Exercises for Applied Analysis

Sheet 10

43. Compute the 3-dimensional Lebesgue measure of the simplex

$$S := \{(x, y, z) \in \mathbb{R}^3 : x, y, z \geq 0,\ x + y + z \leq 1\}.$$

44. In this exercise we calculate the probability to catch the tram at the train station when arriving with bus number 3. This is possible if and only if the delay of the bus is strictly less than the delay of the tram (and if one changes with the speed of light). Denote by X the delay of the tram and by Y the delay of the bus (in minutes). The distribution of the random vector (X, Y) has the following density with respect to λ_2:

$$f(x, y) := \frac{1}{6} \exp\Big(-\frac{3x + 2y}{6}\Big) \mathbb{1}_{[0,\infty)}(x) \mathbb{1}_{[0,\infty)}(y).$$

Compute the probability of the following events.

(a) It is possible to catch the tram.

(b) Together, tram and bus have more than 6 minutes delay.

(c) The delay of the tram is more than three times greater than the delay of the bus.

45. Compute – if well-defined – the iterated integrals

$$\int_{\Omega_1} \int_{\Omega_2} f(x, y)\, \mathrm{d}\mu_2(y)\, \mathrm{d}\mu_1(x) \quad \text{and} \quad \int_{\Omega_2} \int_{\Omega_1} f(x, y)\, \mathrm{d}\mu_1(x)\, \mathrm{d}\mu_2(y)$$

for the following measure spaces $(\Omega_i, \Sigma_i, \mu_i)$, $i = 1, 2$, and measurable functions $f : \Omega_1 \times \Omega_2 \to \mathbb{R}$. Moreover, decide whether f is $(\mu_1 \otimes \mu_2)$-integrable.

(a) $\Omega_1 = \Omega_2 = \mathbb{N}$, $\Sigma_1 = \Sigma_2 = \mathscr{P}(\mathbb{N})$, $\mu_1 = \mu_2 = \zeta$,

$$f(k, l) := \begin{cases} 1 & k = l \\ -1 & k = l + 1 \\ 0 & \text{else.} \end{cases}$$

(b) $\Omega_1 = \Omega_2 = \mathbb{R}$, $\Sigma_1 = \Sigma_2 = \mathscr{B}(\mathbb{R})$, $\mu_1 = \mu_2 = \lambda$,

$$f(x, y) := \begin{cases} x \cdot y & |xy| < 1 \\ 0 & \text{else.} \end{cases}$$

(c) $\Omega_1 = \mathbb{N}$, $\Omega_2 = (0, 1)$, $\Sigma_1 = \mathscr{P}(\mathbb{N})$, $\Sigma_2 = \mathscr{B}(0, 1)$, $\mu_2 = \lambda$,

$$\mu_1(A) := \sum_{k \in A} k, \quad f(k, t) := \Big(-\frac{t}{2}\Big)^{k-1}.$$

46. Prove that

$$\text{R–}\int_0^\infty \frac{\sin t}{t}\,\mathrm{d}t = \frac{\pi}{2}$$

by completing the following steps:

(a) Show that

$$\int_0^\infty \exp(-tx)\,\mathrm{d}x = \frac{1}{t}$$

for all $t > 0$.

(b) Use Fubini's theorem to conclude that

$$\int_0^n \frac{\sin t}{t}\,\mathrm{d}t = \int_0^\infty \int_0^n \exp(-tx)\sin(t)\,\mathrm{d}t\,\mathrm{d}x$$

for all $n \in \mathbb{N}$.

(c) Show that

$$\int_0^n \exp(-tx)\sin(t)\,\mathrm{d}t = \frac{1 - \exp(-xn)(\cos n + x\sin n)}{1 + x^2}$$

for all $n \in \mathbb{N}$.

Hint: Use that $\sin t = \operatorname{Im}(e^{it})$ and $\cos t = \operatorname{Re}(e^{it})$.

(d) Use the dominated convergence theorem to prove that

$$\text{R–}\int_0^\infty \frac{\sin t}{t}\,\mathrm{d}t := \lim_{n\to\infty}\int_0^n \frac{\sin t}{t}\,\mathrm{d}t = \int_0^\infty \frac{1}{1+x^2}\,\mathrm{d}x = \frac{\pi}{2}.$$