University of Ulm
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Exercises Applied Analysis: Sheet 2

1. Show that every Cauchy sequence in $\mathbb{C}$ converges.

Hint: Use that every Cauchy sequence in $\mathbb{R}$ converges.

2. We consider the sequence (x_n) with $x_n := \frac{(1+2(-1)^n)n^3}{n^3-4(-1)^n}$ and the set $\{x_n : n \in \mathbb{N}\} \subset \mathbb{R}$.

(a) Determine the supremum and the infimum of the set.

(b) What is $\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$?

3. Let X be a nonempty set and let $(x_n) \subset X$ be a sequence. Suppose that every subsequence of (x_n) has a subsequence which converges to x. Show that (x_n) already converges to x.

Hint: Proof by contradiction.

4. Let $(x_n) \subset \mathbb{R}$ be a sequence.

(a) Show that if $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$, then (x_n) converges, and the limit is equal to the value of the lim sup.

(b) Suppose that $(a_n), (b_n) \subset \mathbb{R}$ are sequences with $a_n \to x$ and $b_n \to x$ as $n \to \infty$ and $a_n \le x_n \le b_n$ for all $n \in \mathbb{N}$. Show, using the definition of convergence of a sequence, that $x_n \to x$ as $n \to \infty$.

(c) Suppose (x_n) is defined recursively by $x_1 = 10$ and $x_{n+1} = \frac{1}{2}x_n + 3$. Show that (x_n) converges and calculate $\lim_{n\to\infty} x_n$.
Hint: First show that (x_n) is monotone decreasing and bounded from below. Use these properties to show that (x_n) converges.

5. Let $d \in \mathbb{N}$. We define mappings $\|\cdot\|_1, \|\cdot\|_\infty \colon \mathbb{K}^d \to \mathbb{R}$ by

$$\|(x_1, \ldots, x_d)\|_1 := \sum_{n=1}^{d} |x_n|, \qquad \|(x_1, \ldots, x_d)\|_\infty := \sup_{n\in\{1,\ldots,d\}} |x_n|.$$

(a) Show that $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are norms on $\mathbb{K}^d$.

(b) Show that

$$\|(x_1 \cdot y_1, \ldots, x_d \cdot y_d)\|_1 \le \|(x_1, \ldots, x_d)\|_\infty \cdot \|(y_1, \ldots, y_d)\|_1$$

for all $(x_1, \ldots, x_d), (y_1, \ldots, y_d) \in \mathbb{K}^d$.

(c) Show that $\|x\|_\infty \le \|x\|_1 \le d\|x\|_\infty$ for all $x \in \mathbb{K}^d$.