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Wintersemester 2008/2009
Ulm, den 21.10.2008

Applied analysis

Exercise sheet 2

Exercise 5. Let l^∞ be the space of all real bounded sequences, i.e. $l^\infty := \{(x_n)_{n\in\mathbb{N}},\ x_n \in \mathbb{R},\ \sup_{n\in\mathbb{N}} |x_n| < \infty\}$. Check that it is a vector space with the usual definition of addition and scalar-multiplication. Define a metric (distance) by

$$d(x, y) := \sup_{n\in\mathbb{N}} |x_n - y_n|.$$

Check that this indeed defines a metric. Decide whether (l^∞, d) is separable, complete, compact.

Exercise 6. Take an arbitrary nonempty set M and define the discrete metric on M by $d(x, y) := 1$, if $x \neq y$ and $d(x, y) := 0$ otherwise. Determine the convergent sequences in (M, d). Is (M, d) complete? Is it separable? Is it compact? (What do these properties depend on?)

Exercise 7. Let (M, d) be a metric space.

a: Choose $x \in M$. Show that a set $U \subset M$ is a neighbourhood of x if and only if $x \in \operatorname{Int} U$. (In particular, every open set containing x is a neighbourhood of x).

b: Show that every convergent sequence is Cauchy.

c: Let A, B be closed nonempty subsets of $\mathbb{R}$. Define a new set $A + B := \{x \in \mathbb{R},\ \exists\, a \in A\, \exists\, b \in B \text{ s.t. } x = a + b\}$. Is $A + B$ closed? (If not, find a counterexample). What happens if one of the sets is bounded?

Exercise 8. Let X, Y, Z be metric spaces.

a: Show that $f : X \to Y$ is continuous in $x \in X$ if and only if

$$\forall\, \varepsilon > 0\, \exists\, \delta > 0 \text{ such that } d_X(x, y) \leq \delta \text{ implies } d_Y(f(x), f(y)) \leq \varepsilon.$$

b: Assume that $f : X \to Y$ and $g : Y \to Z$ are continuous. Prove that the composition mapping $g \circ f : X \to Z$ is continuous.

c: (*) Find two metric spaces X, Y and a continuous mapping $f : X \to Y$ s.t. there exists a nonempty open $A \subset X$ with $f(A)$ not open in Y. (Compare with the characterization of continuity Th. 1.1.36).