UNIVERSITY OF ULM

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Exercises Applied Analysis: Sheet 12

1. Give an example of a function $f\colon \mathbb{R}^2 \to [0,1]$ such that $f(\cdot, x)$ and $f(x, \cdot)\colon \mathbb{R} \to [0,1]$ are Borel measurable for all $x \in \mathbb{R}$, but f is not Borel measurable.

[Hint: consider the function $f(x,y) = \begin{cases} \mathbb{1}_A(x) & \text{for } x = y, \\ 0 & \text{for } x \neq y, \end{cases}$ where $A \subset \mathcal{P}(\mathbb{R}) \setminus \mathcal{B}(\mathbb{R})$.]

2. We consider the measure spaces $(\mathbb{R}, \mathcal{B}(\mathbb{R}); \lambda)$ and $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \zeta)$. Determine $\mathcal{P}(\mathbb{N}) \otimes \mathcal{B}(\mathbb{R})$ and $\zeta \otimes \lambda$.

3. Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space and let X_1, X_2 be random variables. We say that $\Sigma_1, \Sigma_2 \subset \Sigma$ are independent if $\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1)\mathbb{P}(A_2)$ for all $A_1 \in \Sigma_1$ and $A_2 \in \Sigma_2$. Moreover, we say that X_1 and X_2 are independent if $\sigma(X_1)$ and $\sigma(X_2)$ are independent.

(a) Show that X_1 and X_2 are independent if and only if $\mathbb{P}_{(X_1,X_2)} = \mathbb{P}_{X_1} \otimes \mathbb{P}_{X_2}$.

(b) Conclude that $\mathbb{E}(X_1 X_2) = \mathbb{E}X_1 \mathbb{E}X_2$.

4. Calculate the following integrals, if they exist.

(a) $\displaystyle\int_0^1 \int_x^1 \frac{y}{y^3+1} \,\mathrm{d}\lambda(y)\,\mathrm{d}\lambda(x)$

(b) $\displaystyle\int_{\{(k,l)\in\mathbb{N}^2 : l \leq k\}} \frac{1}{2^k} \,\mathrm{d}(\zeta\otimes\zeta)(k,l)$

(c) $\displaystyle\int_{[0,1]^2} \frac{x-y}{(x+y)^3} \,\mathrm{d}(\lambda\otimes\lambda)(x,y)$

(d) $\displaystyle\int_{-1}^1 \int_{-1}^1 \frac{x(y+2y^2)}{e^y + |y| + y^2 + |\sin y|} \,\mathrm{d}\lambda(y)\,\mathrm{d}\lambda(x)$