University of Ulm

Discussion: Friday, 14.11.2014

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Winter term 2014/15

Exercises Applied Analysis: Sheet 4

1. Let K be a compact set in a normed space $(X, \|\cdot\|)$ and $f\colon K \to \mathbb{R}$ continuous. Show that f attains its minimum and maximum on K.

2. Let K be a compact set in a normed space $(X, \|\cdot\|)$. Then for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ and $x_1, \dots, x_N \in K$ such that $K \subset \bigcup_{k=1}^{N} B(x_k, \varepsilon)$. Deduce that K has a countable dense subset.

3. (a) Let $(X, \|\cdot\|)$ be a normed space. Prove that $\{x \in X : \|x\| \leq 1\}$ is closed in X.

(b) Let $A := \{x \in \ell^\infty : |x_k| < 1 \text{ for all } k \in \mathbb{N}\}$. Is A open in ℓ^∞?

(c) Let $F := \{x \in \ell^p : \|x\|_p \leq 1\}$. Show that F is bounded and closed, but not compact.

4. Let $(X, \|\cdot\|)$ be a normed space and $A_1, A_2, A_3, \dots \subset X$.

(a) If $B_n = \bigcup_{k=1}^{n} A_k$, prove that $\overline{B_n} = \bigcup_{k=1}^{n} \overline{A_k}$, for all $n \in \mathbb{N}$.

(b) If $B = \bigcup_{k=1}^{\infty} A_k$, prove that $\overline{B} \supset \bigcup_{k=1}^{\infty} \overline{A_k}$.
Show, by an example, that this inclusion can be proper.

5. (a) Let $\mathbf{c_0} := \{(x_k) : (x_k) \text{ converges to } 0 \text{ in } \mathbb{K}\}$. Show that $\mathbf{c_0}$ is a vector subspace of ℓ^∞ that is closed in ℓ^∞. Deduce that $(\mathbf{c_0}, \|\cdot\|_\infty)$ is a Banach space. Show that $\mathbf{c_0}$ is not contained in any ℓ^p with $1 \leq p < \infty$.

(b) Let $\mathbf{c_{00}} := \{(x_k) : x_k \in \mathbb{K}, \{n \in \mathbb{N} : x_n \neq 0\} \text{ is finite}\}$. Show that the $\mathbf{c_{00}}$ is contained but not closed in ℓ^p for any $p \in [1, \infty]$.