Dr. Markus Biegert **Wintersemester 2008/2009**
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Applied analysis

Exercise sheet 3

Exercise 9. Let (X, d), (Y, e) be metric spaces.

a: Show that $d : X \times X \to \mathbb{R}$ is continuous.

b: Let $A \subset X$. Show that $d(., A) : X \to \mathbb{R}$ is continuous.

c: Let $(x^n)_n$ be a sequence in $X \times Y$. Prove that $x^n = (x_1^n, x_2^n)$ converges to $x = (x_1, x_2)$ in $X \times Y$ if and only if $(x_1^n)_n$ converges to x_1 in X and $(y_2^n)_n$ converges to y_2 in Y.

Exercise 10.

a: Let $(E, \|\cdot\|_E)$ be a normed $\mathbb{K}$-space. Show that (E, d_E) is a metric space, where d_E is the corresponding distance on E.

b: Let E be a $\mathbb{K}$-vector space and let d be a distance on E such that d is translation invariant and homogeneous, i.e. $d(x + z, y + z) = d(x, y)$ for all $x, y, z \in E$ and $d(\lambda x, \lambda y) = |\lambda| d(x, y)$ for all $x, y \in E$ and $\lambda \in \mathbb{K}$. Define a norm $\|\cdot\|_E$ on E such that $d(x, y) = \|x - y\|_E$ (and check the norm properties).

Exercise 11.

a: Let $E := \mathbb{R}^2$, $\|x\|_1 := |x_1| + |x_2|$, $\|x\|_2 := \sqrt{x_1^2 + x_2^2}$ and $\|x\|_\infty := \max\{|x_1|, |x_2|\}$. Show that these norms are equivalent.

b: Let E be a $\mathbb{K}$-vector space, $\|\cdot\|_1$ and $\|\cdot\|_2$ be two equivalent norms on E and let $x_n, x \in E$. Show that $x_n \to x$ in $(E, \|\cdot\|_1)$ iff $x_n \to x$ in $(E, \|\cdot\|_2)$.

Problem 12. (*) Prove Theorem 1.1.43 (Banach's fixed point theorem) using Corollary 1.1.44 (Banach's FPT for contractions).