Maximal Regularity does not extrapolate

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Operator Semigroups meet Complex Analysis, Harmonic Analysis and Mathematical Physics (Herrnhut)

Maximal regularity

Setting: $-A$ generator of C_0-semigroup $(T(t))_{t \geq 0}$ on Banach space X.

Definition

$-A$ has *maximal regularity* if for $f \in L^p((0,T);X)$ the mild solution $u(t) = \int_0^t T(t-s)f(s)\,ds$ of

$$\begin{cases} \dot{u}(t) + Au(t) & = f(t) \\ u(0) & = 0 \end{cases}$$

satisfies $u \in W^{1,p}((0,T);X) \cap L^p((0,T);D(A))$.

The connection with harmonic analysis

1. Differentiation: $u'(t) = -\int_0^t AT(t-s)f(s)\,ds + f(t)$

   maximal regularity $\Leftrightarrow$ boundedness of conv. with $\|AT(t)\| \sim \dfrac{1}{t}$

2. Fourier transform: we need boundedness of Fourier multiplier

   $$m(u) := \mathcal{F}(-AT(t)) = -AR(iu, -A) = iuR(iu, -A) - \mathrm{Id}$$

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   $$m(u) := \mathcal{F}(-AT(t)) = -AR(iu,-A) = iuR(iu,-A) - \mathrm{Id}$$

3. $X = H$ Hilbert space, $(T(t))_{t\geq 0}$ bounded holomorphic (on sector):

   $$\|iuR(iu,-A)\| \text{ bounded.}$$

   Use (operator-valued) Mihlin's multiplier theorem $\Rightarrow$ $-A$ has maximal regularity (De Simon).

$\mathcal{R}$-boundedness and multipliers

Problem: Operator-valued Mihlin characterizes Hilbert spaces (G. Pisier).

Theorem (L. Weis)

X UMD-space, $m \in C^1(\mathbb{R}\setminus\{0\}, B(X))$, $p \in (1,\infty)$. Assume that

$$\{m(t) : t \in \mathbb{R}\setminus\{0\}\} \text{ and } \{tm'(t) : t \in \mathbb{R}\setminus\{0\}\}$$

are $\mathcal{R}$-bounded. Then $Tf = \mathcal{F}^{-1}(m(\cdot)\hat{f}(\cdot))$ extends to $T \in \mathcal{B}(L^p(X))$.

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- X UMD: Hilbert transform bounded in $L^p(X)$ ($p \in (1, \infty)$).
- $\mathcal{R}$-boundedness: $r_k(t) = \operatorname{sign} \sin(2^k \pi t)$ realization of Rademachers

$$\left\| \sum_{k=1}^{n} r_k m(t_k) x_k \right\|_{L^p([0,1];X)} \leq C \left\| \sum_{k=1}^{n} r_k x_k \right\|_{L^p([0,1];X)}.$$

A characterization of maximal regularity

-A generator of bounded holomorphic C_0-semigroup on X.

$$\{itR(it, -A) : t \in \mathbb{R} \setminus \{0\}\} \ \mathcal{R}\text{-bounded} \Leftrightarrow \{T(z) : z \in \Sigma_\delta\} \ \mathcal{R}\text{-bounded}.$$

Theorem (L. Weis)

(i) *-A has maximal regularity* $\Rightarrow \{T(z) : z \in \Sigma_\delta\}$ *$\mathcal{R}$-bounded.*

(ii) *If X is UMD, then the converse holds.*

The maximal regularity problem

- -A has maximal regularity $\Rightarrow$ -A generates holomorphic C_0-semigroup on X.
- $X = H$ Hilbert space: $-A$ has maximal regularity $\Leftrightarrow$ $-A$ generates holomorphic C_0-semigroup.

Problem (Maximal regularity problem)

Which Banach spaces have this property (MRP)?

The maximal regularity problem

- -A has maximal regularity ⇒ -A generates holomorphic C_0-semigroup on X.
- $X = H$ Hilbert space: $-A$ has maximal regularity ⇔ $-A$ generates holomorphic C_0-semigroup.

Problem (Maximal regularity problem)

Which Banach spaces have this property (MRP)?

- $L^\infty[0,1]$ has (MRP).
- Kalton-Lancien: (MRP) characterizes Hilbert spaces in the class of Banach spaces with an unconditional basis.

The maximal regularity problem

- Kalton & Lancien use abstract results on perfectly homogeneous bases.
- No explicit counterexample has been known on $L^p[0,1]$ ($p \in (1,\infty) \setminus \{2\}$).

Schauder multipliers as semigroup generators

Definition

A sequence $(e_n)_{n \in \mathbb{N}} \subset X$ is called *Schauder basis* if every $x \in X$ has a unique expansion

$$x = \sum_{n=1}^{\infty} a_n e_n. \qquad (a_n \in \mathbb{C})$$

If the above series converge unconditionally, $(e_n)_{n \in \mathbb{N}}$ is called an *unconditional basis*.

For $\gamma_{n+1} \geq \gamma_n$, $-A$ generates a holomorphic C_0-semigroup, where

$$A\left(\sum_{n=1}^{\infty} a_n e_n\right) = \sum_{n=1}^{\infty} \gamma_n a_n e_n.$$

We use: $\gamma_n = 2^n$.

An explicit counterexample

- $X \not\simeq \ell^1, \ell^2, c_0$: There exist a normalized unconditional basis $(\tilde{e}_n)_{n\in\mathbb{N}}$ of X, a permutation $\pi : \mathbb{N} \to \mathbb{N}$ and $(a_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ with

$$\sum_{n=1}^{\infty} a_n \tilde{e}_{\pi(2n)} \text{ exists, but } \sum_{n=1}^{\infty} a_n \tilde{e}_{2n-1} \text{ does not (or vice versa).}$$

- $f_n = \begin{cases} \tilde{e}_n, & n \text{ odd} \\ \tilde{e}_{\pi(n)} + \tilde{e}_{n-1}, & n \text{ even} \end{cases}$ is Schauder basis for X.
- We take

$$A\Big(\sum_{n=1}^{\infty} a_n f_n\Big) = \sum_{n=1}^{\infty} 2^n a_n f_n.$$

An explicit counterexample

- $g := \sum_{n=1}^{\infty} r_n a_n \tilde{e}_{\pi(2n)}$ converges (unconditionality).
- $\mathcal{R}$-boundedness of $\{ T(t) : t \in [0,1] \}$ would imply boundedness of

$$\mathcal{T} : \sum_{n=1}^{\infty} r_n x_n \mapsto \sum_{n=1}^{\infty} r_n T(q_n) x_n$$

on closed span of Rademachers for $(q_n)_{n \in \mathbb{N}} \subset [0,1]$.

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  on closed span of Rademachers for $(q_n)_{n \in \mathbb{N}} \subset [0,1]$.
- Take $q_n = \frac{\log 2}{2^{2n-1}}$. Short calculation:

$$\mathcal{T}(g) = \frac{1}{4} \sum_{n=1}^{\infty} a_n r_n \tilde{e}_{\pi(2n)} - a_n r_n \tilde{e}_{2n-1}$$

  Thus by unconditionality, $\sum_{n=1}^{\infty} a_n \tilde{e}_{2n-1}$ converges. Contradiction!

The case of L^p-spaces

$$X_p := (\oplus_{n=1}^{\infty} \ell_2^n)_{\ell^p}$$

is isomorphic to ℓ^p for $p \in (1, \infty)$ (variant of the Schröder-Bernstein argument, Pełczyński's decomposition technique).

- Use the unit standard basis $(\tilde{e}_n)_{n \in \mathbb{N}}$ for counterexamples.
- This can be done consistently in the X_p-scale $(1 < p < \infty)$.
- Embed this in $L^p(\mathbb{R})$ consistently using Rademachers.

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Theorem (SF (2012))

There exists a family $(T_p(t))_{t \geq 0}$ of consistent holomorphic C_0-semigroups on $L^p(\mathbb{R})$ $(p \in (1, \infty))$ with

$$(T_p(t))_{t \geq 0} \text{ has maximal regularity } \Leftrightarrow p = 2.$$

Thank you for your attention!