UNIVERSITY OF ULM

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Exercises Applied Analysis: Sheet 7

1. The proof of Banach's fixed point theorem shows that for any initial function $u \in C(I)$ the sequence $\Phi^n u$ converges to the unique fixed-point u^* of Φ that is the unique solution of (ODE). This allows to approximate the solution u^* by simply iterating Φ. This method is called Picard iteration.

For the ordinary differential equation $u'(t) = tu(t)$ on $[0, 1]$, use Picard iteration to construct the solution of the equation for the initial value $u(0) = 1$.

2. Calculate the value of the limit

$$\lim_{N\to\infty} \sum_{k=1}^{N} \frac{N}{N^2 + k^2}$$

by using the Riemann integral.

3. In the lecture it was shown that $(C[0, 1], \|\cdot\|_{\mathcal{L}^1})$ with $\|f\|_{\mathcal{L}^1} := \int_0^1 |f(s)|\, ds$ is a normed space.

(a) Show that $(C[0, 1], \|\cdot\|_{\mathcal{L}^1})$ is not complete.

(b) Is $(C[0, 1], \|\cdot\|_{\mathcal{L}^1})$ separable?

4. Urysohn's lemma:

Let $(X, \|\cdot\|)$ be a normed space and $A, B \subset X$ be closed sets such that $A \cap B = \emptyset$. Show that there exists a continuous function $f \in C(X)$ such that $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$.

[Hint: First show that $x \mapsto \inf\{\|x - y\| : y \in A\}$ is Lipschitz continuous if $A \neq \emptyset$. For f use a suitable combination of these functions.]