Dr. Markus Biegert
Dipl.-Math. Michal Chovanec

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Applied analysis

Test Exam

Task 1. Consider the Banach space $C([0,1])$ equipped with the sup norm $\|\cdot\|_\infty$. Fix a function $g \in C([0,1])$. Define a mapping $T : C([0,1]) \to C([0,1])$ by

$$(Tf)(x) := \frac{1}{4}\int_0^x f(s/3)\,ds + g(x)$$

a: Check that T is well-defined.

b: Verify that T is a strict contraction on $C([0,1])$.

c: Prove that there exists a unique function $u \in C([0,1])$ such that

$$u(x) = \frac{1}{4}\int_0^x u(s/3)\,ds + g(x)$$

Task 2. Let $(\mathbb{R}^n, \mathcal{A}, \mu)$ be a measure space, $\mathcal{A}$ is the Borel σ-algebra. Show that for any integrable function $f : \mathbb{R}^n \to \mathbb{R}$ and any $R > 0$ the following holds:

$$\lim_{n\to\infty} \int_{B(0,R-\frac{1}{n})} |f|\,d\mu(x) = \|f\|_{L^1(B(0,R),\mathcal{A},\mu)}$$

Task 3. Let $E := C([0,1])$. Show that $\varphi : C([0,1]) \to \mathbb{R}$ given by

$$\varphi(f) = \int_0^1 [3f(t) + 4t^2 f(t)]\,dt$$

is in E' and compute $\|\varphi\|_{E'}$.

Task 4. Let $T : C([0,1]) \to C([0,1])$ be given by

$$(Tf)(x) := \frac{1}{4}\int_0^x f(s/3)\,ds.$$

Show that $S := I - T$ is invertible.

Task 5. Consider the real Hilbert space l^2. Define a set

$$A := \{x = (x_k)_k \in l^2 : x_1 > 0\}$$

and a point $b := (-3, 0, 0, 0, 0, ...)$

a: Check that A is open and convex.

b: Prove that there exists $\varphi \in (l^2)'$ and $\gamma \in \mathbb{R}$ such that

$$\varphi(x) < \gamma \leq \varphi(y)$$

for any $x \in A$.

Task 6. Is the unit ball in $C([-4, 2])$ compact? Prove your assertion.

Task 7. Let $f(t) := e^t$. Show that $\text{abs}(f) = 1$ and $\hat{f}(s) = 1/(s-1)$ for $s > \text{abs}(f)$. ($\hat{f}$ denotes the Laplace transform of f).

Task 8. Let $f : [0, 1] \times [1, \infty) \to \mathbb{R}$ be given by $f(t, x) := \cos(t^2 x)x^{-3}$. Show that the function $F : [0, 1] \to \mathbb{R}$ given by

$$F(t) := \int_1^\infty f(t, x)\, dx$$

is continuous.