

Applied analysis

Exercise sheet 9

Exercise 33. a) Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space, i.e. $\mu(\Omega) < \infty$. Let $(f_n)_n \subset L^\infty(\Omega)$ be a sequence converging in $L^\infty(\Omega)$ to a function $f \in L^\infty(\Omega)$. Show that $(f_n)_n$ is a sequence in $L^1(\Omega)$, $f \in L^1(\Omega)$ and that $(f_n)_n$ converges to f in $L^1(\Omega)$.

b) Let $\mu, \nu : \mathcal{B}(\mathbb{R}^n) \rightarrow \overline{\mathbb{R}}_+$ be measures such that $\mu(Q) = \nu(Q)$ for all left-open cells Q . Show that $\mu = \nu$.

(2+2 points)

Exercise 34. a) Define a function f on $[0, 1] \times [0, 1]$ by $f(t, x) := \begin{cases} (1 + \frac{1}{t})x^{\frac{1}{t}} & \text{if } 0 < t \leq 1 \\ 0 & \text{if } t = 0 \end{cases}$

Is $F(t) := \int_0^1 f(t, x) dx$ a continuous function on $[0, 1]$?

b) What is the first derivative (if it exists!) with respect to t of the following function:

$$F(t, x) := \int_0^{\frac{1}{2}} \frac{\log(tx)}{x \log |\log x|} dx$$

(4+3 points)

Exercise 35. Check that the substitution $\sin x = y$ is not allowed on the whole interval of integration for

$$\int_0^{5\pi} \sin^2 x \cos^2 x dx.$$

Compute the integral!

(2+4 points)

Exercise 36. Prove the following statement: If f is Laplace transformable, then the Laplace transform $\mathcal{L}(f)$ is infinitely many times differentiable on $(\text{abs}(f), \infty)$ and

$$(\mathcal{L}f)^{(n)}(x) = \mathcal{L}[(-t)^n f(t)](x) \quad \forall x > \text{abs}(f).$$

(3 points)