University of Ulm

Discussion: Friday, 23.1.2014

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Winter term 2014/15

Solutions Applied Analysis: Sheet 11

1. Let (Ω, Σ, μ) be a measure space and $f\colon \Omega \to [0, \infty]$ measurable. Define $\nu(A) := \int_A f\, d\mu$ for $A \in \Sigma$. We know that $\nu(A)$ is a measure. Show that $g \in \mathscr{L}^1(\Omega, \Sigma, \nu)$ if and only if $gf \in \mathscr{L}^1(\Omega, \Sigma, \mu)$. Moreover, show that then $\int g\, d\nu = \int gf\, d\mu$.

2. Let $\alpha > 0$.

(a) Show that the following measures $\nu(A) = \int_A f\, \mathrm{d}\lambda$, where λ is the Lebesgue measure and

$$(i.)\ f(x) = \frac{1}{\pi(1+x^2)}, \qquad (ii.)\ f(x) = \mathbb{1}_{[0,\infty)}(x)\alpha e^{-\alpha x},$$

are probability measures.

(b) Find some $c > 0$ (depending on $\alpha > 0$) such that a measure $\nu(A) = \int_A f\, \mathrm{d}\zeta$, where ζ is the counting measure on $\mathbb{N}_0$ and

$$f(k) = c\frac{\alpha^k}{k!}$$

defines a probability measure.

(c) Calculate the expected value of random variables with the above densities f, i.e.

$$\int_{\mathbb{R}} \frac{x}{\pi(1+x^2)}\, \mathrm{d}\lambda(x), \quad \int_{[0,\infty)} x\alpha e^{-\alpha x}\, \mathrm{d}\lambda(x), \text{ and } \int_{\mathbb{N}} kc\frac{\alpha^k}{k!}\, \mathrm{d}\zeta(k),$$

where λ is the Lebesgue measure and ζ the counting measure.

3. Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$. Give a sequence of integrable functions $f_n\colon \mathbb{R} \to [0, 1]$ such that $f_n(x) \to 0$ as $n \to \infty$ for all $x \in \mathbb{R}$ and $\int f_n = 1$ for all $n \in \mathbb{N}$. Does such an example also exist if $\mathbb{R}$ is replaced by $[-1, 1]$?

4. Find a probability space $(\Omega, \Sigma, \mathbb{P})$ that supports a sequence of stochastically independent, identically distributed random variables $(X_n)_{n\in\mathbb{N}}$ such that $\mathbb{P}(X_n = 1) = 3/5$, $\mathbb{P}(X_n = 0) = 1/5$ and $\mathbb{P}(X_n = 2) = 1/5$. Describe the construction of such a sequence for your choice of $(\Omega, \Sigma, \mathbb{P})$. Prove that the first two functions in your sequence are indeed stochastically independent.