University of Ulm

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Exercises Applied Analysis: Sheet 6

1. Suppose X is a normed space and $K \subset X$ is compact.

(a) Show that a sequence (f_n) in $C(K)$ converges uniformly to $f \in C(K)$ if and only if for every sequence (x_n) and x in K with $x_n \to x$ one has $f_n(x_n) \to f(x)$.

(b) Show that this can fail if K is not assumed to be compact.

2. Let Ω be a set. Show that $(\mathcal{F}_{\mathrm{b}}(\Omega), \|\cdot\|_\infty)$ is a Banach space.

3. Let $\mathscr{A}$ be the set of all even real polynomial functions on $[0,1]$; i.e., p is a polynomial such that $p(-x) = p(x)$.

(a) Show that every $p \in \mathscr{A}$ is of the form $p(x) = c_1 + c_2x^2 + c_3x^4 + \dots c_nx^{2n}$, where $c_1, \dots c_n \in \mathbb{R}$ and $n \in \mathbb{N}$.

(b) Show that $\mathscr{A}$ is dense in $(C[0,1], \|\cdot\|_\infty)$.

(c) Now let $\mathscr{A}$ be the set of all even real polynomial functions on $[-1,1]$.
Is $\mathscr{A}$ dense in $(C[-1,1], \|\cdot\|_\infty)$?

4. Let $(X, \|\cdot\|)$ be a Banach space and let $M \subset X$ be compact. Suppose $\varphi\colon M \to M$ satisfies $\|\varphi(x) - \varphi(y)\| < \|x - y\|$ for all $x, y \in M$. Show that φ has a unique fixed point in M.

5. We are interested in $x, y \in [-1,1]$ which solve

$$50x = x^2 + y^2 + x + 1$$
$$50y = x^3 + y^2 + y.$$

(a) Use Banach's fixed point theorem to prove the existence of a unique solution.

(b) Approximate the solution with an error $< 10^{-4}$ using fixed point iteration starting at $x_0 = y_0 = 0$.