Universität Ulm

Discussion of solutions:
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Dr. M. Kunze
M. Gerlach
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Exercises for Applied Analysis

Sheet 11

47. Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space and $X, Y \in L^2(\Omega, \Sigma, \mathbb{P})$.

(a) Prove that if X and Y are independent, then $\operatorname{Cov}(X, Y) = 0$.

(b) Let (X, Y) be equidistributed on the unit disc

$$\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}.$$

Show that $\operatorname{Cov}(X, Y) = 0$ but X and Y are not independent.

48. Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space, X and Y be independent random variables such that X has continuous distribution, i.e. $\mu_X(\{a\}) = 0$ for all $a \in \mathbb{K}$.

Prove that $\mathbb{P}(X = Y) = 0$.

49. Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space and $(X_n)_{n \in \mathbb{N}}$ be a sequence of independent random variables on Ω.

(a) Prove that the sequence X_n either converges almost surely or diverges almost surely.
Hint: Convince yourself that

$$\{\omega \in \Omega : X_n(\omega) \text{ converges}\} = \bigcap_{k \in \mathbb{N}} \bigcup_{n_0 \geq k} \bigcap_{n,m \geq n_0} \left\{ |X_n - X_m| \leq \frac{1}{k} \right\}$$

and check that it belongs to the tail-σ-algebra.

(b) Assume that $X_n \to X$ a.s. Prove that X is a.s. constant.
Hint: Show that X is measurable w.r.t. the tail-σ-algebra.

50. Decide for the following normed spaces $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ and the linear maps $T : E \to F$ whether $T \in \mathscr{L}(E, F)$. If so, determine $\|T\|_{\mathscr{L}(E,F)}$.

(a) $E = F = C([0, 1])$, $\|f\|_E = \|f\|_F = \|f\|_\infty$,

$$(Tf)(t) := \int_0^t f(s)\, \mathrm{d}s$$

(b) $E = C^1([0, 1])$, the space of continuously differentiable functions $f : [0, 1] \to \mathbb{R}$, $F = C([0, 1])$, $\|f\|_E = \|f\|_F = \|f\|_\infty$ and $Tf = f'$.

(c) $E = (L^p(\Omega, \Sigma, \mu), \|\cdot\|_p)$ for some measure space (Ω, Σ, μ), $(F, \|\cdot\|_F) = (\mathbb{K}, |\cdot|)$ and

$$Tf = \int_\Omega fg\, \mathrm{d}\mu$$

where $g \in L^q(\Omega, \Sigma, \mu)$ and $\frac{1}{p} + \frac{1}{q} = 1$.
Hint: The function $f := \operatorname{sign} g \cdot |g|^{q-1}$ belongs to L^p.