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Exercise Sheet 14

Applied Analysis

Discussion on Thursday 6-2-2014 at 16ct

Exercise 1 (*Construction of random variables*) (5+5+10)

We construct now a random variable with a given cumulative distribution function $F\colon \mathbb{R} \to [0,1]$. Here F is an arbitrary given monotonically increasing function, with

$$\lim_{x\to\infty} F(x) = 1, \quad \lim_{x\to-\infty} F(x) = 0$$

and such that F is right-continuous (compare this to Sheet 9 Exercise 1).

(a) Show that $([0,1], \mathcal{B}([0,1]), \lambda)$ is a probability space, $X\colon [0,1] \to \mathbb{R}$ given by $X(\omega) = \omega$ is a random variable and calculate the cumulative distribution function F_X.

(b) Let us define the **quantile function** $Q_F\colon [0,1] \to \overline{\mathbb{R}}$ by

$$Q_F(p) = \inf\{x \in \mathbb{R}\colon p \le F(x)\}.$$

Prove that Q_F is measurable.

(c) Prove that $Y = Q_F \circ X$ (see (a) and (b)) is an almost surely finite random variable on the probability space $([0,1], \mathcal{B}([0,1]), \lambda)$ and that Y has F as its cumulative distribution function (i.e. $F = F_Y$).

Exercise 2 (*Fubini's theorem*) (5+5+5+5)

Calculate the following integrals, if they exist.

(a) $\displaystyle\int_0^1 \int_x^1 \frac{y}{y^3+1}\, d\lambda(y)\, d\lambda(x)$

(b) $\displaystyle\int_{\{(k,l)\in\mathbb{N}^2\colon\ l\le k\}} \frac{1}{2^k}\, d(\zeta\otimes\zeta)(k,l)$

(c) $\displaystyle\int_{-1}^1 \int_{-1}^1 \frac{x(y+2y^2)}{e^y + |y| + y^2 + |\sin y|}\, d\lambda(y)\, d\lambda(x)$

(d) $\displaystyle\int_{[0,1]^2} \frac{x-y}{(x+y)^3}\, d\lambda^2(x,y)$