# Exercises to Applied Analysis

sheet : 8

**1.** Let $(\Omega, \mathscr{A}, \mu)$ be a measure space and let $f : \Omega \to \overline{\mathbb{R}}_+$ be a non-negative measurable function such that $\int |f| \, d\mu = 0$. **Show** that $f = 0$ $\mu$-almost everywhere. [2]

**2.** Let $\Omega := \mathbb{N}$, $\mathscr{A} := \mathscr{P}(\Omega)$ and $\mu$ be the counting measure on $\Omega$. **Show** that for a function $f : \Omega \to \mathbb{R}$ one has that $f \in L^1(\Omega, \mathscr{A}, \mu)$ if and only if the series $\sum_{n=1}^{\infty} f(n)$ is absolutely convergent. Moreover, **prove** that in this case [6=3+3]

$$\int f \, d\mu = \sum_{n=1}^{\infty} f(n).$$

**3.** Let $\Omega \subset \mathbb{R}$ be an open set, $\mathscr{B}(\Omega)$ be the Borel $\sigma$-algebra on $\Omega$ and let $f \in L^1(\Omega, \mu)$ for a measure $\mu$ on $(\Omega, \mathscr{B}(\Omega))$. **Show** that $F : [0, \infty) \to \mathbb{R}$ given by [6=3+3]

$$F(t) := \int_{\Omega} \exp(-t|x|) f(x) \, d\mu(x)$$

is continuous. Moreover, **decide** whether or not $F$ is differentiable on $(0, \infty)$.

**4.** [Generalized Hölder Inequality] Let $(\Omega, \mathscr{A}, \mu)$ be a measure space and $p, q, r \in [1, \infty]$ be such that $1/p + 1/q = 1/r$ (with $1/\infty := 0$). **Show** that for $f \in L^p(\Omega)$, $g \in L^q(\Omega)$ one has that $fg \in L^r(\Omega)$ and [6]

$$\|fg\|_r \leq \|f\|_p \, \|g\|_q \, .$$

Hint: The case $r = 1$ was proved in the lecture (Theorem 2.6.3) and can be used to prove the generalized Hölder inequality.

**5.** Let $(\Omega, \mathscr{A}, \mu)$ be a finite measure space, that is, $(\Omega, \mathscr{A}, \mu)$ is a measure space and $\mu(\Omega) < \infty$. Let $1 \leq p \leq q \leq \infty$. **Show** that $L^q(\Omega) \subset L^p(\Omega)$ and that there exists a constant $C \geq 0$ (depending on $p$ and $q$) such that [5]

$$\|f\|_p \leq C \, \|f\|_q \qquad \forall f \in L^q(\Omega).$$

Hint: Use the Hölder inequality with appropriate functions.