University of Ulm

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Exercises Applied Analysis: Sheet 8

1. Let (Ω_2, Σ_2) be a measurable space and $\mathscr{A}$ be a generator of Σ_2. Let Ω_1 be a nonempty set and $\mathscr{F}$ be a set of maps from Ω_1 to Ω_2. Using the principle of good sets, show that $\sigma(\mathscr{F})$ is generated by

$$\{f^{-1}[A] : f \in \mathscr{F}, A \in \mathscr{A}\}.$$

2. Show that the set of half-open intervals $[a, b)$ with $a, b \in \mathbb{Q}$ and $a < b$ generates the Borel σ-algebra on $\mathbb{R}$.

3. Let (Ω_1, Σ_1) and (Ω_2, Σ_2) be σ-algebras. We define the product σ-algebra $\Sigma_1 \otimes \Sigma_2$ on $\Omega_1 \times \Omega_2$ by $\sigma(\{A \times B : A \in \Sigma_1,\ B \in \Sigma_2\})$.

(a) Show that $\mathscr{B}(\mathbb{R}^2) = \mathscr{B}(\mathbb{R}) \otimes \mathscr{B}(\mathbb{R})$.

(b) Suppose $\mathscr{A}_j$ is a generator of Σ_j, $j = 1, 2$. Show that

$$\Sigma_1 \otimes \Sigma_2 = \sigma(\{A \times B : A \in \mathscr{A}_1,\ B \in \mathscr{A}_2\}).$$

[Hint: Use the principle of good sets twice.]

(c) Conclude that $\mathscr{B}(\mathbb{R}^2)$ is generated by $[a_1, b_1) \times [a_2, b_2)$ with $a_1, b_1, a_2, b_2 \in \mathbb{Q}$.

4. Let us suppose that a function $f\colon \mathbb{R} \to \mathbb{R}$ is given. We equip the codomain with the Borel-σ-algebra $\mathscr{B}(\mathbb{R})$. Describe (without a proof) the σ-algebra $\sigma(f)$ in the following situations:

1. $f(x) = \operatorname{sgn}(x)$, where $\operatorname{sgn}(x) = 1$ for $x > 0$, $\operatorname{sgn}(x) = -1$ for $x < 0$ and $\operatorname{sgn}(0) = 0$;
2. $f(x) = x^3$,
3. $f(x) = |x|$.