University of Ulm

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Solutions Applied Analysis: Sheet 1

1. (a) Prove that $\mathcal{P}(\mathbb{N})$ is uncountable.

Solution: Note, a set X is countable if and only if there exists a surjective mapping $f\colon \mathbb{N} \to X$. We show that any mapping from $\mathbb{N}$ to $\mathcal{P}(\mathbb{N})$ is not surjective.

Suppose $f\colon \mathbb{N} \to \mathcal{P}(\mathbb{N})$ is a mapping. Let A be the set $\{n \in \mathbb{N} : n \notin f(n)\}$. Then $A \in \mathcal{P}(\mathbb{N})$ and $A \neq f(n)$ for all $n \in \mathbb{N}$. Hence, f is not surjective.

(b) Is the set of functions from $[0,1]$ to $\mathbb{N}$ countable?

Solution: The set of functions

$$\{f_x\colon [0,1] \to \{1,2\}, f(a) = 1 \text{ for } a \neq x \text{ and } f(x) = 2 : x \in [0,1]\}$$

is isomorphic to $[0,1]$ and a subset of all functions from $[0,1]$ to $\mathbb{N}$. Hence the set of functions from $[0,1]$ to $\mathbb{N}$ is uncountable.

(c) What about the set of functions from $\{0,1\}$ to $\mathbb{N}$?

Solution: The above set is isomorphic to $\mathbb{N} \times \mathbb{N}$ via the mapping $f \mapsto (f(0), f(1))$ and hence countable.

2. (a) Prove that in any field F one has $a \cdot 0 = 0$ for all $a \in F$.

Solution: We have $a \cdot 0 = a \cdot (0 + 0) = a \cdot 0 + a \cdot 0$. If we add $-a \cdot 0$ on both sides we obtain $0 = a \cdot 0$.

(b) Show that the identity element and the inverse element of a fixed $g \in G$ is unique in a group G.

Solution: Let e_1, e_2 be neutral elements in G, then

$$e_1 = e_1 \circ e_2 = e_2.$$

Suppose f and h are inverse elements of g, then

$$f = f \circ e = f \circ (g \circ h) = (f \circ g) \circ h = e \circ f = f.$$

3. Let a and b be logical expressions. Use truth tables to show De Morgan's laws:

(a) $\neg(a \vee b) = (\neg a) \wedge (\neg b)$

Solution:

a	b	$a \vee b$	$\neg(a \vee b)$	$\neg a$	$\neg b$	$\neg a \wedge \neg b$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

(b) $\neg(a \wedge b) = (\neg a) \vee (\neg b)$.

Solution:

a	b	$a \wedge b$	$\neg(a \vee b)$	$\neg a$	$\neg b$	$\neg a \vee \neg b$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

4. Let X, Y be nonempty sets and let $f\colon X \to Y$ be a mapping.

(a) Show that f is injective if and only it has a left-inverse; i.e., there exists a function $g\colon Y \to X$ with $g \circ f = Id_X$.

Solution: Suppose f has a left-inverse g. Let $x_1, x_2 \in X$ with $f(x_1) = f(x_2)$. We have

$$x_1 = g \circ f(x_1) = g(f(x_1)) = g(f(x_2)) = g \circ f(x_2) = x_2.$$

Hence f is injective.

Let f be injective and $x_0 \in X$. We define $g\colon Y \to X$ by $g(y) = x$, where $x \in f^{-1}(\{y\})$ if the set $f^{-1}(\{y\})$ is nonempty and $x = x_0$ else. Thus for $x \in X$,

$$g \circ f(x) \in f^{-1}(\{f(x)\}) = \{\tilde{x} \in X : f(\tilde{x}) = f(x)\} = \{x\},$$

since f is injective. Hence, $g \circ f(x) = x$, i.e. g is a left-inverse of f.

(b) Show that f is surjective if and only it has a right-inverse; i.e., there exists a function $g\colon Y \to X$ with $f \circ g = Id_Y$.

Solution: Suppose f has a right-inverse g. Let $y \in Y$ and set $x = g(y)$, then

$$f(x) = f(g(y)) = f \circ g(y) = y.$$

Hence f is surjective.

Let f be surjective. We define $g\colon Y \to X$ by $g(y) = x$, where $x \in f^{-1}(\{y\})$. Note that the set $f^{-1}(\{y\})$ is nonempty for all $y \in Y$ by surjectivity of f. Thus for $y \in Y$,

$$f \circ g(y) \in \{f(x) : x \in f^{-1}(\{y\})\} = \{y\}.$$

Hence, $f \circ g(y) = y$, i.e. g is a left-inverse of f.

5. Let $f\colon \mathbb{N} \to \mathbb{R}$. Negate the following logical expression:

$$\forall \varepsilon > 0 \; \exists n_0 \in \mathbb{N} : k, n \geq n_0 \Rightarrow |f(n) - f(k)| < \varepsilon.$$

Solution:

$$\exists \varepsilon > 0 : \forall n_0 \in \mathbb{N} \; \exists k, n \geq n_0 : |f(n) - f(k)| \geq \varepsilon.$$

6. Show using the principle of induction: It holds for all $n \in \mathbb{N}$ that

(a) $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$,

Solution: Proof by induction:

(IA) $n = 1$: $\sum_{k=1}^{1} \frac{1}{k(k+1)} = \frac{1}{1(1+1)} = \frac{1}{1+1}$.

(IS) $n \to n+1$: Suppose $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$ for some $n \in \mathbb{N}$ (IV). We obtain

$$\begin{aligned}\sum_{k=1}^{n+1} \frac{1}{k(k+1)} &= \sum_{k=1}^{n} \frac{1}{k(k+1)} + \frac{1}{(n+1)((n+1)+1)} \\ &\underset{(IV)}{=} \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \frac{n(n+2)+1}{(n+1)(n+2)} = \frac{(n+1)^2}{(n+1)(n+2)} = \frac{n+1}{n+2}\end{aligned}$$

Now the claim follows by the principle of Induction.

(b) $(b-a) \sum_{k=0}^{n} a^k b^{n-k} = b^{n+1} - a^{n+1}$.

Solution: Proof by induction:

(IA) $n = 1$: $(b-a) \sum_{k=0}^{1} a^k b^{1-k} = (b-a)(a^0 b^1 + a^1 b^0) = (b-a)(b+a) = b^{1+1} - a^{1+1}$.

(IS) $n \to n+1$: Suppose $(b-a)\sum_{k=0}^{n} a^k b^{n-k} = b^{n+1} - a^{n+1}$ for some $n \in \mathbb{N}$ (IV). We obtain

$$b^{n+2} - a^{n+2} = b(b^{n+1} - a^{n+1}) + (b-a)a^{n+1} \underset{(IV)}{=} b(b-a)\sum_{k=0}^{n} a^k b^{n-k} + (b-a)a^{n+1}$$
$$= (b-a)\left(\sum_{k=0}^{n} a^k b^{n+1-k} + a^{n+1}\right) = (b-a)\sum_{k=0}^{n+1} a^k b^{n+1-k}.$$

Now the claim follows by the principle of Induction.

7. Prove that in a theory with contradiction, any formula can be deduced. Contradiction means one has $\varphi \wedge \neg\varphi$. Written without $\wedge$ this becomes $\neg(\varphi \to \varphi)$.

Solution:

$$\begin{aligned}
& \neg(\varphi \to \varphi) \\
\text{(P2)} \quad & (\neg(\varphi \to \varphi)) \to (\neg\psi \to (\neg(\varphi \to \varphi))) \\
\text{(MP)} \quad & \neg\psi \to (\neg(\varphi \to \varphi)) \\
\text{(P4)} \quad & (\neg\psi \to (\neg(\varphi \to \varphi))) \to ((\varphi \to \varphi) \to \psi) \\
\text{(MP)} \quad & (\varphi \to \varphi) \to \psi \\
\text{(P1)} \quad & (\varphi \to \varphi) \\
\text{(MP)} \quad & \psi
\end{aligned}$$

8.* Five men with different nationalities and with different jobs live in consecutive houses on a street. The houses are painted different colors. The men have different pets and have different favorite drinks. Determine who owns a zebra and whose favorite drink is mineral water (which is one of the favorite drinks) given these clues:

1. The Englishman lives in the red house.
2. The Spaniard owns a dog.
3. The Japanese man is a painter.
4. The Italian drinks tea.
5. The Norwegian lives in the first house on the left.
6. The green house is on the right of the white one.
7. The photographer breeds snails.
8. The diplomat lives in the yellow house.
9. Milk is drunk in the middle house.
10. The owner of the green house drinks coffee.
11. The Norwegian's house is next to the blue one.
12. The violinist drinks orange juice.
13. The fox is in a house next to that of the physician.
14. The horse is in a house next to that of the diplomat.

Make a table where the rows represent the men and columns represent the colors of their houses, their jobs, their pets, and their favorite drinks and use logical reasoning to determine the correct entries in the table.