ulm university universität uulm

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# 2. Mock Exam

## Applied Analysis

Discussion on Wednesday 12-2-2014 at 16 ct

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100% corresponds to 100 points (you can achieve 120 points). In the final exam you are allowed to use a double-sided handwritten A4 sheet. This is intended to be solved in 120 minutes.

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**Exercise 1** (*Basic properties of metric spaces*) (5+5+5+5)

Given a metric space $(M, d)$. All the spaces are equipped with the usual metrics if no metric is specified.

(a) Define the property “complete” for a metric space $(M, d)$.

(b) Given a subset $N \subset M$. Which implication is true (no proofs required)?

i. $N$ compact $\Rightarrow$ $N$ is complete.
ii. $N$ compact $\Rightarrow$ $N$ is separable.
iii. $N$ compact $\Rightarrow$ $N$ is totally bounded.
iv. $N$ compact $\Rightarrow$ $N$ is closed.
v. $M$ is compact and $N$ is closed $\Rightarrow$ $N$ is compact.

(c) Which of the following sets are compact (no proof required)?

i. $[0,1] \times \{(x,y) \in \mathbb{R}^2 : \ |x| + |y| \leq 1\} \subset \mathbb{R}^3$.
ii. $[0,1] \times [0,1] \times [0,1) \subset \mathbb{R}^3$.
iii. $\mathbb{R}$ with the discrete metric.
iv. A finite set $X \subset M$ for an arbitrary metric space $(M, d)$.
v. The closed unit ball $\overline{B(0,1)}$ of $\ell^\infty$.

(d) Prove your claim in (c)iv. **or** (c)v. (give enough details!).

**Exercise 2** (*Integrable and measurable functions*) (10+10)

(a) Is the function $f\colon [0,1]^2 \to \mathbb{R}$

$$f(x,y) = \begin{cases} (x-y)^2 & , \text{for } |x|^2 y + |y| < 1 \\ 0 & , \text{for } |x|^2 y + |y| \geq 1 \end{cases}$$

i. continuous (with the Euclidean metric on domain and codomain)?
ii. measurable (with the Borel-$\sigma$-algebra on domain and codomain)?

Give a complete argument!

(b) Which of the following functions are integrable (prove your claim)?

i. $f(x) = x(1+x^2)^{-1}$ on the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$.
ii. $f(n) = \frac{1}{2^n}$ on the measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \zeta)$.

**please turn over!**

**Exercise 3** (*Calculating some Lebesgue integrals*) (5+20)

(a) State the theorem of Fubini and the theorem of Tonelli.

(b) Calculate the following Lebesgue integrals respectively limit of Lebesgue integrals (You don't have to prove that the functions are integrable! You can assume this).

i. $\displaystyle\int_{\mathbb{R}} f \, d\mu$ ii. $\displaystyle\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{xy^2}{\left(1+e^{x^2}\right)\left(1+y^8\right)} \, d\lambda(y) \, d\lambda(x)$

iii. $\displaystyle\lim_{n\to\infty} \int_{\mathbb{R}} x^{-2} \cdot \mathbb{1}_{[1,\infty)}(x) \cdot \exp\left(-n^{-1}x\right) \, d\lambda(x)$

Here $\lambda$ is the Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $\mu = 4\delta_0 + 3\lambda$ is a measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and $f \colon \mathbb{R} \to \mathbb{R}$ is given by

$$f(x) = \begin{cases} 1 & \text{, for } x = 0 \\ 2 & \text{, for } x = 1 \\ x & \text{, for } x \in (0,1) \\ 0 & \text{, otherwise.} \end{cases}$$

**Exercise 4** (*Linear maps and Hilbert spaces*) (5+5+5)

(a) Every Hilbert space has a canonical norm.

i. Define this norm.

ii. Is the Hilbert space complete in this norm?

(b) We define a function $T \colon L^2([0,1]) \to \mathbb{R}$ by

$$Tf = \int_{[0,1]} f \, d\lambda$$

Prove that $T$ is linear **and** well-defined.

(c) Show that $T$ is continuous.

**Exercise 5** (*Principle of good sets*) (5+10+5)

Let $\mu$ be a finite measure on $([0,1], \mathcal{B}([0,1]))$. We will show by using the principle of good sets that for all $A \in \mathcal{B}([0,1])$ and all $\epsilon > 0$ there exist an open set $U \subset [0,1]$ and a compact set $C \subset [0,1]$ such that $C \subset A \subset U$ and $\mu(U \backslash C) < \epsilon$. Let us define

$$\mathcal{G} = \{A \in \mathcal{B}([0,1]) | \forall \epsilon > 0 \ \exists U \supset A \text{ open subset of } [0,1] \ \exists C \subset A \text{ compact}: \ \mu(U \backslash C) < \epsilon\}.$$

(a) Prove that every set of the form $[a,b] \subset [0,1]$ lies in $\mathcal{G}$.

(b) Show that $\mathcal{G}$ is a Dynkin system.

(c) Show the claim mentioned above (i.e. for all $A \in \mathcal{B}(\mathbb{R})$ and all $\epsilon > 0$ there exists some open set $U \subset [0,1]$ and a compact set $C \subset [0,1]$ such that $C \subset A \subset U$ and $\mu(U \backslash C) < \epsilon$.).

**please turn over!**

**Exercise 6** (*Multiple Choice*) (20*)

Decide which of the following statements are true (no proof needed). For every correct answer you get +2 points and for every wrong answer −1 point. The points of this exercise will be rounded up to zero, if the total number is negative.

(a) $\mathbb{Q} \times \{0, 1, 3\} \times \mathbb{Z}$ is countable.
☐ true ☐ false

(b) $\mathbb{R}$ is countable.
☐ true ☐ false

(c) A (non-empty) subset of a countable set is countable.
☐ true ☐ false

(d) $\mathcal{P}(A)$ is uncountable if $A$ is not finite.
☐ true ☐ false

(e) $\{1, 2, 3, 4, \mathbb{R}\}$ is countable.
☐ true ☐ false

(f) Every continuous function $f : \mathbb{R} \to \mathbb{R}$ is integrable.
☐ true ☐ false

(g) Every Cauchy sequence in $L^2(\Omega, \Sigma, \mu)$ has a almost everywhere convergent subsequence. By $(\Omega, \Sigma, \mu)$ we mean here an arbitrary measure space.
☐ true ☐ false

(h) Given a Hilbert space $H$. Then

$$\|x\|_H = \sup_{\|\varphi\|_{H^*} \leq 1} |\varphi(x)|$$

holds for all $x \in H$.
☐ true ☐ false

(i) Every normed vector space $V$ which obeys the parallelogram identity is a Hilbert space (i.e. there exists a inner product which makes $V$ into a Hilbert space and the norm induced by the inner product equals the given norm).
☐ true ☐ false

(j) Given a Hilbert space. Then for every finite-dimensional subspace one can find an orthogonal projection on this subspace.
☐ true ☐ false