}^2$ and $b = (b_1, b_2) \in \mathbb{R}^2$. So $\left([-1,1]^2, d_\infty\right)$ is a complete metric space (either because it is compact and therefore complete, or because it is a closed subset of the complete metric space $(\mathbb{R}^2, d_\infty)$).

Moreover let us define a function

$$f\colon [-1,1]^2 \to [-1,1]^2$$

by

$$f\colon (x,y) \mapsto \left(\frac{1}{10}\left(x^2 + y\right), \frac{1}{10}\left(x^2 + y + 5\right)\right).$$

This is a strict contraction:

$$\begin{aligned} d_\infty\left(f(a), f(b)\right) &= \max\left(\frac{1}{10}\left|(a_1 - b_1)(a_1 + b_1) + (a_2 - b_2)\right|, \frac{1}{10}\left|(a_1 - b_1)(a_1 + b_1) + (a_2 - b_2)\right|\right) \\ &= \frac{1}{10}\left|(a_1 - b_1)(a_1 + b_1) + (a_2 - b_2)\right| \\ &\leq \frac{1}{10}\left|a_1 - b_1\right|\left|a_1 + b_1\right| + \frac{1}{10}\left|a_2 - b_2\right| \leq \frac{1}{10} d_\infty(a,b)\left|a_1 + b_1\right| + \frac{1}{10} d_\infty(a,b) \\ &\leq \frac{3}{10} d_\infty(a,b) \end{aligned}$$

for every $a = (a_1, a_2), b = (b_1, b_2) \in [-1,1]^2$. So we can use Banach's fixed point theorem to conclude that f has a unique fixed point in $[-1,1]^2$. And a fixed point of f is nothing else then a solution of the above equation system.

ad (c):

Using the iteration

$$a_n = f\left(a_{n-1}\right)$$

for $n \in \mathbb{N}$ with $a_0 = (0,0)$, we know two basic error estimations ($a^* = (x^*, y^*)$ the unique fixed point):

$$d_\infty\left(a^*, a_n\right) \leq \frac{L^n}{1-L} d_\infty\left(a_1, a_0\right)$$

or (which we use now)

$$d_\infty(a^*, a_n) \le \frac{L}{1-L} d_\infty(a_n, a_{n-1})$$

for $n \in \mathbb{N}$. From part (b) we get $L = 0.3$ and therefore $L(1-L)^{-1} = \frac{3}{7}$. In particular we have (which is easier to handle)

$$d_\infty(a^*, a_n) \le \frac{L}{1-L} d_\infty(a_n, a_{n-1}) < \frac{1}{2} d_\infty(a_n, a_{n-1})$$

So we get:

$$\begin{aligned} a_0 = (0,0), &\ \ d_\infty(a^*, a_0) = ? \\ a_1 = (0, 0.5), &\ \ d_\infty(a^*, a_1) < \frac{1}{2} d_\infty(a_1, a_0) = 0.25 \\ a_2 = (0.05, 0.55), &\ \ d_\infty(a^*, a_2) < \frac{1}{2} d_\infty(a_2, a_1) = 0.025 \\ a_3 = (0.05525, 0.5525), &\ \ d_\infty(a^*, a_3) < \frac{1}{2} d_\infty(a_3, a_2) = 0.0025 \end{aligned}$$

Hence we get $x^* = 0.05525 \pm 0.0025$ and $y^* = 0.5525 \pm 0.0025$ (We remark that the first two decimal digits are correct).

Exercise 4 (*Countability*) (6+3)

(a) Which of the following sets are countable? (no proof required)

i. $[0, 1]$

ii. $\mathbb{Z} \times \{0, 1, 2, 3, 4, 5\}$

iii. $\{1, 2, 3, 4, 5, 6\}$

iv. $\mathbb{Q}$

v. $\mathcal{P}(\mathbb{Z}) = \{A : A \subset \mathbb{Z}\}$ the power set of $\mathbb{Z}$

vi. $\mathcal{P}_f(\mathbb{N}) := \{A : A \subset \mathbb{N} \text{ is finite}\}$

(b) If $A \neq \emptyset$ is uncountable, prove that B with $A \subset B$ is uncountable too.

Solution of Exercise 4:

ad (a):

ad i.:

not countable

ad ii.:

countable

ad iii.:

countable

ad iv.:

countable

ad v.:

not countable

ad vi.:

countable

ad (b):

We prove this by contradiction. So let us suppose that B is countable. Then we know from one of the exercises, that a subset of a countable set is again countable. So A is countable. This is a contradiction to the assumption, that A uncountable. So B has to be uncountable too.

Exercise 5 (*Linear bounded maps*) (2+5)
If $(x_k) \in \ell^p$ and $(y_k) \in l^q$, then $(x_k y_k) \in \ell^1$ (no proof needed). Here $p, q \in [1, \infty]$ are such that $p^{-1} + q^{-1} = 1$. So for a fixed $(y_k) \in \ell^q$ we get a well-defined function

$$T\colon \ell^p \to \ell^1, \quad T\colon (x_k) \mapsto (x_k y_k).$$

(a) Show that T is linear.
(b) Show that T is bounded.

Solution of Exercise 5:
ad (a):

- Given $x = (x_k)_{k\in\mathbb{N}} \in \ell^p$ and $z = (z_k)_{k\in\mathbb{N}} \in \ell^p$, then $T(x+z) = Tx + Tz$. Indeed

$$T(x+z) = T\left((x_k + z_k)_{k\in\mathbb{N}}\right) = \left((x_k + z_k)y_k\right)_{k\in\mathbb{N}} = (x_k y_k)_{k\in\mathbb{N}} + (z_k y_k)_{k\in\mathbb{N}} = Tx + Tz.$$

- Given $x = (x_k)_{k\in\mathbb{N}} \in \ell^p$ and $\lambda \in \mathbb{K}$, then $T(\lambda x) = \lambda T x$. Indeed

$$T(\lambda x) = T\left((\lambda x_k)_{k\in\mathbb{N}}\right) = \left((\lambda x_k) y_k\right)_{k\in\mathbb{N}} = \left(\lambda (x_k y_k)\right)_{k\in\mathbb{N}} = \lambda \left(x_k y_k\right)_{k\in\mathbb{N}} = \lambda T x.$$

ad (b):
Boundedness means in our case, that we have to find some constant $C > 0$ (independent of x) such that

$$\|Tx\|_1 \le C \|x\|_p$$

holds for every $x \in \ell^p$. Given some arbitrary $x = (x_k)_{k\in\mathbb{N}} \in \ell^p$, then

$$\|Tx\|_1 = \left\|(x_k y_k)_{k\in\mathbb{N}}\right\|_1 = \sum_{k=1}^{\infty} |x_k y_k| \,.$$

Now we use Hölder's inequality and get

$$\|Tx\|_1 = \sum_{k=1}^{\infty} |x_k y_k| \le \|x\|_p \|y\|_q.$$

So we have found our constant (which is independent of x)

$$C = \|y\|_q.$$

Exercise 6 (*Measurable functions and σ-algebras*) (3+5+5+3+5)

(a) List all σ-algebras on $\Omega = \{5, 6, 7\}$.

(b) Let $f: \Omega_1 \to \Omega_2$ be a function and Σ_2 a σ-algebra on Ω_2. Define $\sigma(f)$.

(c) Let the function $f: \{1,2,3,4,5\} \to \{1,2,3,4\}$ given by $f(1) = 1$, $f(2) = f(3) = 2$, $f(4) = f(5) = 3$. We equip the codomain with the σ-algebra $\Sigma = \sigma(\{\{3\},\{4\}\})$. Write down all the elements of $\sigma(f)$.

(d) Let $f: \Omega_1 \to \Omega_2$ be a function, Σ_1 a σ-algebra on Ω_1 and Σ_2 a σ-algebra on Ω_2. Define when f is Σ_1/Σ_2-measurable.

(e) In the situation of part (c): How many functions $g: \{1,2,3,4,5\} \to \{1,2,3,4\}$ are $\sigma(f)/\Sigma$-measurable? Give a detailed argumentation.

Solution of Exercise 6:

ad (a):

$$\mathcal{P}(\Omega) = \{\emptyset, \{5\}, \{6\}, \{7\}, \{5,6\}, \{6,7\}, \{5,7\}, \{5,6,7\}\}$$
$$\{\emptyset, \{5,6,7\}\}$$
$$\{\emptyset, \{5\}, \{6,7\}, \{5,6,7\}\}$$
$$\{\emptyset, \{6\}, \{5,7\}, \{5,6,7\}\}$$
$$\{\emptyset, \{7\}, \{5,6\}, \{5,6,7\}\}$$

ad (b):

One possible (there are other possible definitions!) definition is:

$$\sigma(f) = \left\{ f^{-1}(A) \colon A \in \Sigma_2 \right\}$$

ad (c):

$$\sigma(f) = \sigma\left(\left\{ f^{-1}\left(\{3\}\right), f^{-1}\left(\{3\}\right) \right\}\right) = \sigma\left(\{\{4,5\}, \emptyset\}\right) = \{\emptyset, \{1,2,3\}, \{4,5\}, \{1,2,3,4,5\}\}$$

ad (d):

f is Σ_1/Σ_2-measurable, iff

$$f^{-1}(A) \in \Sigma_1$$

for all $A \in \Sigma_2$.

ad (e):

g is measurable, iff

$$g\left(\{1,2,3\}\right) \subset \{3\}, \ \subset \{4\} \text{ or } \subset \{1,2\}$$

and

$$g\left(\{4,5\}\right) \subset \{3\}, \ \subset \{4\} \text{ or } \subset \{1,2\}.$$

There are $1+1+8 = 10$ possible ways to map to define g on $\{1,2,3\}$ and $1+1+4 = 6$ ways to define g on $\{4,5\}$. So we conclude, that there are 60 measurable functions.

Exercise 7 (*Calculating Riemann integrals*) (5+8)

(a) Suppose that $f\colon [a,c] \to \mathbb{R}$ is Riemann integrable. If $b \in (a,c)$ is given, then

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

Prove this by using the definition of the Riemann integral (You can assume that f is Riemann integrable on $[b,c]$ and $[a,b]$ too).

(b) Calculate the following Riemann integrals:

i. $\int_{-1}^{1} e^{x^2+3x^4} x\,dx$ ii. $\int_0^1 x^2 e^{x^3}\,dx$

iii. $\int_0^1 (x^2+4x)\,dx$ iv. $\int_{-1}^{2} f(x)\,dx$

Here $f\colon [-1,2] \to \mathbb{R}$ is given by

$$f\colon x \mapsto \begin{cases} -x+1 & \text{, for } x<0 \\ 0 & \text{, for } x=0 \\ x-1 & \text{, for } x>0. \end{cases}$$

Solution of Exercise 7:

ad (a):

We find a partition

$$\pi^{(1,n)} = \left(t_0^{(1,n)}, \dots, t_{N(1,n)}^{(1,n)}\right)$$

of $[a,b]$ and a partition

$$\pi^{(1,n)} = \left(t_0^{(1,n)}, \dots, t_{N(2,n)}^{(1,n)}\right)$$

of $[b,c]$ with mesh size $< \frac{1}{n}$ and two vectors

$$\xi^{(1,n)} = \left(\xi_1^{(1,n)}, \dots, \xi_{N(1,n)}^{(1,n)}\right)$$

and

$$\xi^{(2,n)} = \left(\xi_1^{(2,n)}, \dots, \xi_{N(2,n)}^{(2,n)}\right)$$

of sample points for $\pi^{(1,n)}$ respectively $\pi^{(2,n)}$. By definition of the Riemann integral (and Riemann integrablity)

$$\text{R-}\int_a^b f(x)\,dx = \lim_{n\to\infty} S\left(f, \pi^{(1,n)}, \xi^{(1,n)}\right)$$

and

$$\text{R-}\int_b^c f(x)\,dx = \lim_{n\to\infty} S\left(f, \pi^{(2,n)}, \xi^{(2,n)}\right).$$

It is easy to see that

$$\pi^{(n)} := \left(t_0^{(1,n)}, \dots, t_{N(1,n)}^{(1,n)} = t_0^{(2,n)}, \dots, t_{N(2,n)}^{(2,n)}\right)$$

is a partition of mesh size $< \frac{1}{n}$ and

$$\xi^{(1,n)} = \left(\xi_1^{(1,n)}, \dots, \xi_{N(1,n)}^{(1,n)}, \xi_1^{(2,n)}, \dots, \xi_{N(2,n)}^{(2,n)}\right)$$

a vector of sample points for $\pi^{(n)}$. Moreover we get from the definition of Riemann sums

$$S\left(f, \pi^{(n)}, \xi^{(n)}\right) = S\left(f, \pi^{(1,n)}, \xi^{(1,n)}\right) + S\left(f, \pi^{(2,n)}, \xi^{(2,n)}\right).$$

The Riemann integrablity of f on $[a, c]$ shows that the left-hand side converges to the Riemann integral on $[a, c]$. Hence taking the limits we conclude

$$\begin{aligned}\text{R-}\int_a^c f(x)\,dx &= \lim_{n\to\infty} S\left(f, \pi^{(n)}, \xi^{(n)}\right) = \lim_{n\to\infty} S\left(f, \pi^{(1,n)}, \xi^{(1,n)}\right) + S\left(f, \pi^{(2,n)}, \xi^{(2,n)}\right) \\ &= \text{R-}\int_a^b f(x)\,dx + \text{R-}\int_b^c f(x)\,dx.\end{aligned}$$

ad (b):

ad i.:

The function $x \mapsto e^{x^2+3x^4} x$ is odd. Moreover the integration limits are symmetric around zero. So

$$\int_{-1}^{1} x^2 e^{x^3}\,dx = 0.$$

ad ii.:

$$\int_0^1 x^2 e^{x^3}\,dx = \left[\frac{1}{3}e^{x^3}\right]_{x=0}^{x=1} = \frac{1}{3}(e-1)$$

ad iii.:

$$\int_0^1 (x^2+4x)\,dx = \left[\frac{1}{3}x^3 + 2x\right]_{x=0}^{x=1} = \frac{7}{3}$$

ad iv.:

$$\int_{-1}^{2} f(x)\,dx = \int_{-1}^{1} f(x)\,dx + \int_1^2 f(x)\,dx = \int_1^2 (x-1)\,dx = \left[\frac{1}{2}x^2 - x\right]_{x=1}^{x=2} = \frac{1}{2}$$

Here we used, that f is odd and therefore

$$\int_{-1}^{1} f(x)\,dx = 0$$

Exercise 8 (*Independent events*) (5+10)

Let (Ω, Σ, μ) be a probability space. Two sets $A, B \in \Sigma$ are called (stochastically) independent, iff

$$\mu(A \cap B) = \mu(A)\mu(B).$$

Let us suppose that $A \in \Sigma$ and $\mathcal{E} \subset \Sigma$ is given. We say that A is independent of $\mathcal{E}$, iff A, B are independent for all $B \in \mathcal{E}$.

(a) Find a concrete example of the above situation such that A is independent of $\mathcal{E}$ but A is not independent of $\sigma(\mathcal{E})$.

(b) Let us suppose that $\mathcal{E}$ is stable under intersections. Prove that the following properties are equivalent:

i. A and $\mathcal{E}$ are independent.

ii. A and $\sigma(\mathcal{E})$ are independent.

Solution of Exercise 8:

ad (a):

We define the probability space (Ω, Σ, μ) with

$$\Omega = \{1, 2, 3, 4\},\ \Sigma = \mathcal{P}(\Omega) \text{ and } \mu(A) = \frac{|A|}{4}.$$

This defines clearly a probability space. Moreover let us set

$$A = \{2, 3\} \text{ and } \mathcal{E} = \{\{1, 2\}, \{2, 4\}\}.$$

One can immediately see that A is independent of $\mathcal{E}$. Indeed we have

$$\mu(A \cap \{1, 2\}) = \mu(\{2\}) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = \mu(A)\,\mu(\{1, 2\})$$

and

$$\mu(A \cap \{2, 4\}) = \mu(\{2\}) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = \mu(A)\,\mu(\{2, 4\}).$$

So A is independent of $\mathcal{E}$, but A is not independent of $\sigma(\mathcal{E})$. The last claim follows from $\{2\} \in \sigma(\mathcal{E})$ and

$$\mu(A \cap \{2\}) = \mu(\{2\}) = \frac{1}{4} \neq \frac{1}{2} \cdot \frac{1}{4} = \mu(A)\,\mu(\{2\}).$$

ad (b):

The implication "ii. $\Rightarrow$ i." is obvious (but don't forget to write that down). So we concentrate now on the implication "i. $\Rightarrow$ ii."

We want to use the **principle of good sets**. Our good sets are given by

$$\mathcal{G} = \{B \in \Sigma \colon \mu(A \cap B) = \mu(A)\mu(B)\}.$$

In a first step we prove that $\mathcal{G}$ is a Dynkin system.

First step - $\mathcal{G}$ is a Dynkin system:

- $\emptyset \in \mathcal{G}$ because $\mu(A \cap \emptyset) = \mu(\emptyset) = 0 = \mu(A) \cdot \mu(\emptyset)$.
- If $B \in \mathcal{G} \Rightarrow B^c \in \mathcal{G}$. Indeed

$$\mu(A \cap B^c) = \mu(A) - \mu(A \cap B) = \mu(A) - \mu(A)\mu(B) = \mu(A)(1 - \mu(B)) = \mu(A)\mu(B^c).$$

- If $A_n \in \mathcal{G}$ for every $n \in \mathbb{N}$ are given disjoint sets, then we have to show that $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{G}$. But this is can be derived as follows:

$$\begin{aligned}
\mu\left(A \cap \bigcup_{n\in\mathbb{N}} A_n\right) &= \mu\left(\bigcup_{n\in\mathbb{N}} (A_n \cap A)\right) \\
&= \sum_{n=1}^{\infty} \mu\left(A \cap A_n\right) && A \cap A_n \text{ pairwise disjoint} \\
&= \sum_{n=1}^{\infty} \mu(A)\mu\left(A_n\right) \\
&= \mu(A)\left(\sum_{n=1}^{\infty} \mu\left(A_n\right)\right) \\
&= \mu(A) \cdot \mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) && A_n \text{ pairwise disjoint.}
\end{aligned}$$

This shows that $\mathcal{G}$ is a Dynkin system. In the next step we show $\mathcal{E} \subset \mathcal{G}$.
Step 2 - the inclusion $\mathcal{E} \subset \mathcal{G}$:
This is precisely our assumption i..
Final step - the inclusion $\sigma(\mathcal{E}) \subset \mathcal{G}$:
From the second step we conclude $\mathcal{E} \subset \mathcal{G}$. From the definition of the $d(\mathcal{E})$ and our first step, we conclude $d(\mathcal{E}) \subset \mathcal{G}$. One of our assumptions is that $\mathcal{E}$ is stable under intersections. So we conclude from Dynkin's π-λ theorem $\sigma(\mathcal{E}) = d(\mathcal{E}) \subset \mathcal{G}$.
But $\sigma(\mathcal{E}) \subset \mathcal{G}$ is a reformulation of ii., so the claim follows.

Exercise 9 (*Multiple Choice*) (15*)

Decide which of the following statements are true (no proof needed). For every correct answer you get +1 point and for every wrong answer −1 point. The points of this exercise will be rounded up to zero, if the total number is negative.

(a) The trigonometric polynomials are dense in $(C([0,2\pi]), \|\cdot\|_\infty)$.
☐ true ☐ false

(b) The polynomials are dense in $(C([0,2\pi]), \|\cdot\|_\infty)$.
☐ true ☐ false

(c) $(C_b(M), \|\cdot\|_\infty)$ is a Polish space if (M,d) is a metric space.
☐ true ☐ false

(d) $(C(M), \|\cdot\|_\infty)$ is a Polish space if (M,d) is a compact metric space.
☐ true ☐ false

(e) $\mathcal{B}(\mathbb{R}) \neq \mathcal{P}(\mathbb{R})$.
☐ true ☐ false

(f) $A, B \in \mathcal{B}(\mathbb{R})$, then $A \times B \in \mathcal{B}(\mathbb{R}^2)$.
☐ true ☐ false

(g) $\mathcal{B}(\mathbb{R})$ is generated as a σ-algebra by all finite intervals (a,b) with $a < b$.
☐ true ☐ false

(h) $\mathcal{B}(\mathbb{R}^2)$ is generated as a σ-algebra by all open sets in $\mathbb{R}^2$.
☐ true ☐ false

(i) Given two normed spaces $(\mathbb{R}^N, \|\cdot\|)$ and $(\mathbb{R}^N, \|\cdot\|')$. Then the compact subsets of the two metric spaces coincide.
☐ true ☐ false

(j) If (M,d) and (M,d') are metric spaces on same set M. Let us suppose that (x_n) is a convergent sequence in both spaces, then the limits in (M,d) and in (M,d') coincide.
☐ true ☐ false

(k) If (M,d) is a metric space and (x_n) converges to both x and y, then $x = y$.
☐ true ☐ false

(l) The compact subsets in ℓ^2 are precisely the bounded and closed subsets.
☐ true ☐ false

(m) If Σ is a σ-algebra on Ω and $A_i \in \Sigma$ for all $i \in I$ (here I is an arbitrary index set), then $\bigcup_{i\in I} A_i \in \Sigma$.
☐ true ☐ false

(n) If Σ is a σ-algebra on Ω and $A_n \in \Sigma$ for all $n \in \mathbb{N}$, then $\bigcap_{n\in\mathbb{N}} A_n \in \Sigma$.
☐ true ☐ false

(o) The Lebesgue measure λ on $\mathbb{R}$ assigns to every $A \subset \mathbb{R}$ a "length" $\lambda(A) \geq 0$.
☐ true ☐ false

Solution of Exercise 9:

(a) The trigonometric polynomials are dense in $(C([0,2\pi]), \|\cdot\|_\infty)$.
☐ true ☒ false

(b) The polynomials are dense in $(C([0,2\pi]), \|\cdot\|_\infty)$.
☒ true ☐ false

(c) $(C_b(M), \|\cdot\|_\infty)$ is a Polish space if (M,d) is a metric space.
☐ true ☒ false

(d) $(C(M), \|\cdot\|_\infty)$ is a Polish space if (M,d) is a compact metric space.
☒ true ☐ false

(e) $\mathcal{B}(\mathbb{R}) \neq \mathcal{P}(\mathbb{R})$.
☒ true ☐ false

(f) $A, B \in \mathcal{B}(\mathbb{R})$, then $A \times B \in \mathcal{B}(\mathbb{R}^2)$.

☒ true ☐ false

(g) $\mathcal{B}(\mathbb{R})$ is generated as a σ-algebra by all finite intervals (a, b) with $a < b$.

☒ true ☐ false

(h) $\mathcal{B}(\mathbb{R}^2)$ is generated as a σ-algebra by all open sets in $\mathbb{R}^2$.

☒ true ☐ false

(i) Given two normed spaces $(\mathbb{R}^N, \|\cdot\|)$ and $(\mathbb{R}^N, \|\cdot\|')$. Then the compact subsets of the two metric spaces coincide.

☐ true ☒ false

(j) If (M, d) and (M, d') are metric spaces on same set M. Let us suppose that (x_n) is a convergent sequence in both spaces, then the limits in (M, d) and in (M, d') coincide.

☐ true ☒ false

(k) If (M, d) is a metric space and (x_n) converges to both x and y, then $x = y$.

☒ true ☐ false

(l) The compact subsets in ℓ^2 are precisely the bounded and closed subsets.

☐ true ☒ false

(m) If Σ is a σ-algebra on Ω and $A_i \in \Sigma$ for all $i \in I$ (here I is an arbitrary index set), then $\bigcup_{i\in I} A_i \in \Sigma$.

☐ true ☒ false

(n) If Σ is a σ-algebra on Ω and $A_n \in \Sigma$ for all $n \in \mathbb{N}$, then $\bigcap_{n\in\mathbb{N}} A_n \in \Sigma$.

☒ true ☐ false

(o) The Lebesgue measure λ on $\mathbb{R}$ assigns to every $A \subset \mathbb{R}$ a "length" $\lambda(A) \geq 0$.

☐ true ☒ false