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Exercise Sheet 12

Applied Analysis

Discussion on Thursday 23-1-2014 at 16ct

Exercise 1 (*Measure with density*) (3+2+5)

Let (Ω, Σ, μ) be a measure space and $f\colon \Omega \to [0, \infty)$ an integrable function.

(a) We construct a measure, which we call $f \cdot \mu$, on (Ω, Σ) by setting

$$(f \cdot \mu)(A) := \int_A f \, d\mu$$

for every $A \in \Sigma$. Show that $f \cdot \mu$ is indeed a measure (see Exercise 3.65).

(b) Show that $f \cdot \mu$ is a probability measure if and only if

$$\int_\Omega f \, d\mu = 1.$$

(c) Given another measurable function $g\colon \Omega \to [0, \infty)$ show (see Exercise 3.72 and follow the proof of Theorem 3.73)

$$\int_\Omega g \, d(f \cdot \mu) = \int_\Omega gf \, d\mu.$$

We say that a measure ν **has density** $f\colon \Omega \to [0, \infty)$ **with respect to** μ, if $\nu = f \cdot \mu$.

Exercise 2 (*Some probability distributions*) (5+5+5)

(a) Show that the following measures $f \cdot \lambda$ with ($\alpha > 0$ fixed and see the exercise before)

i. $f(x) = \dfrac{1}{\pi(1 + x^2)}$ ii. $f(x) = \mathbb{1}_{[0,\infty)}(x)\alpha e^{-\alpha x}$

are probability measures.

(b) Find some $c > 0$ (depending on $\alpha > 0$) such that a measure $f \cdot \zeta$ which has the density $f(k) = c\frac{\alpha^k}{k!}$ with respect to the counting measure ζ on $\mathbb{N}_0$ defines a probability measure.

Exercise 3 (5+5+5+5)

Calculate the following integrals if they exist ($\alpha > 0$)

(a) $\displaystyle\int_\mathbb{N} \frac{k\alpha^k}{k!} e^{-\alpha} \, d\zeta(k)$ (b) $\displaystyle\int_{[0,\infty)} x\alpha e^{-\alpha x} \, d\lambda(x)$

(c) $\displaystyle\int_\mathbb{R} \frac{x}{\pi(1 + x^2)} \, d\lambda(x)$ (d) $\displaystyle\int_{[0,2]} g \, d\lambda$

Here $g\colon [0, 2] \to \mathbb{R}$ is given by

$$g(x) = \begin{cases} x^2 & \text{, for } x < 1 \\ x(2 - x) & \text{, for } x \geq 1 \end{cases}$$

Remark: In the last exercise of this sheet we interpret these integrals as expected values of certain probability distributions.

please turn over!

Exercise 4 (*Two basic formulas for expected values*) (2+2+10)
Let a probability space $(\Omega, \Sigma, \mathbb{P})$ and a real valued random variable $X\colon \Omega \to \mathbb{R}$ be given. We say that X **has a finite expected value**, if

$$\int_\Omega |X| \, d\mathbb{P} < \infty$$

and call in this case

$$\mathbb{E}X = \int_\Omega X \, d\mathbb{P}$$

the **expected value**.

(a) Prove (use Theorem 3.73) that X has finite expected value, iff

$$\int_\mathbb{R} |x| \, d\mathbb{P}_X(x) < \infty$$

and that in this case one has

$$\mathbb{E}X = \int_\mathbb{R} x \, d\mathbb{P}_X(x).$$

Here $\mathbb{P}_X$ is the push forward of $\mathbb{P}$ under X on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ given by $\mathbb{P}_X(A) = \mathbb{P}\left(X^{-1}(A)\right)$ for all $A \in \mathcal{B}(\mathbb{R})$ (compare this with the lecture).

(b) Let us suppose that X has a density f with respect to μ, i.e. $\mathbb{P}_X = f \cdot \mu$ for some measure μ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and some integrable function $f\colon \mathbb{R} \to \mathbb{R}$. Prove that X has finite expected value, iff

$$\int_\mathbb{R} |x| f(x) \, d\mu(x) < \infty$$

and that in this case one has

$$\mathbb{E}X = \int_\mathbb{R} x f(x) \, d\mu(x).$$

(c) Interpret the integrals in Exercise 3 as the expected values of some random variables. In particular, what is the probability distribution for the last integral (Is it a probability distribution?)?