Structural Properties of Maximal Regularity

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$-A$ generator of bounded analytic C_0-semigroup on Banach space X

Definition (Maximal Regularity)

A has *maximal regularity* if $s \mapsto isR(is, A)$ $(s \neq 0)$ defines a bounded Fourier multiplier on $L_p(\mathbb{R}; X)$ for one (equiv. all) $p \in (1, \infty)$.

- Always true if X is a Hilbert space (use Plancherel's theorem)
- L. Weis: characterization in terms of $\mathcal{R}$-boundedness of $\{isR(is, A) : s \neq 0\}$ on UMD-spaces
- A bounded H^∞-calculus with $\omega_{H^\infty}(A) < \frac{\pi}{2}$ implies maximal regularity of A if X is UMD

Non-trivial positive result for maximal regularity

Theorem (L. Weis)

$-A$ generator of bounded analytic semigroup on L_p-space for $p \in (1, \infty)$ that is positive and contractive on the real line. Then A has maximal regularity.

- Generalizes to $\|T(t)\|_r \leq 1$ for $t \geq 0$ ($\|\cdot\|_r$ regular norm)
- Seems to be the only *generic* positive result known

One may ask for possible generalizations:

Problem

$-A$ generator of bounded analytic semigroup on L_p-space for $p \in (1, \infty)$ that is ~~positive and~~ contractive on the real line. Does A have maximal regularity?

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These questions are the motivating forces.

We do not know an answer to both questions. One may even generalize further:

Problem

$-A$ generator of bounded analytic semigroup on a uniformly convex UMD-space that is contractive on the real line. Does A have maximal regularity?

Problem

$-A$ generator of bounded analytic semigroup on a UMD Banach lattice that is positive on the real line. Does A have maximal regularity?

Theorem (C. Arhancet, S. F., C. Le Merdy)

A sectorial with bounded H^∞-calculus and $\omega_{H^\infty}(A) < \frac{\pi}{2}$ on super-reflexive space X, $-A \sim (T(z))$. There exists an equivalent uniformly convex norm $\|\cdot\|$ on X such that

$$\|T(t)\| \leq 1 \qquad \forall t \geq 0.$$

- X super-reflexive $\iff$ X has equivalent uniformly convex norm (P. Enflo).

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Problem

$-A$ generator of contractive semigroup on uniformly convex space. Does A have a bounded H^∞-calculus?

Theorem (S. F.)

Let $p \neq p \in (1, \infty)$. *There exists a sectorial operator* A *on* $\ell_p(\ell_q)$ *with* $\omega(A) = 0$, $-A \sim (T(z))$, *and* $T(t) \geq 0$ *for all* $t \geq 0$ *such that* A *does not have maximal regularity.*

Positivity is not sufficient on general UMD Banach lattices!

Typical approach to construct counterexamples (Schauder multiplier approach):

- $(f_m)_{m\in\mathbb{N}}$ wisely chosen bad Schauder basis for X
- $(\gamma_m)_{m\in\mathbb{N}}$ sequence of positive non-decreasing real numbers

$$D(A) = \left\{ x = \sum_{m=1}^{\infty} a_m f_m : \sum_{m=1}^{\infty} \gamma_m a_m f_m \text{ exists} \right\}$$

$$A\left(\sum_{m=1}^{\infty} a_m f_m\right) = \sum_{m=1}^{\infty} \gamma_m a_m f_m$$

$-A$ generates analytic semigroup $(T(z))_{z\in\Sigma_{\frac{\pi}{2}}}$.

Our choices for $X_p = (\oplus_{n=1}^{\infty} \ell_2^n)_{\ell_p}$ for $p \in [2, \infty)$

- $(e_m)_{m \in \mathbb{N}}$ standard basis of X_p seen as sequence space

$$f_m = \begin{cases} e_m & m \text{ odd} \\ e_{m-1} + e_{\pi(m)} & m \text{ even} \end{cases}$$

  π permutation of even numbers with $[(e_{\pi(4m+2)})] \simeq \ell_p$ and $[e_{\pi(4m)})] \simeq X_p$.

- $(\gamma_m)_{m \in \mathbb{N}}$ given by $\gamma_1 = 1$ and further recursively by

$$c_m = \frac{\gamma_{m+1} - \gamma_m}{\gamma_m}$$

  for sequence $(c_m)_{m \in \mathbb{N}}$ with $c_m \in (0, 1)$.

$(c_m)_{m\in\mathbb{N}}$ is the relative growth of $(\gamma_m)_{m\in\mathbb{N}}$.

$$c_m = \frac{\gamma_{m+1} - \gamma_m}{\gamma_m}$$

Example

$\gamma_m = p(m)$ for p polynomial of degree n. Then

$$c_m \sim \frac{n}{m}.$$

Example

$\gamma_m = 2^m$. Then

$$c_m = 1.$$

$X_p = (\oplus_{n=1}^{\infty} \ell_2^n)_{\ell_p}$ for $p \in [2, \infty)$

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2}.$$

Theorem (S. F.)

$(c_m)_{m\in\mathbb{N}}$ eventually non-increasing. TFAE:

(i) *A has maximal regularity*

(ii) *A has a bounded H^{∞}-calculus*

(iii) $(c_m)_{m\in\mathbb{N}} \in (\oplus_{n=1}^{\infty} \ell_q^n)_{\ell_\infty}$

- $p = 2$: $(\oplus_{n=1}^{\infty} \ell_q^n)_{\ell_\infty} = \ell_\infty$
- Limit case $p = \infty$: $(\oplus_{n=1}^{\infty} \ell_q^n)_{\ell_\infty} = (\oplus_{n=1}^{\infty} \ell_2^n)_{\ell_\infty}$

In this case maximal regularity is completely understood!

Interesting sequences $c_m = m^{-\alpha}$ for $\alpha \in (0,1)$. Associated $(\gamma_m)_{m\in\mathbb{N}}$ have sub-exponential but super-polynomial growth.

Corollary (S. F.)

Let $I \subset (1,\infty)$ be an interval with $2 \in I$. There exists a family $(T_p(z))_{z\in\Sigma_{\frac{\pi}{2}}}$ of consistent C_0-semigroups on $L_p(\mathbb{R})$ for $p \in (1,\infty)$ with

$$(T_p(z)) \text{ has maximal regularity} \iff p \in I.$$

The extrapolation problem for maximal regularity behaves in the worst way possible.

$X_p = (\oplus_{n=1}^{\infty} \ell_2^n)_{\ell_p}$ for $p \geq 2$

What happens with contractivity?

- $p = 2$ (Hilbert space case): always contractive
- $p = \infty$: ($X_\infty = (\oplus_{n=1}^{\infty} \ell_2^n)_{c_0}$)

$$(c_m) \notin (\oplus_{n=1}^{\infty} \ell_2^n)_{\ell_\infty} \Rightarrow -A \sim (T(t)) \text{ not contractive}$$

  (compare with Lamberton's result)
- $p \in (2, \infty)$: I do not know, but canonical choices give non-contractive semigroups, so one may wonder

$$(c_m) \notin (\oplus_{n=1}^{\infty} \ell_q^n)_{\ell_\infty} \Rightarrow -A \sim (T(t)) \text{ not contractive?}$$

Thank you for your attention!