Exercises to Applied Analysis

sheet : 10

9. Let E and F be normed vector spaces over $\mathbb{K}$ and let $T \in \mathscr{L}(E,F)$.

(a) **Show** that $\|T\|_{\mathscr{L}(E,F)} = \sup\{\|Tx\| : x \in \bar{B}_E(0,1)\}$. [3]

(b) **Show** that $\|Tx\|_F \le \|T\|_{\mathscr{L}(E,F)} \|x\|_E$ for all $x \in E$. [3]

10. (a) Let E and F be normed vector spaces over $\mathbb{K}$ and let $T_n, T \in \mathscr{L}(E,F)$ be such that $T_n \to T$ in $\mathscr{L}(E,F)$, that is, $\|T_n - T\|_{\mathscr{L}(E,F)} \to 0$ as $n \to \infty$. **Show** that $T_n \to T$ strongly, that is, $\|T_n x - Tx\|_F \to 0$ as $n \to \infty$ for all $x \in E$. [3]

(b) **Find** Banach spaces E and F, operators $T_n, T \in \mathscr{L}(E,F)$ such that $s-\lim_n T_n = 0$ and $\|T_n\|_{\mathscr{L}(E,F)} = 1$. **Conclude**, that strong-convergence is weaker than norm convergence (=uniform convergence). [3]

11. Let $\mathbf{c}$ denote the space of all convergent real-valued sequences and let $\mathbf{c}_0$ denote the subspace of $\mathbf{c}$ consisting of sequences converging to 0. Both spaces are equipped with the sup-norm. For $x \in \mathbf{c}$, $x = (x_n)_n$, we let $T : \mathbf{c} \to \mathbf{c}_0$ be given by [4]

$$Tx := (x_\infty, x_1 - x_\infty, x_2 - x_\infty, \dots)$$

where $x_\infty := \lim_n x_n$. **Show** that T is an isomorphism.

12. Let E and F be two normed vector spaces over the same field $\mathbb{K}$ and assume that E and F are isomorphic. **Show** that E is complete if and only if F is complete. [3]

13. Let E be a vector space over $\mathbb{K}$ and let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on E. **Show** that the identity $I : (E, \|\cdot\|_1) \to (E, \|\cdot\|_2)$ is an isomorphism if and only if the norm $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent. [3]

14. Let E be a normed vector space over $\mathbb{K}$ and let $T \in \mathscr{L}(E)$. For $n \in \mathbb{N}_0$ we define T^n by $T^0 := I$, $T^1 := T$ and $T^{n+1} := T \circ T^n$. **Show** that $\|T^k\| \le \|T\|^k$ for all $k \in \mathbb{N}_0$ where $0^0 := 1$. [3]