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Exercises for Applied Analysis

Sheet 12

51. Let $(H, (\cdot \mid \cdot))$ be an inner product space. We say that a sequence $(x_n) \subset H$ *converges weakly* to $x \in H$ and write $x_n \rightharpoonup x$ if $(x_n \mid y) \to (x \mid y)$ for all $y \in H$.

(a) Prove that $x_n \rightharpoonup x$ and $x_n \rightharpoonup x'$ implies that $x = x'$, i.e. the weak limit is unique.

(b) Prove that $x_n \to x$ if and only if $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$.

(c) Give an example of a sequence (x_n) with $x_n \rightharpoonup 0$ but $x_n \not\to 0$.

52. Give an example of

(a) vectors $x, y \in \mathbb{R}^2$ with $\|x+y\|_p^2 + \|x-y\|_p^2 \neq 2\|x\|_p^2 + 2\|y\|_p^2$ for an arbitrary $p \in (1,\infty)\backslash\{2\}$ and conclude that $(\mathbb{R}^2, \|\cdot\|_p)$ is not a Hilbert space.

(b) a Banach space X, a closed linear subspace $K \subset X$ and vectors $x \in X$, $y_1, y_2 \in K$ such that $y_1 \neq y_2$ and $\|x - y_1\| = \|x - y_2\| = \min\{\|x - y\| : y \in K\}$.

Hint: Try $(X, \|\cdot\|) = (\mathbb{R}^2, \|\cdot\|_\infty)$.

(c) a measurable space (Ω, Σ) and two probability measures $\mathbb{P}_1, \mathbb{P}_2$ such that $\mathbb{P}_1 \ll \mathbb{P}_2$ but $\mathbb{P}_2 \not\ll \mathbb{P}_1$.

53. In this exercise, we want to describe the number of free seats in the mensa (aka southern glass palace) between 11:30 a.m. and 1:30 p.m. approximately by a polynomial of degree 2. To this end, Markus and I had to go to lunch for n times and count them. We got n tupels (t, x) of a timestamp t (minutes after 11:30 a.m. we started counting) and the number of free seats x at time t. Let us denote them by $(t_1, x_1), \ldots, (t_n, x_n)$.

Now, we want to determine a vector $\alpha = (\alpha_0, \alpha_1, \alpha_2)^T \in \mathbb{R}^3$ such that the function

$$f_\alpha(t) := \alpha_0 + \alpha_1 t + \alpha_2 t^2$$

fits best to the data we collected. More precisely, we want to minimize the sum of squared errors

$$\sum_{k=1}^{n} (f_\alpha(t_k) - x_k)^2. \tag{1}$$

(a) Denote by $A = (a_{ij})$ the $n \times 3$-matrix

$$\begin{pmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ \vdots & \vdots & \vdots \\ 1 & t_n & t_n^2 \end{pmatrix}$$

and prove that a vector $\alpha \in \mathbb{R}^3$ minimizes the expression in (1) if and only if $A^T A \alpha = A^T x$ where $x = (x_1, \ldots, x_n)^T$.

Hint: Orthogonal projection.

(b) The numbers we noted down are listed in the following table.

time (minutes after 11:30 a.m.)	t	20	42	20	53	7	79
number of free seats	x	21	19	15	75	323	283

Determine a vector $a \in \mathbb{R}^3$ that minimizes the expression in (1).

54. Consider the Lebesgue measure λ and the counting measure ζ on $([0,1], \mathscr{B}([0,1]))$. Show that $\lambda \ll \zeta$ however there is no measurable function $h : [0,1] \to [0,\infty)$ such that $\lambda(A) = \int_A h \, \mathrm{d}\zeta$ for all $A \in \mathscr{B}$.