University of Ulm

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Solutions Applied Analysis: Sheet 9

1. Let Ω be a nonempty set and $f, g\colon \Omega \to \mathbb{R}$. As usual, equip the range $\mathbb{R}$ with the Borel σ-algebra.

Show that the set $\{x \in \Omega : f(x) \leq g(x)\}$ is in the σ-algebra generated by $\{f, g\}$ on Ω. The same holds if $\leq$ is replaced by $=$, $<$ or $>$.

2. Let us suppose that a Borel measurable function $f\colon \mathbb{R} \to \mathbb{R}$ is given. Describe (without a proof) the push forward of the Lebesgue measure λ under f.

1. $f(x) = \operatorname{sgn}(x)$, where $\operatorname{sgn}(x) = 1$ for $x > 0$, $\operatorname{sgn}(x) = -1$ for $x < 0$ and $\operatorname{sgn}(0) = 0$;
2. $f(x) = x^3$,
3. $f(x) = |x|$.

3. Give an example of a Dynkin system that is not a σ-algebra.

4. Let (Ω, Σ) be a measurable space and $\mathscr{A}$ be a generator of Σ that is stable under intersections. Suppose that there exists an increasing sequence $\Omega_n \in \mathscr{A}$ with $\bigcup \Omega_n = \Omega$.

(a) For $n \in \mathbb{N}$ we set $\Sigma_n := \sigma(\{A \cap \Omega_n : A \in \mathscr{A}\}$. Show that $\Omega_n \cap \Sigma = \Sigma_n$.

(b) Let μ and ν are two (σ-finite) measures on (Ω, Σ) such that $\mu(A) = \nu(A) < \infty$ for all $A \in \mathscr{A}$. Show that $\mu = \nu$.
[Hint: Use part (a) and the continuity from below of measures.]

5. Let $F\colon \mathbb{R} \to \mathbb{R}$ be a monotonically increasing function, i.e., $F(x) \leq F(y)$ if $x \leq y$. Define $F_+(t) := \inf\{F(s) : s > t\}$. Show that there exists at most one measure μ on $\mathscr{B}(\mathbb{R})$ such that $\mu\big((a, b]\big) = F_+(b) - F_+(a)$ for all $a, b \in \mathbb{R}$ with $a < b$.

Give two different functions F such that the associated measure is both times δ_0.