ulm university universität uulm

Manfred Sauter
Manuel Bernhard
winter semester 2013/2014

2. Mock Exam

Applied Analysis

Discussion on Wednesday 12-2-2014 at 16 ct

100% corresponds to 100 points (you can achieve 120 points). In the final exam you are allowed to use a double-sided handwritten A4 sheet. This is intended to be solved in 120 minutes.

Contents

Exercise 1 (*Basic properties of metric spaces*) (5+5+5+5)
Given a metric space (M, d). All the spaces are equipped with the usual metrics if no metric is specified.

(a) Define the property "complete" for a metric space (M, d).
(b) Given a subset $N \subset M$. Which implication is true (no proofs required)?

  i. N compact $\Rightarrow$ N is complete.
  ii. N compact $\Rightarrow$ N is separable.
  iii. N compact $\Rightarrow$ N is totally bounded.
  iv. N compact $\Rightarrow$ N is closed.
  v. M is compact and N is closed $\Rightarrow$ N is compact.

(c) Which of the following sets are compact (no proof required)?

  i. $[0, 1] \times \{(x, y) \in \mathbb{R}^2 : \ |x| + |y| \leq 1\} \subset \mathbb{R}^3$.
  ii. $[0, 1] \times [0, 1] \times [0, 1) \subset \mathbb{R}^3$.
  iii. $\mathbb{R}$ with the discrete metric.
  iv. A finite set $X \subset M$ for an arbitrary metric space (M, d).
  v. The closed unit ball $\overline{B(0, 1)}$ of ℓ^∞.

(d) Prove your claim in (c)iv. **or** (c)v. (give enough details!).

Solution of Exercise 1:
ad (a):
A metric space (M, d) is called complete, iff every Cauchy sequence in (M, d) converges in (M, d).
ad (b):
ad i.: true
ad ii.: true
ad iii.: true
ad iv.: true
ad v.: true
ad (c):
ad i.: compact (**not part of solution:** closed and bounded subset of a Euclidean space)
ad ii.: not compact (**not part of solution:** not closed)
ad iii.: not compact (**not part of solution:** infinite set with discrete metric)
ad iv.: compact (**not part of solution:** finite set with discrete metric)
ad v.: not compact (**not part of solution:** A basic counterexample for the false statement: Closed and bounded subsets are always compact in Banach spaces)
ad (d):
We expect here only **one** proof in your solution but give here both.
ad (c)iv.: Given a sequence $(x_n)_{n\in\mathbb{N}}$ in M, then there exists an element $m \in M$, which occurs infinitely often in the sequence (because M is finite). Hence we can choose a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ with $x_{n_k} = m$ for all $k \in \mathbb{N}$. As every constant sequence converges, the sequence $(x_{n_k})_{k\in\mathbb{N}}$ converges in (M, d) to m. Because the sequence $(x_n)_{n\in\mathbb{N}}$ was arbitrary and we found a convergent subsequence, our space (M, d) is compact.
ad (c)v.: Let us suppose that $\overline{B(0, 1)}$ is compact. Then we can find a subsequence $(e_{n_k})_{k\in\mathbb{N}}$ of the sequence $(e_n)_{n\in\mathbb{N}}$ in $\overline{B(0, 1)}$, which is convergent. Hence $(e_{n_k})_{k\in\mathbb{N}}$ is a Cauchy sequence and therefore

$$\|e_{n_k} - e_{n_l}\|_\infty < \frac{1}{2}$$

for k, l large enough. On the other side we get for $k \neq l$

$$\|e_{n_k} - e_{n_l}\|_\infty = 1$$

by definition of the norm. So we get for $k \neq l$ large enough the contradiction

$$1 = \|e_{n_k} - e_{n_l}\|_\infty < \frac{1}{2}.$$

Now we conclude by contradiction that $\overline{B(0,1)}$ is not compact.

Exercise 2 (*Integrable and measurable functions*) (10+10)

(a) Is the function $f\colon [0,1]^2 \to \mathbb{R}$

$$f(x,y) = \begin{cases} (x-y)^2 & \text{, for } |x|^2 y + |y| < 1 \\ 0 & \text{, for } |x|^2 y + |y| \geq 1 \end{cases}$$

i. continuous (with the Euclidean metric on domain and codomain)?

ii. measurable (with the Borel-σ-algebra on domain and codomain)?

Give a complete argument!

(b) Which of the following functions are integrable (prove your claim)?

i. $f(x) = x(1+x^2)^{-1}$ on the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$.

ii. $f(n) = \frac{1}{2^n}$ on the measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \zeta)$.

Solution of Exercise 2:

ad (a):

ad i.: $x_n = 0$ and $y_n = 1 - \frac{1}{n}$, then $(x_n, y_n) \to (0,1)$ for $n \to \infty$ but

$$f(x_n, y_n) = y_n^2 \to 1 \neq 0 = f(0,1)$$

for $n \to \infty$. Hence f is not continuous.

ad ii.: We define the functions

$$g_1\colon \mathbb{R} \to \mathbb{R}, \quad g_1\colon (x,y) \mapsto (x-y)^2,$$

$$g_2\colon \mathbb{R} \to \mathbb{R}, \quad g_2\colon (x,y) \mapsto 0$$

and

$$g_3\colon \mathbb{R} \to \mathbb{R}, \quad g_3\colon (x,y) \mapsto |x|^2 y + |y|.$$

We remark that all these functions are continuous and hence measurable. So for every measurable set $A \subset \mathbb{R}$ we conclude (just translate the logical expression into set-theoretic formulas)

$$f^{-1}(A) = \Big[g_1^{-1}(A) \cap g_3((-\infty, 1))\Big] \cap \Big[g_2^{-1}(A) \cap g_3([1,\infty))\Big]$$

is measurable as the union and intersection of measurable sets. This proves that f is measurable.

ad (b):

ad i.: The function is continuous and hence measurable. The estimate

$$\Big|x(1+x^2)^{-1}\Big| \geq \Big|x(2x^2)^{-1}\Big| = |2x|^{-1}$$

for $x > 1$ shows that the function is not integrable. Indeed (using the monotonicity and the monotone convergence theorem)

$$\begin{aligned} \int_{\mathbb{R}} \Big|x(1+x^2)^{-1}\Big| \, d\lambda(x) &\geq \int_{[1,\infty)} \Big|x(1+x^2)^{-1}\Big| \, d\lambda(x) \geq \int_{[1,\infty]} |2x|^{-1} \, d\lambda(x) \\ &= \lim_{n\to\infty} \int_{[1,n]} |2x|^{-1} \, d\lambda(x) = \lim_{n\to\infty} \mathsf{R}\text{-}\int_1^n |2x|^{-1} \, dx = \lim_{n\to\infty} [2\log|x|]_1^n \\ &= \infty. \end{aligned}$$

ad ii.: As the σ-algebra equals the power set, all functions are measurable. Moreover the monotone convergence theorem shows

$$\int_{\mathbb{N}} \frac{1}{2^n} \, d\zeta(n) = \lim_{m\to\infty} \int_{\{1,\ldots,m\}} \frac{1}{2^n} \, d\zeta(n) = \lim_{m\to\infty} \sum_{n=1}^{m} \frac{1}{2^n} \zeta(\{n\}) = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1 < \infty.$$

Hence f is integrable.

Exercise 3 (*Calculating some Lebesgue integrals*) (5+20)

(a) State the theorem of Fubini and the theorem of Tonelli.

(b) Calculate the following Lebesgue integrals respectively limit of Lebesgue integrals (You don't have to prove that the functions are integrable! You can assume this).

i. $\displaystyle\int_{\mathbb{R}} f\,d\mu$ ii. $\displaystyle\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{xy^2}{(1+e^{x^2})\,(1+y^8)}\,d\lambda(y)\,d\lambda(x)$

iii. $\displaystyle\lim_{n\to\infty}\int_{\mathbb{R}} x^{-2}\cdot \mathbb{1}_{[1,\infty)}(x)\cdot \exp\left(-n^{-1}x\right)\,d\lambda(x)$

Here λ is the Lebesgue measure on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$, $\mu = 4\delta_0 + 3\lambda$ is a measure on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$ and $f\colon \mathbb{R}\to\mathbb{R}$ is given by

$$f(x) = \begin{cases} 1 & \text{, for } x = 0 \\ 2 & \text{, for } x = 1 \\ x & \text{, for } x \in (0,1) \\ 0 & \text{, otherwise.} \end{cases}$$

Solution of Exercise 3:

ad (a):

Fubini: Given two σ-finite measure spaces $(\Omega_1,\Sigma_1,\mu_1)$ and $(\Omega_2,\Sigma_2,\mu_2)$ and a $f\colon \Omega_1\times\Omega_2\to\mathbb{K}$ in $\mathcal{L}^1(\Omega_1\times\Omega_2,\Sigma_1\otimes\Sigma_2,\mu_1\otimes\mu_2)$ Then

- $y\mapsto f(x,y)$ lies in $\mathcal{L}^1(\Omega_2,\Sigma_2,\mu_2)$ for almost all $x\in\Omega_1$,
- $x\mapsto f(x,y)$ lies in $\mathcal{L}^1(\Omega_1\Sigma_1,\mu_1)$ for almost all $y\in\Omega_2$,
- $y\mapsto \int_{\Omega_1} f(x,y)\,d\mu_1(x)$ lies in $\mathcal{L}^1(\Omega_2,\Sigma_2,\mu_2)$,
- $x\mapsto \int_{\Omega_2} f(x,y)\,d\mu_2(y)$ lies in $\mathcal{L}^1(\Omega_1,\Sigma_1,\mu_1)$
- and moreover

$$\int_{\Omega_1}\int_{\Omega_2} f(x,y)\,d\mu_2(y)d\mu_1(x) = \int_{\Omega_1\times\Omega_2} f\ d\mu_1\otimes\mu_2 = \int_{\Omega_2}\int_{\Omega_1} f(x,y)\,d\mu_1(x)d\mu_2(y).$$

Tonelli: Given two σ-finite measure spaces $(\Omega_1,\Sigma_1,\mu_1)$ and $(\Omega_2,\Sigma_2,\mu_2)$ and a $\Sigma_1\otimes\Sigma_2$-measurable **non-negative** function $f\colon\Omega_1\times\Omega_2\to\overline{\mathbb{R}}$. Then

- $y\mapsto f(x,y)$ is a Σ_2-measurable function for all $x\in\Omega_1$,
- $x\mapsto f(x,y)$ is a Σ_1-measurable function for all $y\in\Omega_2$,
- $y\mapsto \int_{\Omega_1} f(x,y)\,d\mu_1(x)$ is Σ_2-measurable,
- $x\mapsto \int_{\Omega_2} f(x,y)\,d\mu_2(y)$ is Σ_1-measurable
- and moreover

$$\int_{\Omega_1}\int_{\Omega_2} f(x,y)\,d\mu_2(y)d\mu_1(x) = \int_{\Omega_1\times\Omega_2} f\ d\mu_1\otimes\mu_2 = \int_{\Omega_2}\int_{\Omega_1} f(x,y)\,d\mu_1(x)d\mu_2(y).$$

ad (b):

ad i.: We give to proofs.

A first proof:

$$\int_{\mathbb{R}} f\,d\mu = 3\int_{\mathbb{R}} f\,d\lambda + 4\int_{\mathbb{R}} f\,d\delta_0$$

As f is almost everywhere equal to the simple function $\mathbb{1}_{\{0\}}$ with respect to δ_0, we get

$$\int_{\mathbb{R}} f \, d\delta_0 = 1.$$

Moreover, f is almost everywhere equal to the function $g\colon x \mapsto x\mathbb{1}_{[0,1]}(x)$ with respect to λ. Hence

$$\int_{\mathbb{R}} f \, d\lambda = \int_{\mathbb{R}} g \, d\lambda = \mathsf{R}\text{-}\int_0^1 x \, dx = \frac{1}{2}x^2\Big|_0^1 = \frac{1}{2}.$$

We conclude

$$\int_{\mathbb{R}} f \, d\mu = 3\int_{\mathbb{R}} f \, d\lambda + 4\int_{\mathbb{R}} f \, d\delta_0 = \frac{11}{2}.$$

A second proof: We use $f(x) = x\mathbb{1}_{(0,1)} + \mathbb{1}_{\{0\}} + 2\mathbb{1}_{\{1\}}$ for $x \in \mathbb{R}$. Hence

$$\int_{\mathbb{R}} f \, d\mu = \int_{(0,1)} x \, d\mu(x) + \int_{\{0\}} 1 \, d\mu + \int_{\{1\}} 2 \, d\mu.$$

The last to integrals have a simple function as integrand, hence

$$\int_{\mathbb{R}} f \, d\mu = \int_{(0,1)} x \, d\mu(x) + \mu(\{0\}) + 2\mu(\{0\}) = \int_{(0,1)} x \, d\mu(x) + 4.$$

Moreover, $\mu = 3\lambda$ on $(0,1)$, so we conclude

$$\int_{\mathbb{R}} f \, d\mu = 4 + 3\int_{(0,1)} x \, d\lambda(x) = 3\left[\frac{1}{2}x^2\right]_0^1 + 4 = \frac{11}{2}.$$

ad ii.: We apply Fubini's theorem

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{xy^2}{\left(1+e^{x^2}\right)\left(1+y^8\right)} \, d\lambda(y) \, d\lambda(x) &= \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{xy^2}{\left(1+e^{x^2}\right)\left(1+y^8\right)} \, d\lambda(x) \, d\lambda(y) \\ &= \int_{\mathbb{R}} 0 \, d\lambda(y) = 0\end{aligned}$$

Here we used, that

$$x \mapsto \frac{xy^2}{\left(1+e^{x^2}\right)\left(1+y^8\right)}$$

is an odd function for fixed $y \in \mathbb{R}$ and therefore

$$\int_{\mathbb{R}} \frac{xy^2}{\left(1+e^{x^2}\right)\left(1+y^8\right)} \, d\lambda(x) = 0.$$

ad iii.: We give two alternative proofs (only one is needed!).
Proof 1: Using the monotone convergence theorem:

$$f_n\colon x \mapsto x^{-2} \cdot \mathbb{1}_{[1,\infty)}(x) \cdot \exp\left(-n^{-1}x\right)$$

is a **monotonically increasing sequence** of measurable **non-negative** functions. So the monotone convergence theorem shows

$$\begin{aligned}\lim_{n\to\infty} \int_{\mathbb{R}} x^{-2} \cdot \mathbb{1}_{[1,\infty)}(x) \cdot \exp\left(-n^{-1}x\right) \, d\lambda(x) &= \int_{\mathbb{R}} \lim_{n\to\infty} x^{-2} \cdot \mathbb{1}_{[1,\infty)}(x) \cdot \exp\left(-n^{-1}x\right) \, d\lambda(x) \\ &= \int_{\mathbb{R}} x^{-2} \cdot \mathbb{1}_{[1,\infty)}(x) \, d\lambda(x) = \lim_{n\to\infty} \int_{[1,n]} x^{-2} \, d\lambda(x) \\ &= \lim_{n\to\infty} \mathsf{R}\text{-}\int_1^n x^{-2} \, dx = \lim_{n\to\infty} \left[-x^{-1}\right]_1^n \\ &= \lim_{n\to\infty} \left(-n^{-1} + 1\right) = 1\end{aligned}$$

We remark, that we used in the second line the monotone convergence theorem again.
Proof 2: Using the dominated convergence theorem: The sequence of functions

$$f_n \colon x \mapsto x^{-2} \cdot \mathbb{1}_{[1,\infty)}(x) \cdot \exp\left(-n^{-1}x\right)$$

is uniformly bounded by a integrable function. Indeed

$$\left|x^{-2} \cdot \mathbb{1}_{[1,\infty)}(x) \cdot \exp\left(-n^{-1}x\right)\right| \leq x^{-2} \cdot \mathbb{1}_{[1,\infty)}(x)$$

and that the bound is integrable follows from

$$\int_{\mathbb{R}} x^{-2}\mathbb{1}_{[1,\infty)}(x)\, d\lambda(x) = \lim_{n\to\infty} \int_{[1,n]} x^{-2}\, d\lambda(x) = \lim_{n\to\infty} \mathsf{R}\text{-}\int_1^n x^{-2}\, dx = \lim_{n\to\infty} \left[-x^{-1}\right]_1^n = 1,$$

where we used the monotone convergence theorem. So an application of the dominated convergence theorem gives

$$\begin{aligned}\lim_{n\to\infty} \int_{\mathbb{R}} x^{-2} \cdot \mathbb{1}_{[1,\infty)}(x) \cdot \exp\left(-n^{-1}x\right)\, d\lambda(x) &= \int_{\mathbb{R}} \lim_{n\to\infty} x^{-2} \cdot \mathbb{1}_{[1,\infty)}(x) \cdot \exp\left(-n^{-1}x\right)\, d\lambda(x) \\ &= \int_{\mathbb{R}} x^{-2} \cdot \mathbb{1}_{[1,\infty)}(x)\, d\lambda(x) = 1.\end{aligned}$$

Exercise 4 (*Linear maps and Hilbert spaces*) (5+5+5)

(a) Every Hilbert space has a canonical norm.

  i. Define this norm.

  ii. Is the Hilbert space complete in this norm?

(b) We define a function $T\colon L^2([0,1]) \to \mathbb{R}$ by

$$Tf = \int_{[0,1]} f \, d\lambda$$

Prove that T is linear **and** well-defined.

(c) Show that T is continuous.

Solution of Exercise 4:

ad (a):

ad i.: This norm $\|\cdot\|\colon H \to [0,\infty)$ is defined by

$$\|x\| = \sqrt{(x|x)}$$

Here $(\cdot|\cdot)$ is the inner product and x is an element in the Hilbert space H.

ad ii.: Literally by the definition of a Hilbert space this is true.

ad (b):

well-defined: Every $f \in L^2([0,1])$ lies in $L^1([0,1])$ too (this is true for general finite measure spaces, e.g. probability spaces). And the integral of $f \in L^1([0,1])$ is well-defined by the lecture (i.e. the integral is independent of the choice of the representative g of $f = [g]$.).

linear: For every $f, g \in L^2([0,1])$ and every $\mu \in \mathbb{K}$ we get

$$T(\mu f + g) = \int_{[0,1]} (\mu f + g) \, d\lambda = \mu \int_{[0,1]} f \, d\lambda + \int_{[0,1]} g \, d\lambda = \mu Tf + Tg.$$

Hence T is linear.

ad (c):

Boundedness means in this context:

$$\exists C > 0 \; \forall f \in L^2([0,1]) : \; |Tf| \leq C \|f\|_{L^2([0,1])}$$

Given $f \in L^2([0,1])$, Hölder's inequality implies

$$\|f\|_{L^1([0,1])} = \|f \cdot 1\|_{L^1([0,1])} \leq \|f\|_{L^2([0,1])} \|1\|_{L^2([0,1])} = \|f\|_{L^2([0,1])}.$$

Hence

$$|Tf| = \left| \int_{[0,1]} f \, d\lambda \right| \leq \int_{[0,1]} |f| \; d\lambda = \|f\|_{L^1([0,1])} \leq \|f\|_{L^2([0,1])}.$$

In other words T is bounded (or equivalently continuous because T is linear).

Exercise 5 (*Principle of good sets*) (5+10+5)
Let μ be a finite measure on $([0,1],\mathcal{B}([0,1]))$. We will show by using the principle of good sets that for all $A\in\mathcal{B}([0,1])$ and all $\epsilon>0$ there exist an open set $U\subset[0,1]$ and a compact set $C\subset[0,1]$ such that $C\subset A\subset U$ and $\mu(U\backslash C)<\epsilon$. Let us define

$$\mathcal{G}=\{A\in\mathcal{B}([0,1])|\forall\epsilon>0\ \exists U\supset A \text{ open subset of } [0,1]\ \exists C\subset A \text{ compact}:\ \mu(U\backslash C)<\epsilon\}\,.$$

(a) Prove that every set of the form $[a,b]\subset[0,1]$ lies in $\mathcal{G}$.
(b) Show that $\mathcal{G}$ is a Dynkin system.
(c) Show the claim mentioned above (i.e. for all $A\in\mathcal{B}(\mathbb{R})$ and all $\epsilon>0$ there exists some open set $U\subset[0,1]$ and a compact set $C\subset[0,1]$ such that $C\subset A\subset U$ and $\mu(U\backslash C)<\epsilon$.).

Solution of Exercise 5:
We remark:

$$\mathcal{G}=\{A\in\mathcal{B}([0,1])|\exists U_n\supset A \text{ open } \exists C_n\subset A \text{ closed}:\ \mu(U_n\backslash C_n)\to 0 \text{ for } n\to\infty\}\,. \qquad (*)$$

ad (a):
We use the compact set $C_n=[a,b]$ and the open set $U_n=(a-n^{-1},b+n^{-1})\cap[0,1]$ (for $n\in\mathbb{N}$). It is clear that $C_n\subset[a,b]\subset U_n$ and

$$\mu(U_n\backslash C_n)\to\mu(\emptyset)=0$$

for $n\to\infty$ by the continuity of the measure (we remark $U_n\backslash C_n\supset U_{n+1}\backslash C_{n+1}$ and $\bigcap_{n\in\mathbb{N}}U_n\backslash C_n=\emptyset$). Hence $[a,b]\in\mathcal{G}$.
ad (b):
$\emptyset\in\mathcal{G}$: Using $C_n=U_n=\emptyset$, it is clear that $\emptyset\in\mathcal{G}$.
$A\in\mathcal{G}\Rightarrow A^c\in\mathcal{G}$: Given U_n and C_n as in $*$ one has

$$\mu\left(C_n^c\backslash U_n^c\right)=\mu(C_n^c)-\mu(U_n^c)=\mu(U_n)-\mu(C_n)=\mu\left(U_n\backslash C_n\right)\to 0$$

for $n\to\infty$. As U_n^c is closed, $C_n^c\subset A^c\subset U_n^c$ and C_n^c is open the claim follows.
$A_n\in\mathcal{G}$ disjoint (for $n\in\mathbb{N}$) $\Rightarrow A:=\bigcup_{n\in\mathbb{N}}A_n\in\mathcal{G}$: [1] Let us fix some arbitrary $\epsilon>0$. By definition we find an open U_n and a compact C_n with $U_n\supset A\supset C_n$ and

$$\mu\left(U_n\backslash C_n\right)=\frac{\epsilon}{2^n}.$$

Let us set $U:=\bigcup_{n\in\mathbb{N}}U_n\supset A$ (which is obviously open) and $C=\bigcup_{n=1}^m C_m\subset A$ (which is obviously closed and therefore compact in $[0,1]$). Here $m\in\mathbb{N}$ will be determined soon. By the continuity of the measure we get

$$\begin{aligned}\mu\left(U\backslash C\right)&=\mu\left(\bigcup_{k=m+1}^{\infty}A_k\cup\bigcup_{k=1}^{\infty}U_k\backslash C_k\right)\le\sum_{k=m+1}^{\infty}\mu(A_k)+\sum_{k=1}^{\infty}\mu\left(U_k\backslash C_k\right)\\&=\mu\left(\bigcup_{n=m+1}^{\infty}A_k\right)+\epsilon\sum_{k=1}^{\infty}2^{-k}=\mu\left(\bigcup_{n=m+1}^{\infty}A_k\right)+\epsilon\to\epsilon\end{aligned}$$

for $m\to\infty$. In the step with the new line we used that the A_n are pairwise disjoint and therefore

$$\mu\left(\bigcup_{n=m+1}^{\infty}A_k\right)=\sum_{k=m+1}^{\infty}\mu(A_k).$$

[1] Actually we can show that $\mathcal{G}$ is a σ-algebra (by using essentially the same proof but the estimates are more technical).

Moreover the step with the inequality we used the σ-subadditivity. Now we choose m so big that

$$\mu\left(U \backslash C\right) \leq 2\epsilon.$$

We remark that $\epsilon > 0$ was arbitrary, so we conclude $A = \bigcup_{n\in\mathbb{N}} A_n \in \mathcal{G}$.
ad (c):
Let $\mathcal{I}$ be the set of all compact subintervals of $[0,1]$. Then (a) can be rephrased as $\mathcal{I} \subset \mathcal{G}$. Hence (b) implies $\mathrm{dyn}(\mathcal{I}) \subset \mathcal{G}$. As $\mathcal{I}$ is a set of generators of $\mathcal{B}([0,1])$ (i.e. $\sigma(\mathcal{I}) = \mathcal{B}([0,1])$) which is stable under intersections (i.e. $A, B \in \mathcal{I}$, then $A \cap B \in \mathcal{I}$) we can apply Dynkin's π-λ theorem. Hence $\sigma(\mathcal{I}) = \mathrm{dyn}(\mathcal{I})$. And by the argument above we conclude $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{I}) = \mathrm{dyn}(\mathcal{I}) \subset \mathcal{G}$. But this is nothing other than the claim.

Exercise 6 (*Multiple Choice*) (20*)

Decide which of the following statements are true (no proof needed). For every correct answer you get +2 points and for every wrong answer −1 point. The points of this exercise will be rounded up to zero, if the total number is negative.

(a) $\mathbb{Q} \times \{0, 1, 3\} \times \mathbb{Z}$ is countable.
☐ true ☐ false

(b) $\mathbb{R}$ is countable.
☐ true ☐ false

(c) A (non-empty) subset of a countable set is countable.
☐ true ☐ false

(d) $\mathcal{P}(A)$ is uncountable if A is not finite.
☐ true ☐ false

(e) $\{1, 2, 3, 4, \mathbb{R}\}$ is countable.
☐ true ☐ false

(f) Every continuous function $f : \mathbb{R} \to \mathbb{R}$ is integrable.
☐ true ☐ false

(g) Every Cauchy sequence in $L^2(\Omega, \Sigma, \mu)$ has a almost everywhere convergent subsequence. By (Ω, Σ, μ) we mean here an arbitrary measure space.
☐ true ☐ false

(h) Given a Hilbert space H. Then

$$\|x\|_H = \sup_{\|\varphi\|_{H^*} \leq 1} |\varphi(x)|$$

holds for all $x \in H$.
☐ true ☐ false

(i) Every normed vector space V which obeys the parallelogram identity is a Hilbert space (i.e. there exists a inner product which makes V into a Hilbert space and the norm induced by the inner product equals the given norm).
☐ true ☐ false

(j) Given a Hilbert space. Then for every finite-dimensional subspace one can find an orthogonal projection on this subspace.
☐ true ☐ false

Solution of Exercise 6:

(a) $\mathbb{Q} \times \{0, 1, 3\} \times \mathbb{Z}$ is countable.
☒ true ☐ false

(b) $\mathbb{R}$ is countable.
☐ true ☒ false

(c) A (non-empty) subset of a countable set is countable.
☒ true ☐ false

(d) $\mathcal{P}(A)$ is uncountable if A is not finite.
☒ true ☐ false

(e) $\{1, 2, 3, 4, \mathbb{R}\}$ is countable.
☒ true ☐ false

(f) Every continuous function $f : \mathbb{R} \to \mathbb{R}$ is integrable.
☐ true ☒ false

(g) Every Cauchy sequence in $L^2(\Omega, \Sigma, \mu)$ has an almost everywhere convergent subsequence. By (Ω, Σ, μ) we mean here an arbitrary measure space.
☒ true ☐ false

(h) Given a Hilbert space H. Then

$$\|x\|_H = \sup_{\|\varphi\|_{H^*} \leq 1} |\varphi(x)|$$

holds for all $x \in H$.

☒ true ☐ false

(i) Every normed vector space V which obeys the parallelogram identity is a Hilbert space (i.e. there exists a inner product which makes V into a Hilbert space and the norm induced by the inner product equals the given norm).

☐ true ☒ false

(j) Given a Hilbert space. Then for every finite-dimensional subspace one can find an orthogonal projection on this subspace.

☒ true ☐ false