University of Ulm

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Exercises Applied Analysis: Sheet 3

1. Let $d \in \mathbb{N}$ and $(X_1, \|\cdot\|_1), \dots, (X_d, \|\cdot\|_d)$ be normed spaces. Make the Cartesian product $Y := X_1 \times \cdots \times X_d$ into a vector space using componentwise addition and scalar multiplication, and equip it with the norm $\|(x_1, \dots, x_d)\| := \sum_{k=1}^d \|x_k\|_{X_k}$.

(a) Show that the component maps $\pi_k \colon Y \to X_k$, $\pi_k(x_1, \dots, x_d) = x_k$ are continuous.

(b) Let $(W, \|\cdot\|_W)$ be a normed space and $A \subset W$. Show that a function $f \colon A \to Y$ is continuous if and only if $\pi_k \circ f \colon A \to X_k$ is continuous for all $k \in \{1, \dots, d\}$.

(c) Let $(Z, \|\cdot\|_Z)$ be a normed space and $f \colon Y \to Z$. Suppose that for all $(x_1, \dots, x_d) \in Y$ and $k \in \{1, \dots, d\}$ one has $f(x_1, \dots, x_{k-1}, \cdot, x_{k+1}, \dots, x_d) \colon X_k \to Z$ is continuous. Show that f doesn't need to be continuous.

2. Show that a normed space $(X, \|\cdot\|)$ is a Banach space if and only if for every sequence $(x_n)_{n \in \mathbb{N}}$ in X one has the implication

$$\sum_{n=1}^{\infty} \|x_n\| < \infty \quad \Rightarrow \quad \left(\sum_{n=1}^{N} x_n \right)_{N \in \mathbb{N}} \text{ converges in } X.$$

Hint: Given a Cauchy sequence $(x_n)_{n \in \mathbb{N}}$, consider a suitable subsequence $(x_{n_k})_{k \in \mathbb{N}}$ and teleskopic sums like $\sum_{k=M}^{N} (x_{n_{k+1}} - x_{n_k})$.

3. For which $1 \le p \le \infty$ are the following sequences in ℓ^p?

(a) $\left(\frac{1}{n}\right)_{n \in \mathbb{N}}$

(b) $(n^{50} e^{-n})_{n \in \mathbb{N}}$

(c) $\left(\frac{1}{\log(n+1)}\right)_{n \in \mathbb{N}}$

4. (a) Let $1 \le p < q \le \infty$. Show that ℓ^p is continuously embedded in ℓ^q.
Hint: Given $x \in \ell^p$ set $y := x\|x\|_p^{-1}$. Show that $|y_n| \le 1$ and apply Hölder's inequality on $\|y\|_p^p = \|(|y_n|^p)_{n \in \mathbb{N}}\|_1$.

(b) Let $1 \le p \le \infty$. Give an example of a sequence $(x_n) \subset \ell^p$ with $\|x_n\|_p \le 1$, such that no subsequence of (x_n) converges.