Universität Ulm
Discussion of solutions:
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Dr. M. Kunze
M. Gerlach
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Exercises for Applied Analysis

Sheet 1

1. Consider the following definitions of (M, d) and decide whether (M, d) is a metric space or not.

(a) Let Ω be a finite set and $M := \mathcal{P}(\Omega)$ its power set, i.e., M is the set of all subsets of Ω. For a set $A \in \mathcal{P}(M)$ we denote by $\#A$ the number of its elements.
Let $d : M \times M \to [0, \infty)$ be given by

$$d(A, B) := \#(A \triangle B) := \#((A \backslash B) \cup (B \backslash A)),$$

i.e., $d(A, B)$ is the number of elements in the so-called symmetric difference of A and B.

(b) Let M be the set of all addresses in Ulm (that can be reached by car). We define $d(x, y)$ as the length of the shortest way from x to y that you can drive by car.

2. Let $d \in \mathbb{N}$, $d \geq 2$. For which $0 < p < 1$ does

$$\|x\|_p := \left(\sum_{k=1}^{d} |x_k|^p \right)^{\frac{1}{p}} \quad (x \in \mathbb{K}^d)$$

define a norm on $\mathbb{K}^d$?

3. (a) Prove Hölder's inequality for $p = 1$ and $q = \infty$.

(b) Prove Minkowski's inequality for $p = 1$ and for $p = \infty$.

4. Let $1 \leq p, q \leq \infty$. Prove that $\|\cdot\|_p$ and $\|\cdot\|_q$ are equivalent norms on $\mathbb{K}^d$.

Hint: Use a direct argument if either $p = \infty$ or $q = \infty$. Otherwise, apply Hölder's inequality with $p' = q/p$ and $q' = q/(q - p)$.

5. On ℓ^∞, the space of all bounded sequences, define

$$\|\mathbf{x}\|_0 := \sum_{k=1}^{\infty} 2^{-k} |x_k|.$$

(a) Show that $\|\cdot\|_0$ defines a norm on ℓ^∞.

(b) Let $\mathbf{x}_n$ be a sequence in ℓ^∞ such that

$$\sup\{\|\mathbf{x}_n\|_\infty : n \in \mathbb{N}\} < \infty,$$

i.e., $\mathbf{x}_n = (x_k^{(n)})_{k \in \mathbb{N}}$ is a $\|\cdot\|_\infty$-bounded sequence of bounded sequences.
Prove that $\mathbf{x}_n$ converges to some $\mathbf{x} = (x_k)_{k \in \mathbb{N}}$ in $(\ell^\infty, \|\cdot\|_0)$ if and only if

$$\lim_{n \to \infty} x_k^{(n)} = x_k \text{ for all } k \in \mathbb{N}.$$

(c) Find $\mathbf{x}$, $\mathbf{x}_n \in \ell^\infty$ such that

$$\lim_{n\to\infty} x_k^{(n)} = x_k \text{ for all } k \in \mathbb{N}$$

but $\|\mathbf{x}_n - \mathbf{x}\|_0 \not\to 0$ as $n \to \infty$.

6. Let $1 \leq p < q \leq \infty$. Prove that $\ell^p \neq \ell^q$.

Hint: You may use that $\sum_{k=1}^\infty k^{-\alpha}$ converges if and only if $\alpha > 1$.