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Solutions Applied Analysis: Sheet 10

1. Let (Ω, Σ, μ) be a probability space. Two sets $A, B \in \Sigma$ are called (stochastically) independent, if and only if

$$\mu(A \cap B) = \mu(A)\mu(B).$$

Let us suppose that $A \in \Sigma$ and $\mathcal{E} \subset \Sigma$ are given. We say that A is independent of $\mathcal{E}$, if and only if A, B are independent for all $B \in \mathcal{E}$.

(a) Find a concrete example of the above situation such that A is independent of $\mathcal{E}$ but A is not independent of $\sigma(\mathcal{E})$.

(b) Let us suppose that $\mathcal{E}$ is stable under intersections. Prove that A and $\mathcal{E}$ are independent if and only if A and $\sigma(\mathcal{E})$ are independent.

2. Let $F\colon \mathbb{R} \to \mathbb{R}$ be a monotonically increasing function, i.e., $F(x) \le F(y)$ if $x \le y$. Define $F_+(t) := \inf\{F(s) : s > t\}$. Show that there exists a measure μ on $\mathscr{B}(\mathbb{R})$ such that $\mu\Big((a, b]\Big) = F_+(b) - F_+(a)$ for all $a, b \in \mathbb{R}$ with $a < b$.

3. Let (Ω, Σ, μ) be a measure space and $f\colon \Omega \to [0, \infty)$ be a measurable function.

(a) Show that $\nu(A) = \int \mathbb{1}_A f \,\mathrm{d}\mu$ defines a measure on (Ω, Σ).

(b) When is the measure ν finite?

4. Suppose μ is the counting measure on the measurable space $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$. Let $f\colon \mathbb{N} \to [0, \infty)$ be a function. Note that f is measurable. Show that f is integrable if and only if $f \in \ell^1$.