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Exercise Sheet 15

Applied Analysis

Discussion on Thursday 13-2-2014 at 16ct

Exercise 1 (*Hilbert spaces*) (5+5+10)

(a) Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space and let $\Sigma' \subset \Sigma$ be another σ-algebra. Show that $L^2(\Omega, \Sigma', \mathbb{P}|_{\Sigma'})$ is a closed subspace of $L^2(\Omega, \Sigma, \mathbb{P})$.

(b) Show that the orthogonal projection of some $X \in L^2(\Omega, \Sigma, \mathbb{P})$ on $L^2(\Omega, \Sigma', \mathbb{P}|_{\Sigma'})$ with $\Sigma' = \{\emptyset, \Omega\}$ is $\mathbb{E}X$.

(c) Given some exponentially distributed random variable X on a probability space $(\Omega, \Sigma, \mathbb{P})$ and set $\Sigma' = \sigma(\{X \leq 1\})$. Show that X lies in $L^2(\Omega, \Sigma, \mathbb{P})$ and calculate the orthogonal projection of X on $L^2(\Omega, \Sigma', \mathbb{P})$.

Exercise 2 (L^p-*spaces*) (5+5+5+5)

(a) Given a probability space $(\Omega, \Sigma, \mathbb{P})$. Then we define for $p, q \in [1, \infty]$ with $p \geq q$

$$T\colon L^p(\Omega, \Sigma, \mathbb{P}) \to L^q(\Omega, \Sigma, \mathbb{P}), \quad T\colon f \mapsto f.$$

Show that T is well-defined, linear and continuous.

(b) Is the following map

$$T\colon L^1(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), \lambda_2) \to L^1(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda), \text{ with } (Tf)(x) = f(x, 0) \text{ for all } x \in \mathbb{R}$$

well-defined?

(c) Let a measure space (Ω, Σ, μ) be given. Then we define for fixed $g \in L^q(\Omega)$

$$T\colon L^p(\Omega, \Sigma, \mu) \to L^1(\Omega, \Sigma, \mu), \quad T\colon f \mapsto f \cdot g.$$

Show that T is well-defined, linear and continuous. Here $p, q \in [1, \infty]$ are arbitrary with $p^{-1} + q^{-1} = 1$.

(d) Which of the following claims are true? Give counterexamples for the wrong statements.

i. The map $\|\|_1\colon \mathcal{L}^1(\Omega, \Sigma, \mu) \to [0, \infty)$ given by

$$\|f\|_1 = \int_\Omega |f| \, d\mu$$

is a norm, that makes $\mathcal{L}^1(\Omega, \Sigma, \mu)$ into a Banach space. Here (Ω, Σ, μ) is an arbitrary measure space.

□ true □ false

ii. $L^p(\Omega, \Sigma, \mu)$ is a Banach space with the norm given by

$$\|f\|_p = \sqrt[p]{\int_\Omega |f|^p \, d\mu}.$$

Here (Ω, Σ, μ) is an arbitrary measure space and $p \in [1, \infty)$.

□ true □ false

iii. Given a sequence $(f_n)_{n\in\mathbb{N}}$ in $L^1(\Omega, \Sigma, \mu)$ which converges in $L^1(\Omega, \Sigma, \mu)$, then $(f_n)_{n\in\mathbb{N}}$ converges almost everywhere in Ω.

□ true □ false

please turn over!

iv. Given a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{L}^1(\Omega, \Sigma, \mu)$ which converges almost everywhere in Ω to some $f \in \mathcal{L}^1(\Omega, \Sigma, \mu)$, then $(f_n)_{n\in\mathbb{N}}$ converges in $L^1(\Omega, \Sigma, \mu)$ to f.

☐ true ☐ false

v. Given a sequence $(f_n)_{n\in\mathbb{N}}$ in $L^p(\Omega, \Sigma, \mu) \cap L^q(\Omega, \Sigma, \mu)$ which converges in $L^p(\Omega, \Sigma, \mu)$ to f and in $L^q(\Omega, \Sigma, \mu)$ to g (for arbitrary $p, q \in [1, \infty]$), then the limits are equal (i.e. $f = g$ almost everywhere in Ω).

☐ true ☐ false