Exercises to Applied Analysis

sheet : 9

6. Compute the following limits: [1 each]

(a) $\lim_{n\to\infty} \int_0^1 \frac{\sqrt{n^3}x}{1+n^2x^2}\,dx.$

(b) $\lim_{n\to\infty} \int_0^1 \frac{nx}{1+n^2x^2}\,dx$

(c) $\lim_{n\to\infty} \int_0^1 nxe^{-nx^2}\,dx$

(d) $\lim_{n\to\infty} \int_0^\infty nxe^{-nx^2}\,dx$

(e) $\lim_{n\to\infty} \int_0^1 nx^{-nx^2}\,dx$

(f) $\lim_{n\to\infty} \int_0^\infty \frac{\ln(n+x)}{n}e^{-x}\,dx$

(g) $\lim_{n\to\infty} \int_0^\infty e^{-xn}\cos(x)\,dx$

(h) $\lim_{n\to\infty} \int_0^1 \frac{\exp\big(-n^2(x^2+1)\big)}{x^2+1}\,dx.$

7. (a) **Show** that $\ell^1 \subset \ell^2 \subset \ell^\infty$. [2]

(b) **Show** that $\ell^\infty \not\subset \ell^2 \not\subset \ell^1$. [2]

(c) **Explain** why (7b) is not a contradiction to Question 5, sheet 8. [2]

8. Let $(\Omega, \mathscr{A}, \mu)$ be a σ-finite measure space and let $f : \Omega \to \mathbb{R}_+ := [0,\infty)$ be a measurable function. Then we consider the product space $\Omega \times \mathbb{R}_+$ with the product σ-algebra $\mathscr{A} \otimes \mathscr{B}(\mathbb{R}_+)$ and the product measure $\mu \otimes \lambda$ where λ is the 1-dimensional Lebesgue on $\mathscr{B}(\mathbb{R}_+)$. We let $G := \{(x,t) \in \Omega \times \mathbb{R}_+ : f(x) \geq t\}$.

(a) **Show** that G is measurable, that is, $G \in \mathscr{A} \otimes \mathscr{B}(\mathbb{R}_+)$. [3]

(b) **Show** that [4]

$$(\mu \otimes \lambda)(G) = \int_0^\infty \mu(P_1(G,t))\,d\lambda(t) = \int \lambda(P_2(G,x))\,d\mu(x)$$

where for $A \in \mathscr{A} \otimes \mathscr{B}(\mathbb{R}_+)$, $x \in \Omega$ and $t \in \mathbb{R}_+$

$$P_2(A,x) := \{s \in \mathbb{R}_+ : (x,s) \in A\} \quad \text{and} \quad P_1(A,t) := \{y \in \Omega : (y,t) \in A\}.$$

Hint: Use Theorem 2.8.6.

(c) **Deduce** that [4]

$$(\mu \otimes \lambda)(G) = \int_0^\infty \mu(\{x \in \Omega : f(x) \geq t\})\,d\lambda(t) = \int f(x)\,d\mu(x).$$