Regularity of Semigroups via the Asymptotic Behaviour at Zero

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Warm-up: Holomorphic Semigroups

Definition

A semigroup $(T(t))_{t \geq 0}$ on a complex Banach space X is called *holomorphic* if it has a holomorphic extension to

$$\Sigma_\phi := \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \phi\} \quad (\phi > 0)$$

which is bounded on

$$\Sigma_\phi \cap \{z \in \mathbb{C} : |z| \leq 1\}.$$

A Classical Characterization via the Behaviour at Zero

Theorem (T. Kato)

Let $(T(t))_{t \geq 0}$ be a C_0-semigroup. Assume that

$$\limsup_{t \downarrow 0} \| T(t) - \mathrm{Id} \| < 2.$$

Then $(T(t))$ extends to a holomorphic semigroup.

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Theorem (A. Pazy)

Let $(T(z))_{z \in \Sigma}$ be a contractive holomorphic C_0-semigroup on a uniformly convex Banach space X. Then

$$\limsup_{t \downarrow 0} \| T(t) - \mathrm{Id} \| < 2.$$

A Counterexample: One needs a Geometric Condition

The Gaussian semigroup

$$T(t)f(x) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} f(x-y)e^{-y^2/4t}\, dy$$

on $X = L^1(\mathbb{R})$ is contractive and holomorphic and satisfies

$$\limsup_{t \downarrow 0} \| T(t) - \mathrm{Id} \| = 2.$$

A Complete Characterization

Theorem

A C_0-semigroup $(T(t))_{t\geq 0}$ is holomorphic if and only if there exists a polynomial P such that

$$\limsup_{t\downarrow 0} \|P(T(t))\| < \|P\|_D \,,$$

where $\|P\|_D := \sup_{|z|\leq 1} |P(z)|$.

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where $\|P\|_D := \sup_{|z|\leq 1} |P(z)|$.

In particular, for $P(z) = (z-1)^N$ we obtain that the semigroup $(T(t))_{t\geq 0}$ is holomorphic if

$$\limsup_{t\downarrow 0} \left\|(T(t)-\mathrm{Id})^N\right\|^{1/N} < 2.$$

Open Question

Open question: A generates $(T(z))$ holomorphic, C_0. Is there always an $N \in \mathbb{N}$ with

$$\limsup_{t \downarrow 0} \left\| (T(t) - \mathrm{Id})^N \right\|^{1/N} < 2?$$

True if: X Hilbert space and generator A has bounded H^∞-calculus.

Application: Holomorphy Extrapolates

Theorem

- $(T_2(t))_{t \geq 0}$ *and* $(T_p(t))_{t \geq 0}$ *two consistent semigroups on* L^2 *and* L^p*.*
- $(T_2(t))_{t \geq 0}$ C_0*, holomorphic and* $(T_p(t))_{t \geq 0}$ *locally bounded.*

Then: Induced semigroups $(T_q(t))$ *on* L^q *are* C_0 *& holomorphic.*

Application: Holomorphy Extrapolates

Theorem

- *$(T_2(t))_{t\geq 0}$ and $(T_p(t))_{t\geq 0}$ two consistent semigroups on L^2 and L^p.*
- *$(T_2(t))_{t\geq 0}$ C_0, holomorphic and $(T_p(t))_{t\geq 0}$ locally bounded.*

Then: Induced semigroups $(T_q(t))$ on L^q are C_0 & holomorphic.

Proof.

- $\exists$ polynomial P of degree n, $\|P\|_D = 1$ and $\rho < 1$, $\delta > 0$ such that

$$\|P(T_2(t))\| \leq \rho \quad \text{for } 0 < t < \delta.$$

- Riesz-Thorin for θ with $\frac{1}{q} = \frac{\theta}{2} + \frac{1-\theta}{p}$ $(M := \sup_{t\in[0,1]} \|T_p(t)\|)$

$$\left\|P^N(T_q(t))\right\| \leq \left\|P^N(T_2(t))\right\|^{\theta} \left\|P^N(T_p(t))\right\|^{1-\theta} \leq \rho^{\theta N}(M(Nn+1))^{1-\theta}$$

□

Remarks on the Extrapolation Result

The same proof works for more general interpolation methods of exponent $\theta \in (0, 1)$:
Replace the L^2 and L^p by interpolation couple (X_1, X_2) of Banach spaces and L^q by real interpolation space $(X_1, X_2)_{\theta,q}$ for $\theta \in (0, 1)$ and $q \in [1, \infty)$.

Generalizations to $\mathcal{R}$-Boundedness / Maximal Regularity

Theorem (S.F.)

Let $(T(t))_{t \geq 0}$ be a strongly continuous semigroup on a Banach space X with $\mathcal{R}\{T(t) : 0 < t < 1\} < \infty$. Then $(T(t))$ is $\mathcal{R}$-analytic if and only if there exists a polynomial P such that

$$\lim_{\epsilon \downarrow 0} \mathcal{R}\{P(T(t)) : 0 < t < \epsilon\} < \|P\|_D .$$

Extrapolation of $\mathcal{R}$-Boundedness / Maximal Regularity

Theorem (S.F.)

- *X_1 and X_2 B-convex.*
- *$(T_1(t))_{t\geq 0}$ and $(T_2(t))_{t\geq 0}$ two consistent semigroups on interpolation couple (X_1, X_2).*
- *$\mathcal{F}$ interpolation functor of exponent $\theta \in (0, 1)$.*
- *$X = \mathcal{F}((X_1, X_2))$.*
- *$(T_1(t))_{t\geq 0}$ C_0 and $\mathcal{R}$-analytic.*
- *$\mathcal{R}\{T_2(t) : 0 < t < 1\} < \infty$.*

Then: Interpolated semigroup $(T(z))_{z\in\Sigma}$ on X is $\mathcal{R}$-analytic.

Application: X_1 Hilbert space and $(T_2(t))$ has Gaussian estimates.

Application: Zero-Two Laws

Kato's characterization can be used to prove a zero-two law for C_0-groups. Let $(U(t))_{t \in \mathbb{R}}$ be a C_0-group with

$$\limsup_{t \downarrow 0} \| U(t) - \mathrm{Id} \| < 2.$$

Then $(U(t))$ is holomorphic, a fortiori immediately norm continuous. So

$$\| U(t) - \mathrm{Id} \| = \| U(-1)(U(t+1) - U(1)) \| \xrightarrow[t \to 0]{} 0.$$

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Our generalization allows us to prove a 0-2 law for cosine families.

Definition

A cosine family C is a strongly continuous mapping $C : \mathbb{R} \to \mathcal{L}(X)$ satisfying $C(0) = \mathrm{Id}$ and $2C(t)C(s) = C(t+s) + C(t-s)$ $(t, s \in \mathbb{R})$.

A Zero-Two Law for Cosine Families

Theorem (S.F.)

Let $(C(t))_{t\in\mathbb{R}}$ *be a cosine family on a UMD space* X*. Then*

$$\limsup_{t\downarrow 0} \|C(t) - \mathrm{Id}\| \in \{0\} \cup [2, \infty).$$

A Zero-Two Law for Cosine Families

Theorem (S.F.)

Let $(C(t))_{t\in\mathbb{R}}$ be a cosine family on a UMD space X. Then

$$\limsup_{t\downarrow 0} \|C(t) - \mathrm{Id}\| \in \{0\} \cup [2, \infty).$$

Proof.

- Assume $\|C(t) - \mathrm{Id}\| \leq \rho$ for $\rho < 2$ and $t < \delta$.
- Fattorini: we may assume $C(t) = \frac{U(t)+U(-t)}{2}$, $(U(t))_{t\in\mathbb{R}}$ C_0-group.
- Let $P(z) = \frac{1}{2}(z-1)^2$. We estimate $\left\|P^N(U(t))\right\|$ by

$$\|U(Nt)\| \left\| \left[\frac{U(t)+U(-t)}{2} - \mathrm{Id} \right]^N \right\| \leq (M^{1/N} e^{\omega t} \rho)^N \leq \tilde{\rho}^N < 2^N.$$

Thus $(U(t))$ is norm continuous and $\lim_{t\downarrow 0} \|C(t) - \mathrm{Id}\| = 0$. □

References

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Thank you for your attention!